AALTO UNIVERSITY
Systems analysis laboratory

Mat-2.4136 Special Topics in Decision Making: Aggregation functions
Matteo Brunelli

Exercise 4: Aggregation based on normalized monotone measures

1. Given a capacity μ on N its *dual* is defined by $\mu^*(A) = 1 - \mu(A^c)$, for all $A \subseteq N$ with $A^c = N \setminus A$. A capacity is *self-dual* if and only if it is equal to its dual, i.e. $\mu(A) + \mu(A^c) = 1$

   - Show that the set of self-dual measures is convex
   - Show that the set of additive measures is convex

2. Consider the following measure on the set of criteria $\{1, 2, 3\}$

$$\begin{aligned}
&\mu(\emptyset) = 0 \\
&\mu(\{1\}) = 0.3 \quad \mu(\{2\}) = 0.4 \quad \mu(\{3\}) = 0.2 \\
&\mu(\{1,2\}) = 0.7 \quad \mu(\{1,3\}) = 0.8 \quad \mu(\{2,3\}) = 0.6 \\
&\mu(\{1,2,3\}) = 1
\end{aligned}$$

   Which is the most important criterion? You need to calculate the Shapley values $\phi(1), \phi(2), \phi(3)$.
   Do 1 and 2, positively or negatively interact? Calculate the interaction index $I_{1,2}$.

3. The Choquet integral of $\mathbf{x}$ w.r.t. a capacity μ is

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)}) \mu(\{\sigma(i), \ldots, \sigma(n)\})$$

   where σ permutes $\{1, \ldots, n\}$ such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$ and $x_{\sigma(0)} = 0$.
   Consider the Choquet integral of $\mathbf{x} = (x_1, x_2, x_3, x_4)$. For what capacity does it reduce to the:

   - Median function?
   - Arithmetic mean of the greatest 2 components of $\mathbf{x}$
   - OWA function with weights $\mathbf{w} = (0.1, 0.2, 0.4, 0.3)$

4. The Sugeno integral of $\mathbf{x}$ w.r.t. a capacity μ is

$$\mathcal{S}_\mu \mathbf{x}) = \bigvee_{i=1}^{n} \left(x_{\sigma(i)} \wedge \mu(\{\sigma(i), \ldots, \sigma(n)\}) \right)$$

   where σ permutes $\{1, \ldots, n\}$ such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$.

Calculate the Sugeno integral of the following vector $\mathbf{x} = (0.8, 0.2, 0.4, 0.6)$ considering a normalized monotone measure with

$$\mu(\{1\}) = 0.05 \quad \mu(\{2\}) = 0.1 \quad \mu(\{3\}) = 0.12 \quad \mu(\{4\}) = 0.07$$
$$\mu(A) = 0.4 \ \forall A \subseteq \{1, 2, 3, 4\} \text{ with } |A| = 2$$
$$\mu(\{1, 2, 3\}) = 0.7, \quad \mu(\{1, 2, 4\}) = 0.8, \quad \mu(\{1, 3, 4\}) = 0.5, \quad \mu(\{2, 3, 4\}) = 0.6,$$

Interpret graphically what happens.
Hint: plot the points $x_{\sigma(i)}$ (non-decreasing sequence) and $\mu(\{x_{\sigma(i)}, \ldots, x_{\sigma(n)}\}$ (non-increasing sequence) against the values $1, 2, 3, 4$. This should help you to visualize what it does, so that you do not forget the formula.

5. Prove the truth or the falsity of the following:

$$\mathcal{S}_\mu(\mathbf{x}) = median(x_1, \ldots, x_n, \mu(\{x_{\sigma(2)}, \ldots, x_{\sigma(n)}\}), \mu(\{x_{\sigma(3)}, \ldots, x_{\sigma(n)}\}), \ldots, \mu(\{x_{\sigma(n)}\}))$$

Hint: The previous exercise helps a lot to solve this. However, I shall not accept geometric proof but only non-geometric ones with logical argumentations.

6. The idea of the final evaluation in this course, is that there are three criteria 1=exam, 2=homeworks, 3=presentations. Each of them will take values in $[0, 1]$, e.g. $x_1, x_2, x_3 \in [0, 1]$. Please come out with a proposal to use the Choquet or the Sugeno integral to evaluate yourself. You must be able to justify it by stating the Shapley values of the three criteria and their interactions.