MS-E2112 Multivariate Statistical Analysis (5cr)
Lecture 7: Multiple Correspondence Analysis

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Multiple Correspondence Analysis

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Multiple correspondence analysis (MCA) is an extension of bivariate correspondence analysis to more than 2 variables.

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Example: Gender, Civil Status and Education

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In this lecture, we consider an example where we examine dependencies of categorical variables gender, civil status and education.

Frequency Tables

Contingency Tables

We consider a sample of size n described by P qualitative variables $Y_1, ..., Y_P$. The variable Y_p has K_p modalities (categories), and $\sum_{p=1}^{P} K_p$ is the total number of the categories. The number of individuals having the modality l of the variable Y_p is denoted by n_{pl}. We set a variable $x_{ipl} = 1$ if individual i has modality l of Y_p, and we set $x_{ipl} = 0$ otherwise. Now

$$\sum_{l=1}^{K_p} n_{pl} = n,$$

and

$$\sum_{p=1}^{P} \sum_{l=1}^{K_p} n_{pl} = nP.$$

Table of Dummy Variables

The table of K_p dummy variables associated with variable Y_p.

	1	2	$\cdots$	K_p	
1	x_{1p1}	x_{1p2}	$\cdots$	x_{1pK_p}	1
2	x_{2p1}	x_{2p2}	$\cdots$	x_{2pK_p}	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	x_{np1}	x_{np2}	$\cdots$	x_{npK_p}	1
	n_{p1}	n_{p2}	$\cdots$	n_{pK_p}	n

Table: Table of dummy variables

Complete Disjunctive Table

Now we introduce the $n \times K$ table/matrix $X = [X_1, \ldots, X_P]$, called the complete disjunctive table.

	X_1			$\cdots$	X_P			
	X_{11}	$\cdots$	X_{1K_1}	$\cdots$	X_{P1}	$\cdots$	X_{PK_P}	$\sum_{p=1}^{P}\sum_{l=1}^{K_p} x_{ipl}$
1	x_{111}	$\cdots$	x_{11K_1}	$\cdots$	x_{1P1}	$\cdots$	x_{1PK_P}	P
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	x_{i11}	$\cdots$	x_{i1K_1}	$\cdots$	x_{iP1}	$\cdots$	x_{iPK_P}	P
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	x_{n11}	$\cdots$	x_{n1K_1}	$\cdots$	x_{nP1}	$\cdots$	x_{nPK_P}	P
$\sum_{i=1}^{n} x_{ipl}$	n_{11}	$\cdots$	n_{1K_1}	$\cdots$	n_{P1}	$\cdots$	n_{PK_P}	nP

Table: Complete disjunctive table

Example: Gender, Civil Status and Education

We consider a sample of 4 individuals and 3 variables — $n = 4, P = 3$.

- Variable X_1 gender has two modalities/categories — male (1) and female (2).
- Variable X_2 civil status has three modalities — single (1), married (2), divorced/widowed (3).
- Variable X_3 education has two modalities — low education (1), at least high school diploma (2).

Now $K = K_1 + K_2 + K_3 = 2 + 3 + 2 = 7$.

Example: Gender, Civil Status and Education

We display the gender, civil status and education data as a complete disjunctive table.

	X_{11}	X_{12}	X_{21}	X_{22}	X_{23}	X_{31}	X_{32}	$\sum_{p=1}^{7}\sum_{l=1}^{K_p} x_{ipl}$
1	0	1	1	0	0	1	0	3
2	0	1	1	0	0	0	1	3
3	1	0	0	0	1	1	0	3
4	0	1	0	1	0	0	1	3
$\sum_{i=1}^{n} x_{ipl}$	1	3	2	1	1	2	2	12

Table: Complete disjunctive table

- The first individual is female, single, and has low education.
- The third individual is male, divorced/widowed, and has low education.

Multiple Correspondence Analysis

Bivariate correspondence analysis is now applied to the complete disjunctive table!

Relative Frequency Tables

From the complete disjunctive table, it is straightforward to compute the associated relative frequency table (F), where the elements of the complete disjunctive table are divided by the total sum nP leading to

$$f_{ipl} = \frac{x_{ipl}}{nP} \;\; (i = 1, \ldots, n; p = 1, \ldots, P; l = 1, \ldots, K_p).$$

The marginal relative frequencies are computed as

$$f_{i..} = \frac{1}{n} \; (i = 1, \ldots, n) \text{ and } f_{.pl} = \frac{n_{pl}}{nP} \; (p = 1, \ldots, P; l = 1, \ldots, K_p).$$

Example: Gender, Civil Status and Education

We display the gender, civil status and education data as a complete disjunctive table.

	X_{11}	X_{12}	X_{21}	X_{22}	X_{23}	X_{31}	X_{32}	$f_{i..}$
1	0	$\frac{1}{12}$	$\frac{1}{12}$	0	0	$\frac{1}{12}$	0	$\frac{1}{4}$
2	0	$\frac{1}{12}$	$\frac{1}{12}$	0	0	0	$\frac{1}{12}$	$\frac{1}{4}$
3	$\frac{1}{12}$	0	0	0	$\frac{1}{12}$	$\frac{1}{12}$	0	$\frac{1}{4}$
4	0	$\frac{1}{12}$	0	$\frac{1}{12}$	0	0	$\frac{1}{12}$	$\frac{1}{4}$
$f_{.pl}$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	1

Table: Relative frequency table

Row Profiles

Row Profiles

The idea behind MCA, like in bivariate correspondence analysis, is to apply PCA on one hand to the row profiles, and on the other hand to the column profiles of the relative frequencies table F. The coordinate pl of the row profile $l_i(1 \times K)$ associated with individual i is given as

$$(l_i)_{pl} = \frac{f_{ipl}}{f_{i..}} = \frac{x_{ipl}}{P}, \qquad i = 1, \ldots, n.$$

The n row profiles weighted equally $(1/n)$ compose a point cloud in $\mathbb{R}^K$ with a center given by the relative marginal profile

$$G_l = (\frac{n_{11}}{nP}, \ldots, \frac{n_{1K_1}}{nP}, \ldots, \frac{n_{P1}}{nP}, \ldots, \frac{n_{PK_P}}{nP}).$$

Example: Gender, Civil Status and Education

The row profiles of the gender, civil status and education data is given as follows.

	X_{11}	X_{12}	X_{21}	X_{22}	X_{23}	X_{31}	X_{32}	
1	0	$\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{1}{3}$	0	1
2	0	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	$\frac{1}{3}$	1
3	$\frac{1}{3}$	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0	1
4	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	1

Table: Row profiles

Row Profiles

Intuitively, the distance between two individuals is small if they have many modalities in common, and the distance between the individual i and the center increases as the modalities taking by the individual i becomes rare ($x_{ipl} = 1$ for n_{pl} small).

Row Profiles

More formally, the chi-square distances between two row profiles l_{i_1} and l_{i_2} can be given as

$$d^2(l_{i_1}, l_{i_2}) = \sum_{p=1}^{P} \sum_{l=1}^{K_P} \frac{1}{f_{.pl}} ((l_{i_1})_{pl} - (l_{i_2})_{pl})^2$$

$$= \frac{n}{P} \sum_{p=1}^{P} \sum_{l=1}^{K_P} \frac{1}{n_{pl}} (x_{i_1 pl} - x_{i_2 pl})^2.$$

Example: Gender, Civil Status and Education

The distance between individual 1 and individual 2 is

$$(\frac{n}{P}\sum_{p=1}^{P}\sum_{k=1}^{K_P}\frac{1}{n_{pl}}(x_{i_1pl}-x_{i_2pl})^2)$$

$$=(\frac{4}{3}\sum_{p=1}^{3}\sum_{k=1}^{K_P}\frac{1}{n_{pl}}(x_{i_1pl}-x_{i_2pl})^2)$$

$$=(\frac{4}{3}(1(0-0)^2+\frac{1}{3}(1-1)^2+\frac{1}{2}(1-1)^2+1(0-0)^2+1(0-0)^2+\frac{1}{2}(1-0)^2+\frac{1}{2}(0-1)^2))$$

$$=\frac{4}{3}\approx 1.33.$$

Column Profiles

Column Profiles

The coordinate i of the column profile c_{pl} $(n \times 1)$ associated with the modality l of Y_p is given as

$$(c_{pl})_i = \frac{f_{ipl}}{f_{.pl}} = \frac{x_{ipl}}{n_{pl}}, \qquad p = 1, \ldots, P; l = 1, \ldots, K_p.$$

The weight of each column profiles is proportional to its frequency and given by $f_{.pl} = \frac{n_{pl}}{nP}$. The K column profiles compose a point cloud in $\mathbb{R}^n$ with the center given by the relative marginal profile $G_c = (\frac{1}{n}, \ldots, \frac{1}{n})$.

Example: Gender, Civil Status and Education

The column profiles of the gender, civil status and education is given as follows.

	X_{11}	X_{12}	X_{21}	X_{22}	X_{23}	X_{31}	X_{32}
1	0	$\frac{1}{3}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0
2	0	$\frac{1}{3}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
3	1	0	0	0	1	$\frac{1}{2}$	0
4	0	$\frac{1}{3}$	0	1	0	0	$\frac{1}{2}$
	1	1	1	1	1	1	1

Table: Column profiles

Column Profiles

Intuitively, the χ^2 distance between two modalities is small if the same individuals take these two modalities together, and the distance between the modality l of Y_p and the center increases as the modality becomes more rare (n_{pl} small).

Column Profiles

More formally, the chi-square distances between two column profiles $c_{p_1 l_1}$ and $c_{p_2 l_2}$ can be given as

$$d^2(c_{p_1 l_1}, c_{p_2 l_2}) = \sum_{i=1}^{n} \frac{1}{f_{i..}}((c_{p_1 l_1})_i - (c_{p_2 l_2})_i)^2$$

$$= n \sum_{i=1}^{n} (\frac{x_{ip_1 l_1}}{n_{p_1 l_1}} - \frac{x_{ip_2 l_2}}{n_{p_2 l_2}})^2.$$

Example: Gender, Civil Status and Education

The distance between modality 1 of Y_1 (male) and modality 2 of Y_2 (married) is

$$\sum_{i=1}^{n} \frac{1}{f_{i..}}((c_{p_1 l_1})_i - (c_{p_2 l_2})_i)^2$$

$$= 4((0-0)^2 + (0-0)^2 + (1-0)^2 + (0-1)^2) = 8$$

Attraction Repulsion Indices

Attraction Repulsion Indices

With categorical variables, it is usual to test, whether there is a significant association between the variables, with the chi-square test of independence. It is also interesting to compare the association at the level of the modalities instead of the variables. Let $n_{p_1 l_1, p_2 l_2}$ be the number of individuals having the modality l_1 of the variable Y_{p_1} and the modality l_2 of the variable Y_{p_2}. Now the attraction repulsion index $d_{p_1 l_1, p_2 l_2}$ between the modality l_1 of the variable Y_{p_1} and the modality l_2 of the variable Y_{p_2} is given by

$$d_{p_1 l_1, p_2 l_2} = \frac{n_{p_1 l_1, p_2 l_2}/n}{n_{p_1 l_1}/n n_{p_2 l_2}/n} = \frac{n_{p_1 l_1, p_2 l_2}}{\frac{n_{p_1 l_1} n_{p_2 l_2}}{n}}.$$

Attraction Repulsion Indices

It is clear that if the attraction repulsion index is larger than one, the individuals are more inclined to take both modalities simultaneously than under the hypothesis of independence. And vice-versa, if the attraction repulsion index is smaller than one, the individuals are less inclined to take both modalities simultaneously than under the hypothesis of independence. The aim of the MCA is to produce graphical display in lower dimension which reproduce, without losing too much information, the associations between the modalities through the attraction repulsion index.

Attraction Repulsion Indices

The attraction repulsion index $d_{i,pl}$ between the individual i and the modality l of the variable Y_p is defined as follows.

$$d_{i,pl} = \frac{f_{ipl}}{f_{i..}f_{.pl}} = \frac{x_{ipl}}{n_{pl}/n}.$$

Now, clearly

$$d_{i,pl} = 0,$$

if $x_{ipl} = 0$ and

$$d_{i,pl} = \frac{n}{n_{pl}},$$

if $x_{ipl} = 1$. Thus, if the individual i does not have the modality l of the variable Y_p, then the attraction repulsion index $d_{i,pl}$ is equal to 0, and if the individual i does have the modality l of Y_p, then the attraction repulsion index $d_{i,pl}$ increases as the l of Y_p becomes rare.

Multiple Correspondence Analysis

Multiple Correspondence Analysis

To obtain a representation in lower dimension, PCA is applied on the two data clouds: the rows and column profiles. A transformation of the profiles is necessary to center the variables, and to be able to use euclidian distances instead of χ^2 distances. Finally, PCA is applied to the transformed row and column profiles:

$$(l_i^\circ)_{pl} = \frac{(l_i)_{pl}}{\sqrt{f_{.pl}}} - \sqrt{f_{.pl}} \text{ and } (c_{pl}^\circ)_i = \frac{(c_{pl})_i}{\sqrt{f_{i..}}} - \sqrt{f_{i..}}.$$

The solution of the problem of maximization associated with the PCA on the transformed row and column profiles is given respectively by the eigenvalues and the eigenvectors of the matrices $V(K \times K)$ and $W(n \times n)$ where

$$V = T^T T \text{ and } W = TT^T \text{ where } T_{i,pl} = \frac{x_{ipl} - n_{pl}/n}{\sqrt{Pn_{pl}}}.$$

Multiple Correspondence Analysis

The principal components for the individuals are derived from the eigenvectors of the matrix V, and the principal components for the modalities from the eigenvectors of the matrix W.

Let $H = rank(V) = rank(W)$. The new system of coordinates (scores) for the individuals is defined through the H principal components

$$\phi_{h,i} = \sum_{k=1}^{K} u_{h,k}(l_i^{\circ})_k \qquad h = 1, \ldots, H,$$

where $u_{h,k}$ is the kth element of the eigenvector associated with the hth largest eigenvalues of V.

The new system of coordinates (scores) for the modalities is defined through the principal components

$$\psi_{h,pl} = \sum_{i=1}^{n} v_{h,i}(c_{pl}^{\circ})_i \qquad h = 1, \ldots, H.$$

Contribution of the Modalities

Contribution of the modality l of Y_p on the variance of the new variable ψ_h is given by

$$C(pl,h) = \frac{f_{.pl}\psi_{h,pl}^2}{\lambda_h} = \frac{n_{pl}\psi_{h,pl}^2}{nP\lambda_h}.$$

Global contribution of the variable Y_p is given by

$$C(p,h) = \sum_{l=1}^{K_p} C(pl,h).$$

Graphical Presentation

Comparison of the Modalities

The attraction repulsion index

$$d_{p_1 l_1, p_2 l_2} = 1 + \sum_{h=1}^{H} \psi_{h,p_1 l_1} \psi_{h,p_2 l_2}.$$

The graphical output of MCA is the approximation of the previous formula using few dimensions. Suppose that the modalities are well represented in two dimensions, then we can plot the two first principal components and interpret the proximity between the points on the first principal plan with the following approximation

$$d_{p_1 l_1, p_2 l_2} \approx 1 + \sum_{h=1}^{2} \psi_{h,p_1 l_1} \psi_{h,p_2 l_2}.$$

Comparison of the Individuals

The proximity between two individuals i_1 and i_2 is defined as

$$d_{i_1,i_2} = 1 + \sum_{h=1}^{H} \phi_{h,i_1}\phi_{h,i_2}.$$

Two individuals are close if they have in general the same modalities.
Now d_{i_1,i_2} can be approximated by

$$d_{i_1,i_2} \approx 1 + \sum_{h=1}^{2} \phi_{h,i_1}\phi_{h,i_2}.$$

Simultaneous Comparison

The attraction repulsion index

$$d_{i,pl} = 1 + \sum_{h=1}^{H} \frac{1}{\sqrt{\lambda_h}} \phi_{h,i} \psi_{h,pl},$$

and thus again

$$d_{i,pl} \approx 1 + \sum_{h=1}^{2} \frac{1}{\sqrt{\lambda_h}} \phi_{h,i} \psi_{h,pl}.$$

Simultaneous Comparison

The components are often standardized defining

$$\hat{\phi}_{1,j} = \frac{1}{\sqrt{\lambda_1}} \phi_{1,j}$$

and

$$\hat{\phi}_{2,j} = \frac{1}{\sqrt{\lambda_2}} \phi_{2,j}.$$

Then

$$d_{i,pl} \approx 1 + \sum_{h=1}^{2} \hat{\phi}_{h,i} \psi_{h,pl},$$

and the final graphical representation can be given simultaneously as a double biplot.

Example

Example of MCA: Extended Gender, Civil Status and Education Data

Disclaimer: This example data set is randomly generated. Please do not draw real life conclusions from it.

	X_{11}	X_{12}	X_{21}	X_{22}	X_{23}	X_{31}	X_{32}	$\sum_{p=1}^{7}\sum_{l=1}^{K_p} x_{ipl}$
1	0	1	1	0	0	1	0	3
2	0	1	1	0	0	0	1	3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
25	1	0	0	0	1	0	1	3
$\sum_{i=1}^{n} x_{ipl}$	16	9	9	6	10	14	11	

Table: Complete disjunctive table

Example of MCA

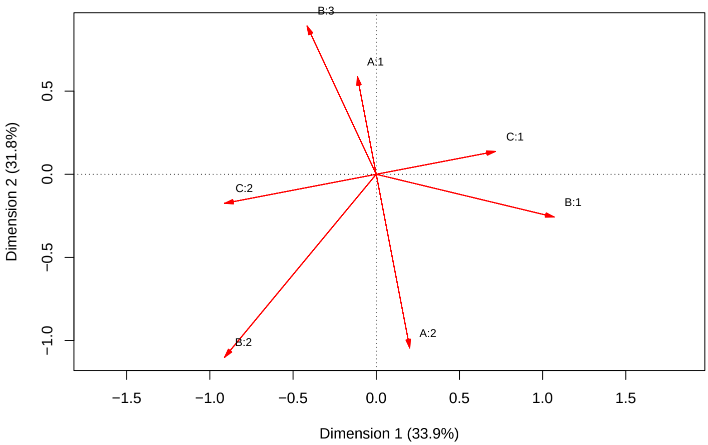

Figure: Result of MCA (A1=male, A2=female, B1=single, B2=married, B3=divorced/widoved, C1=low education, C2=at least high school diploma.)

Some Remarks

Some Remarks

When performing MCA, it is better to take into account variables that have more or less the same number of modalities. (The number of modalities have an effect on analysis.) It is also advised to avoid having rare modalities. (Rare modalities have a big impact on analysis, and that makes MCA quite nonrobust method.) One can preprocess the data by grouping modalities if necessary.

Next Week

Next week we will talk about canonical correlation analysis.

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