GIS-E3010
Least-Squares Methods in Geoscience
Lecture 2/2018

- Observation equation model
- Derivation of general solution
- Examples

Why redundant observations

- **Uncertainties in measurements**
  - Instruments
  - Circumstances
  - Observer
  - Methods
  - Difference between a mathematical model and reality
  - The purpose of measurements
  - Economical reasons

  - Redundant observations

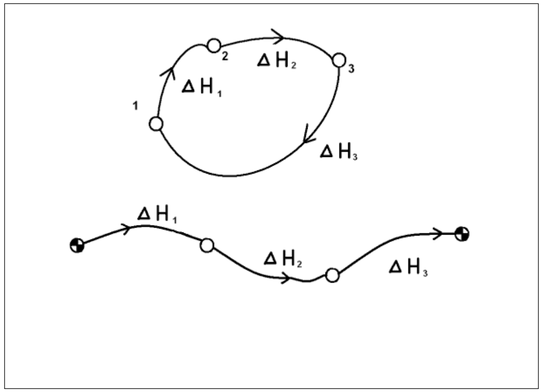
If we have more observation than necessary, we need adjustment

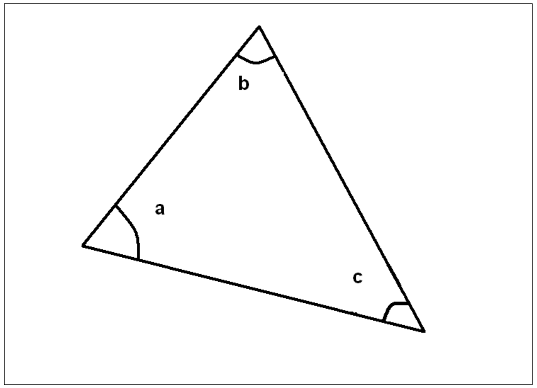
Questions in adjustment calculus and design of measurements

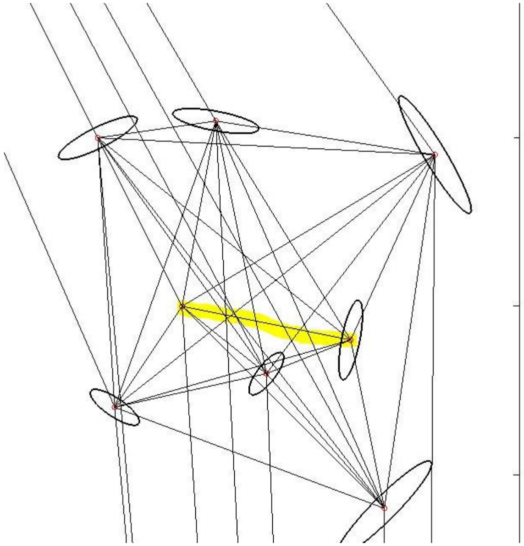
- Repeated observations do not give the same answer or redundant observations are not consistent
- Can we detect blunders, gross errors, outliers from our data
- Can we detect systematic errors
- What is the best and the most reliable way to take into account all observations and to get the best final results
- How can we prevent the corruption of the results due to the non-detected outliers

Examples

- From height differences to heights
- From angles to angles (or shape of the triangle)
- From angles, distances and GPS-vectors to the vector between two points

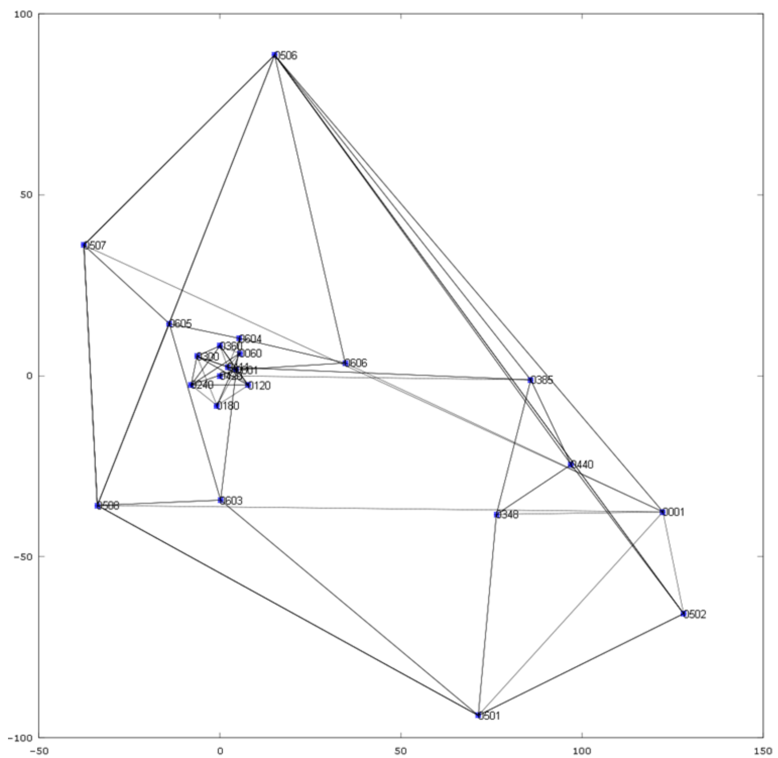


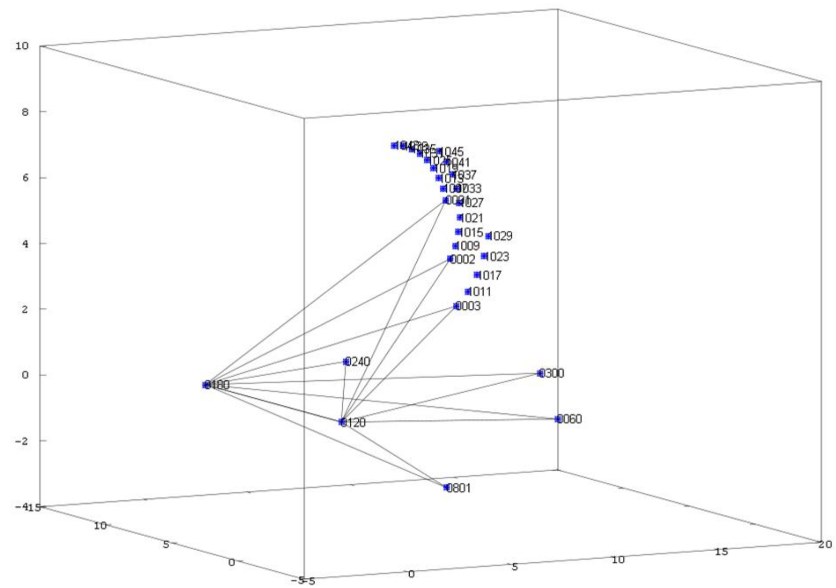




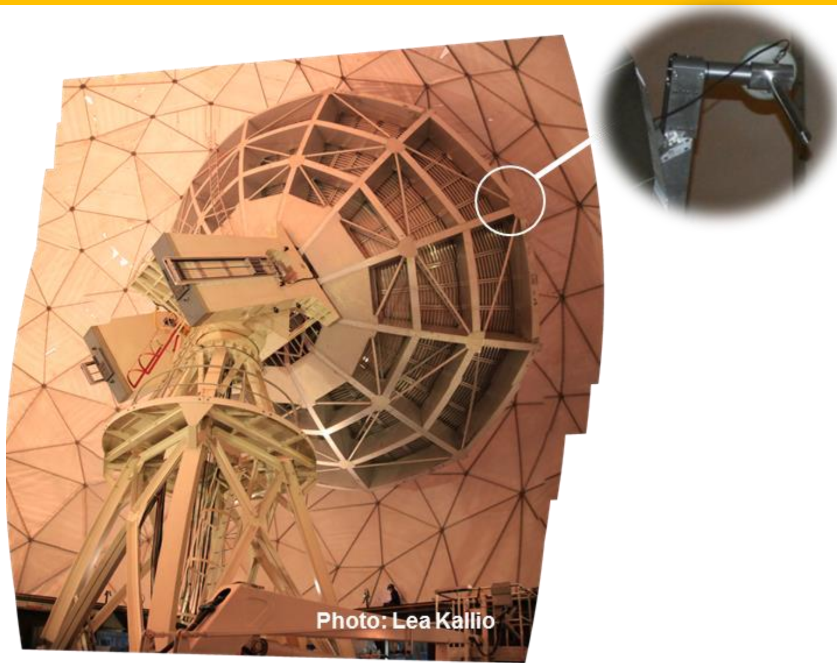
Examples 2

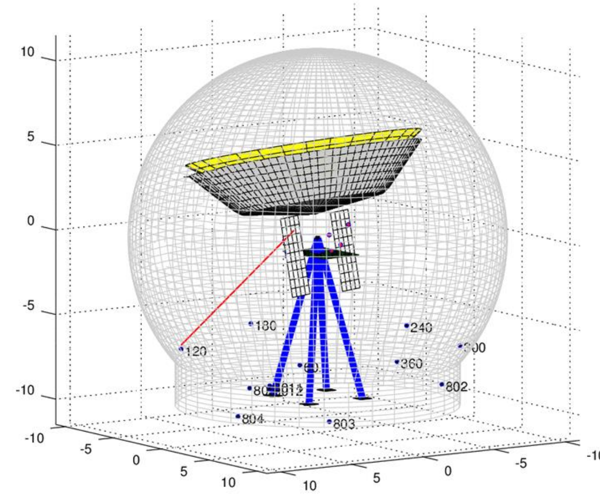
- From angles, distances, height differences and GPS-vectors to 3D- coordinates
- From angles to 3D-coordinates

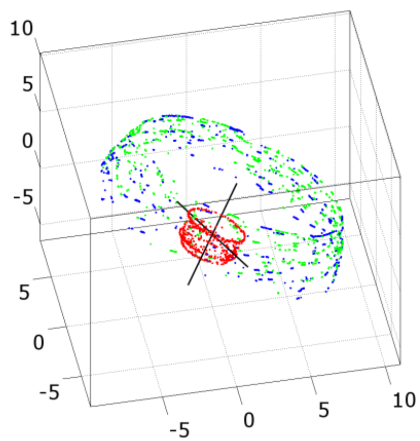




Still one example







- From angles and distances to 3D coordinates (red) and from 3D coordinates to the reference point and axis directions of the telescope
- From GPS phase observations to 3D coordinates (blue and green) and from 3D coordinates to reference points and axis direction of the telescope

Models

- Observation equation model

  (Gauss-Markov)

$$f_i(x_1, x_2, \ldots, x_u, \ell_i) = 0$$

$$Ax - y = v$$

- Condition equation model

$$f_i(\ell_1, \ell_2, \ldots, \ell_n) = 0$$

$$Bv - y = 0$$

- General or mixed model

  (Gauss-Helmert)

$$f_i(x_1, x_2, \ldots, x_u, \ell_1, \ell_2, \ldots, \ell_n) = 0$$

$$A(x - x_0) + Bv - y = 0$$

Notation

x	is unknown parameters
u	is number of unknown parameter
n	is number of observations
A	is design matrix (coefficients of unknown parameters)
y	is y-vector, opposite number of calculated minus observed (in linear model observations and possible constants, when approximate values of parameters are zeros)
ℓ	is observation
v	is residual vector, adjusted minus observed
f	is functional model, the relation between observations and unknown parameters
P	is weight matrix

Observation equation model, linear model

$$\begin{cases} a_{10} + a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3 + \cdots + a_{1u} \cdot x_u - \ell_1 = 0 \\ a_{20} + a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3 + \cdots + a_{2u} \cdot x_u - \ell_2 = 0 \\ a_{30} + a_{31} \cdot x_1 + a_{32} \cdot x_2 + a_{33} \cdot x_3 + \cdots + a_{3u} \cdot x_u - \ell_3 = 0 \\ \vdots \\ a_{n0} + a_{n1} \cdot x_1 + a_{n2} \cdot x_2 + a_{n3} \cdot x_3 + \cdots + a_{nu} \cdot x_u - \ell_n = 0 \end{cases}$$

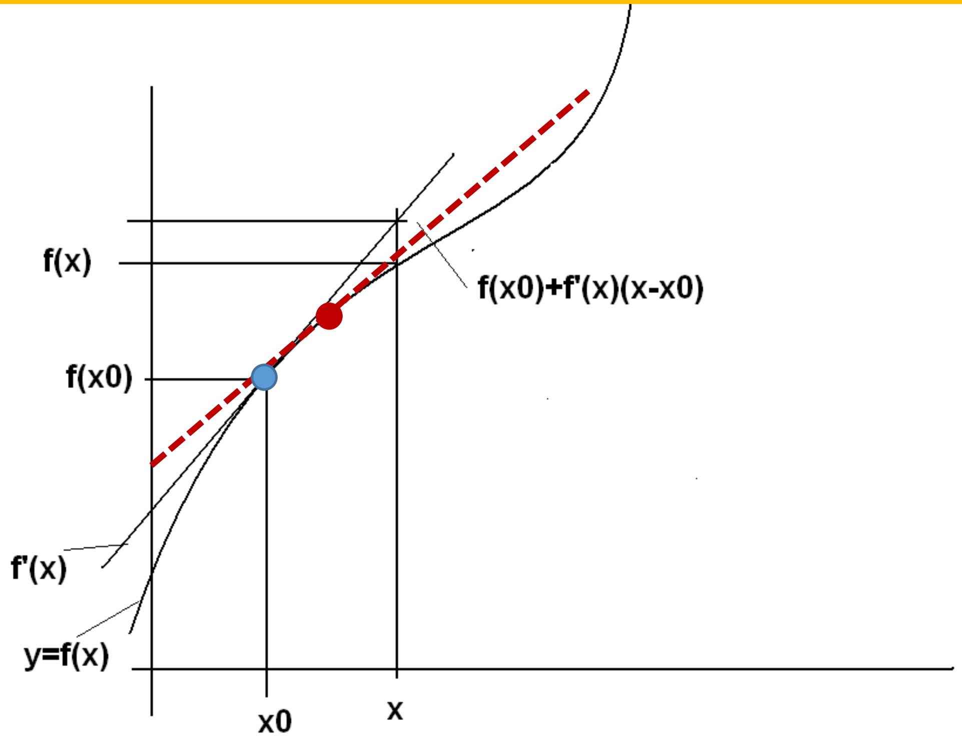
$$\begin{pmatrix} a_{10} \\ a_{20} \\ a_{30} \\ \vdots \\ a_{n0} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1u} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2u} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3u} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nu} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_u \end{pmatrix} - \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \vdots \\ \ell_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{cases} a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3 + \cdots + a_{1u} \cdot x_{u1} \boxed{- \ell_{1_{obs}} + a_{10}} = v_1 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3 + \cdots + a_{2u} \cdot x_{u2} \boxed{- \ell_{2_{obs}} + a_{20}} = v_2 \\ a_{31} \cdot x_1 + a_{32} \cdot x_2 + a_{33} \cdot x_3 + \cdots + a_{3u} \cdot x_{u3} \boxed{- \ell_{3_{obs}} + a_{30}} = v_3 \\ \vdots \\ a_{n1} \cdot x_1 + a_{n2} \cdot x_2 + a_{n3} \cdot x_3 + \cdots + a_{nu} \cdot x_u \boxed{- \ell_{n_{obs}} + a_{n0}} = v_n \end{cases}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1u} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2u} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3u} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nu} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_u \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} == \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$$

$$Ax - y = v$$

Linearization with Taylor



$$f(x) = \boxed{f(x_0) + \frac{d}{dx} f(x_0)\Delta x} + \frac{1}{2!}\frac{d^2}{dx^2} f(x_0)\Delta x^2 + \cdots + \frac{1}{(q-1)!}\frac{d^{(q-1)}}{dx^{(q-1)}} f(x_0)\Delta x^{(q-1)} + R_q(\theta, \Delta x)$$

$$R_q(\theta, \Delta x) = \frac{1}{(q)!}\frac{d^{(q)}}{dx^{(q)}} f(\theta)\Delta x^q$$

Linearization

$$F(x,\ell) = F(x_0,\ell_0) + \frac{\partial F}{\partial x}(x - x_0) + \frac{\partial F}{\partial \ell}(\ell - \ell_0) = 0$$

Approximate value + correction

$$-y + A(x - x_0) + Bv = 0$$

$$A = \frac{\partial F(x_0,\ell_0)}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_u} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_u} \\ & \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_u} \end{pmatrix} \qquad B = \frac{\partial F(x_0,\ell_0)}{\partial \ell} = \begin{pmatrix} \frac{\partial f_1}{\partial \ell_1} & \frac{\partial f_1}{\partial \ell_2} & \cdots & \frac{\partial f_1}{\partial \ell_n} \\ \frac{\partial f_2}{\partial \ell_1} & \frac{\partial f_2}{\partial \ell_2} & \cdots & \frac{\partial f_2}{\partial \ell_n} \\ & \vdots & & \\ \frac{\partial f_n}{\partial \ell_1} & \frac{\partial f_n}{\partial \ell_2} & \cdots & \frac{\partial f_n}{\partial \ell_n} \end{pmatrix}$$

Observation equation model, nonlinear model differnce of y-vector and observation ℓ

$$
\begin{aligned}
f_i(x_1, x_2, \ldots, x_u, \ell_i) \approx\ & f_i(x_{1_0}, x_{2_0}, \ldots, x_{u_0}, \ell_{i_0}) \\
& + \frac{\partial f_i(x_{1_0}, x_{2_0}, \ldots, x_{u_0}, \ell_{i_0})}{\partial x_1} \cdot (\hat{x}_1 - x_{1_0}) + \frac{\partial f_i(x_{1_0}, x_{2_0}, \ldots, x_{u_0}, \ell_{i_0})}{\partial x_2} \cdot (\hat{x}_2 - x_{2_0}) + \cdots + \frac{\partial f_i(x_{1_0}, x_{2_0}, \ldots, x_{u_0}, \ell_{i_0})}{\partial x_u} \cdot (\hat{x}_u - x_{u_0}) \\
& + \frac{\partial f_i(x_{1_0}, x_{2_0}, \ldots, x_{u_0}, \ell_{i_0})}{\partial \ell_i} \cdot (\hat{\ell}_i - \ell_{i_0}) \\
& = 0
\end{aligned}
$$

If ℓ_i can be directly expressed with parameters *x*, the equation above is

$$f_i = g(x_1, x_2, \cdots, x_u) - \ell_i = 0$$

Thus the last partial derivative is -1.

By substituting

$$\hat{\ell}_i - \ell_{i_{obs}} = v_i \quad \text{and} \quad -f_i(x_{1_0}, x_{2_0}, \ldots, x_{u_0}, \ell_{i_0}) = y_i$$

We obtain $\boxed{A(x - x_0) - y = v}$

Linearized model

$$
\left\{
\begin{aligned}
& f_i(x_1, x_2, \dots, x_u, \ell_1) \approx f_1(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{1_0}) \\
& + \frac{\partial f_1(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{1_0})}{\partial x_1} \cdot (\hat{x}_1 - x_{1_0}) + \frac{\partial f_1(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{1_0})}{\partial x_2} \cdot (\hat{x}_2 - x_{2_0}) + \cdots + \frac{\partial f_1(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{1_0})}{\partial x_u} \cdot (\hat{x}_u - x_{u_0}) \\
& + \frac{\partial f_1(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{1_0})}{\partial \ell_1} \cdot (\hat{\ell}_1 - \ell_{1_0}) = 0 \\[2ex]
& f_2(x_1, x_2, \dots, x_u, \ell_i) \approx f_2(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{2_0}) \\
& + \frac{\partial f_2(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{2_0})}{\partial x_1} \cdot (\hat{x}_1 - x_{1_0}) + \frac{\partial f_2(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{2_0})}{\partial x_2} \cdot (\hat{x}_2 - x_{2_0}) + \cdots + \frac{\partial f_2(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{2_0})}{\partial x_u} \cdot (\hat{x}_u - x_{u_0}) \\
& + \frac{\partial f_2(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{2_0})}{\partial \ell_2} \cdot (\hat{\ell}_2 - \ell_{2_0}) = 0 \\[2ex]
& \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\[2ex]
& f_n(x_1, x_2, \dots, x_u, \ell_i) \approx f_n(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{n_0}) \\
& + \frac{\partial f_n(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{n_0})}{\partial x_1} \cdot (\hat{x}_1 - x_{1_0}) + \frac{\partial f_i(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{n_0})}{\partial x_2} \cdot (\hat{x}_2 - x_{2_0}) + \cdots + \frac{\partial f_n(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{n_0})}{\partial x_u} \cdot (\hat{x}_u - x_{u_0}) \\
& + \frac{\partial f_n(x_{1_0}, x_{2_0}, \dots, x_{u_0}, \ell_{n_0})}{\partial \ell_n} \cdot (\hat{\ell}_n - \ell_{n_0}) = 0
\end{aligned}
\right.
$$

General solution: deterministic derivation

$$\boldsymbol{Ax - y = v}$$

$$
\begin{aligned}
& v^T P v = \min \\
\Rightarrow\quad & (Ax - y)^T P (Ax - y) = \min \\
\Rightarrow\quad & (x^T A^T P - y^T P)(Ax - y) = \min \\
\Rightarrow\quad & x^T A^T P A x - x^T A^T P y - y^T P A x + y^T P y = \min \\
\Rightarrow\quad & 2x^T A^T P A - y^T P A - y^T P A = 0 \\
\Rightarrow\quad & 2x^T A^T P A - 2y^T P A = 0 \\
\Rightarrow\quad & x^T A^T P A = y^T P A \\
\Rightarrow\quad & A^T P A x = A^T P y
\end{aligned}
$$

Normaaliyhtälöt

Normal equations

If we have linear form $u = x^T A y$, then

$\frac{\partial u}{\partial x} = y^T A^T$ and $\frac{\partial u}{\partial y} = x^T A$

For quadratic form $q = x^T A x$,

$\frac{\partial q}{\partial x} = 2x^T A$

$$x = (A^T P A)^{-1} A^T P y$$

Solution of normal equations

Weighting of observations

- Weight matrix P is inverse of the covariance matrix of the observations
- Variance factor ${\sigma_0}^2$ is the variance of an observation which has the weight 1

$$P = {\sigma_0}^2 \Sigma^{-1}$$

Variance factor can be chosen

The solution does not depend on the choise of the variance factor ${\sigma_0}^2$

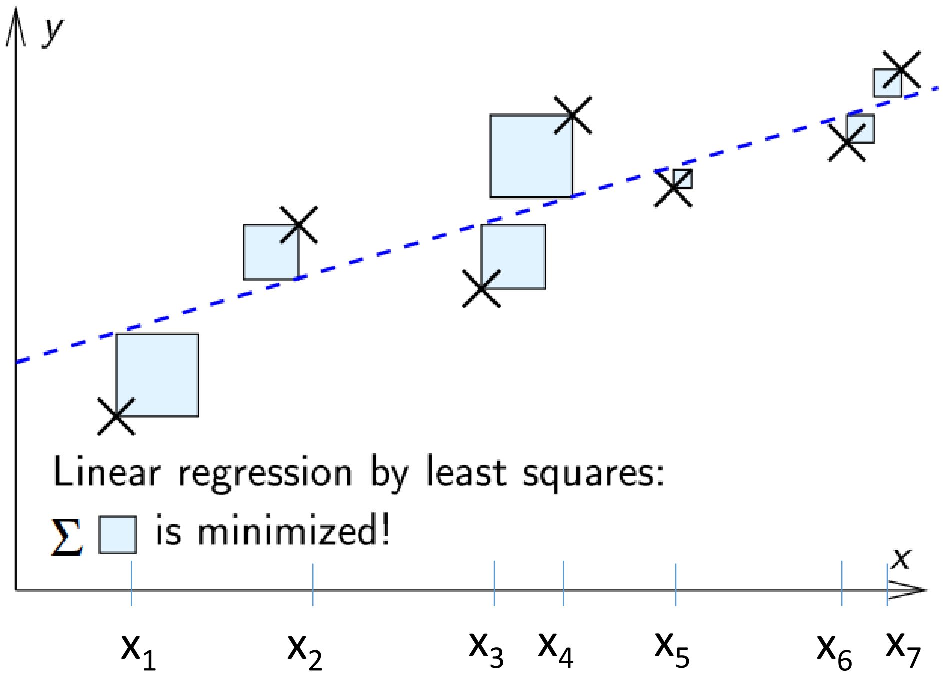
$$x = \left(A^T {\sigma_0}^2 \Sigma^{-1} A\right)^{-1} A^T {\sigma_0}^2 \Sigma^{-1} y$$

$$= \frac{1}{{\sigma_0}^2} \left(A^T \Sigma^{-1} A\right)^{-1} A^T {\sigma_0}^2 \Sigma^{-1} y$$

Exercise: arithmetic mean

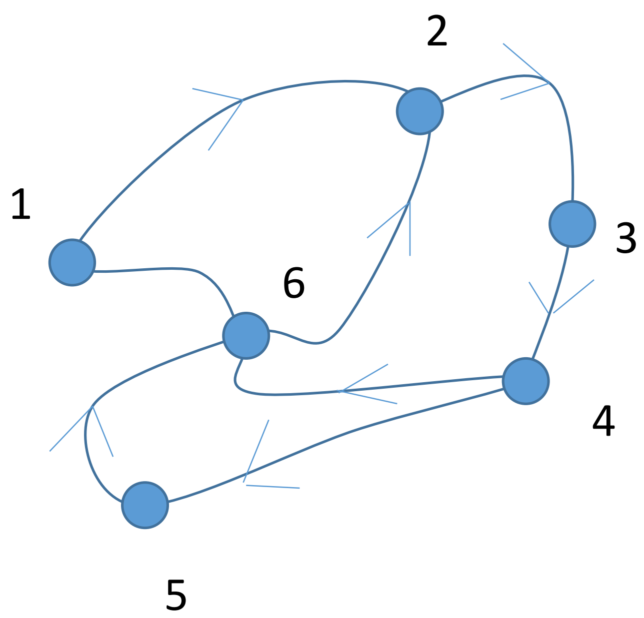
- What is number of equations in observation equation model?
- What is number of unknown parameters?
- Functional model?
- A-matrix?
- y-vector?
- Normal equations?
- LSQ solution?

Exercise: Linear regression



- How many observations?
- How many unknown parameters?
- Functional model ?
- A-matrix?
- y-vector?

Exercise: levelling network

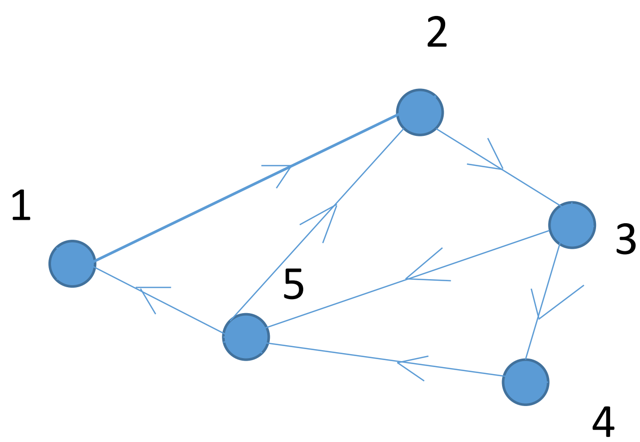


- Height differences between points has been observed as shown in the left
- Arrows show the direction

- How many equation?
- What are observations?
- How many unknown parameters?
- What are unknown parameters?
- Functional model?
- A-matrix?
- y-vector?

Exercise: GPS network

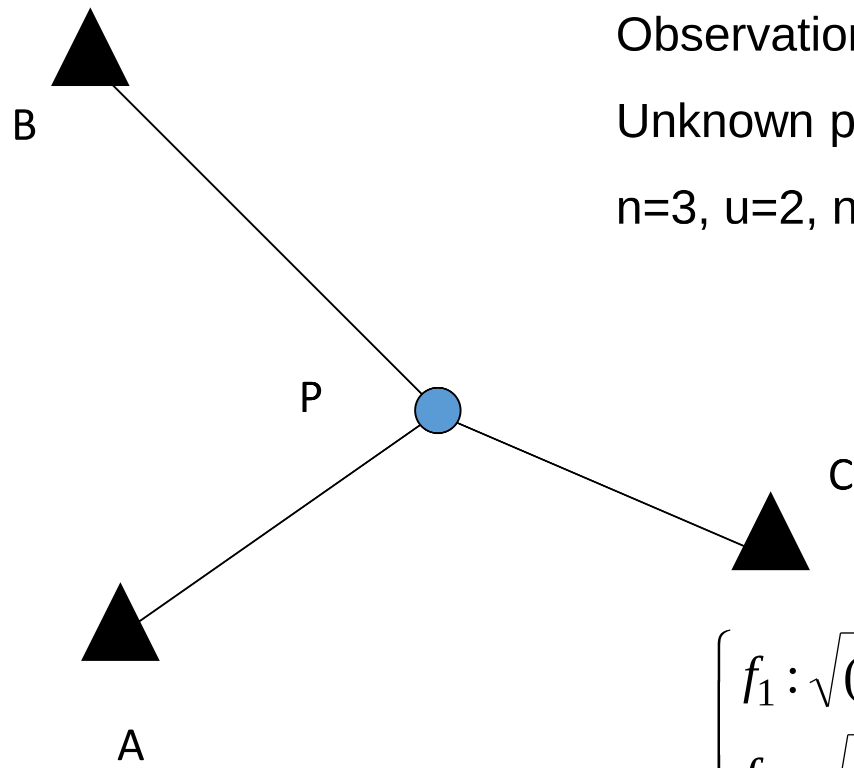
- The observations, coordinate differences, are results of the baseline processing (from phase double difference observations to coordinate differences between the points)
- Also the covariance matrices of the coordinate differences are saved in baseline processing



- Coordinate differences $\Delta X, \Delta Y, \Delta Z$ between points has been observed as shown in the left
- Arrows show the direction

- How many equation?
- What are observations?
- How many unknown parameters?
- What are unknown parameters?
- Functional model?
- A-matrix?
- y-vector?

Non-linear functional models, trilateration



Observations:distances s

Unknown parameters: x,y

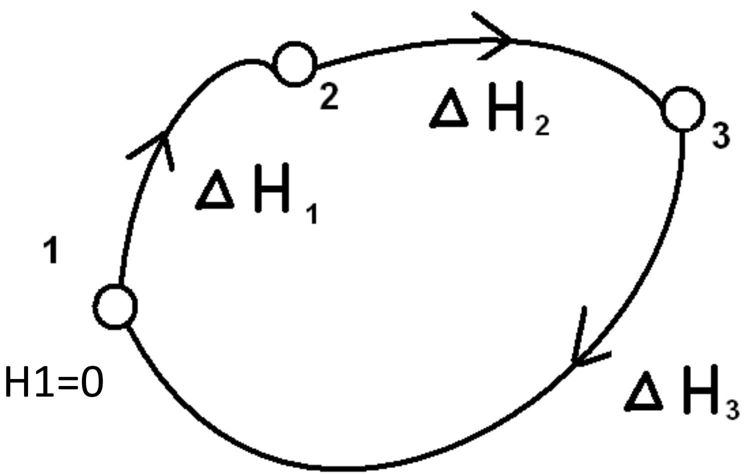
n=3, u=2, n-u=1

$$\begin{cases} f_1: \sqrt{(x_A - x_P)^2 + (y_A - y_P)^2} - s_{PA} = 0 \\ f_2: \sqrt{(x_B - x_P)^2 + (y_B - y_P)^2} - s_{PB} = 0 \\ f_3: \sqrt{(x_C - x_P)^2 + (y_C - y_P)^2} - s_{PC} = 0 \end{cases}$$

Least squares estimate , BLUE, Maximum likelihood

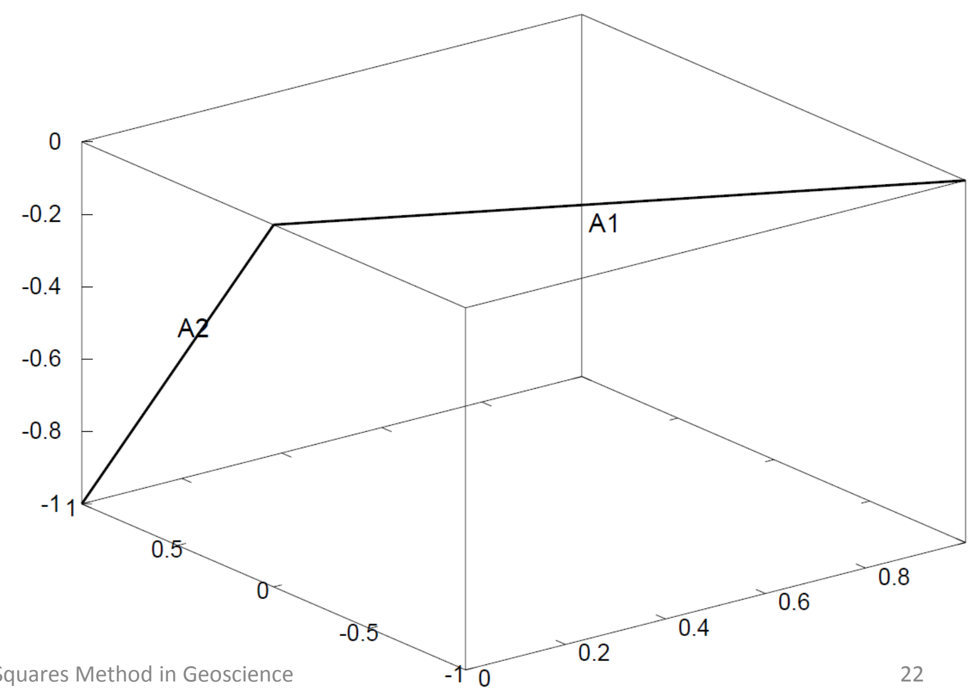
- Least squares estimation
  - No assumptions of the probability distribution of vector of observations
  - Based on the minimizing the quadratic form $(Ax - y)^T P(Ax - y)$
- LSQ estimate is BLUE (Best Linear Unbiased Estimatation) if
  - Linear: LSQ estimate is linear $\mathrm{x} = (A^T PA)^{-1} A^T Py$
  - Unbiased: LSQ estimate is unbiased $E(\hat{x}) = x$ for $\forall x$
  - Best: the variance of estimated $\hat{x}$ is minimum when $P = {\sigma_0}^2 \Sigma^{-1}$
- ML estimate is BLUE if the probability distribution of observation is $y \sim N(Ax, \Sigma_y)$ and
- ML estimate is LSQ if it is BLUE and $P = {\sigma_0}^2 \Sigma^{-1}$

LSQ estimate is ortogonal projection

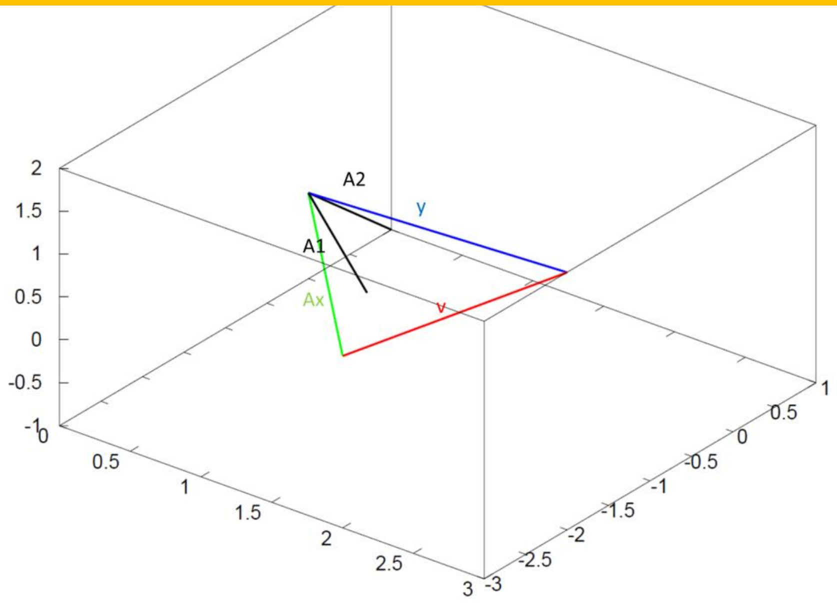


$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \qquad y = \begin{pmatrix} \Delta H_{12} \\ \Delta H_{23} \\ \Delta H_{31} \end{pmatrix}$$

The columns of A matrix span the two dimensional space. Estimated $\hat{y}$ is in in this space



LSQ is orthogonal projection



$$\hat{x} = (A^T A)^{-1} A^T y$$

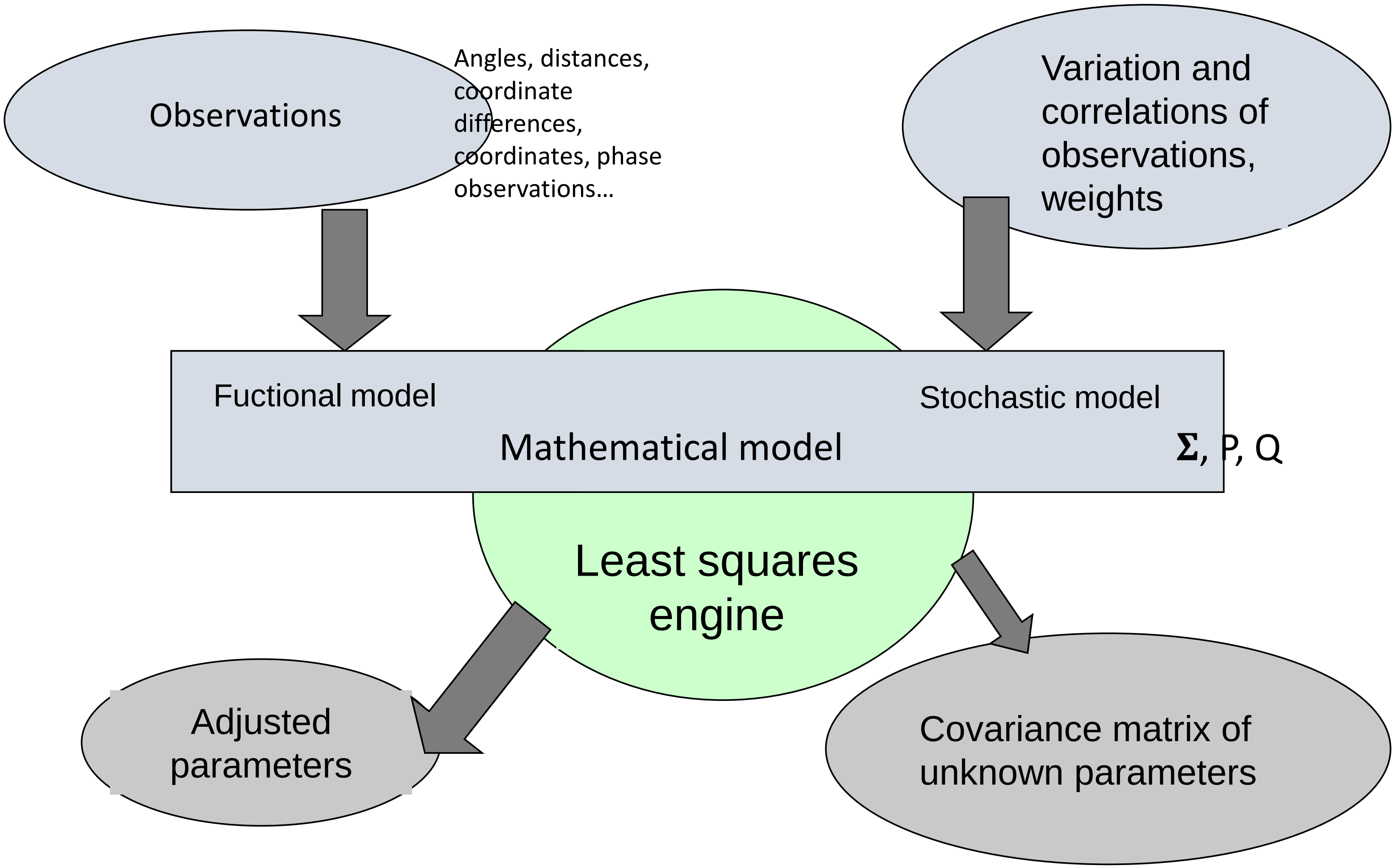
$$\hat{y} = A\hat{x} = A(A^T A)^{-1} A^T y$$

$$\widehat{e} = y - \widehat{y} = y - A\hat{x} = y - A(A^T A)^{-1} A^T y = (I - A(A^T A)^{-1} A^T)y$$

$$\widehat{y}^T \widehat{e} = 0$$

$$\widehat{y} \perp \widehat{e}$$

Least squares process



Examples of functional models

$$\Phi_i^k(t) = \rho_i^k(t) \times \frac{f}{c} + \left(h^k(t) - h_i(t)\right) \times f + ion_i^k(t) + trop_i^k(t) - N_i^k + \varepsilon$$

GPS phase observation

$$\tau_{obs} = -\frac{1}{c} b_t \cdot Y \cdot X \cdot U \cdot N \cdot P \cdot k_c$$

$$+\tau_{y.abb} + \tau_{d.abb} + \tau_{rel.}$$

$$+\tau_{tides} + \tau_{o.load.} + \tau_{a.load} + \tau_{h.load}$$

$$+\tau_{ion} + \tau_{instr.} + \tau_{atm.dry} + \tau_{atm.wet} + \tau_{clock}$$

VLBI time delay

$$X_0 + R_{\alpha,a}(E - X_0) + R_{\alpha,a} R_{\beta,e} p - X = 0$$

Local tie, reference point of VLBI telescope

$$\alpha = tan^{-1}\left(\frac{-sin\lambda \cdot \Delta u + cos\lambda \cdot \Delta v}{-sin\varphi cos\lambda \cdot \Delta u - sin\varphi sin\lambda \cdot \Delta v + cos\varphi \cdot \Delta w}\right)$$

$$\beta = sin^{-1}\left(\frac{cos\varphi cos\lambda \cdot \Delta u + cos\varphi sin\lambda \cdot \Delta v + sin\varphi \Delta w}{\sqrt{\Delta u^2 + \Delta v^2 + \Delta w^2}}\right)$$

$$s = \sqrt{\Delta u^2 + \Delta v^2 + \Delta w^2}$$

Azimuth, elevation angle, distance

Litterature

- Kallio 1998:Tasoituslasku
- Cooper 1987: Control Surveys in Civil Engineering
- Leick 1995:GPS Satellite Surveying
- Hirvonen 1965: Tasoituslasku
- Mikhail 1976: Observations and Least Squares
- Teunissen 2003: Adjustment theory an introduction