GIS-E3010 Least-Squares Methods in Geoscience

Lecture 5

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Learning objectives

- To understand and apply
  - The solution of exterior orientation parameters
  - The solution of relative orientation parameters
  - The solution of forward intersection parameters (3D point)

Exterior orientation

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Exterior orientation

- Known/measured object coordinates *XYZ*
- Measure image coordinates (*xy)* of known points
- Solve six parameters of orientation (the location of the projection center and the rotation angles of the image)
  - Interior orientation is expected to be known, and corrected
- Collinearity equations $$\begin{cases} x = -c\dfrac{r_{11}(X-X_0)+r_{12}(Y-Y_0)+r_{13}(Z-Z_0)}{r_{31}(X-X_0)+r_{32}(Y-Y_0)+r_{33}(Z-Z_0)} = f_x \\ y = -c\dfrac{r_{21}(X-X_0)+r_{22}(Y-Y_0)+r_{23}(Z-Z_0)}{r_{31}(X-X_0)+r_{32}(Y-Y_0)+r_{33}(Z-Z_0)} = f_y \end{cases}$$
- There are 2 equations per point, and we have 6 unknowns => at least 3 points are needed (3x2=6)

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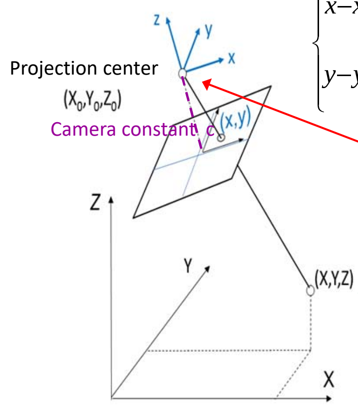
Collinearity equation



$$\begin{cases} x-x_0=-c\dfrac{r_{11}(X-X_o)+r_{12}(Y-Y_o)+r_{13}(Z-Z_o)}{r_{31}(X-X_o)+r_{32}(Y-Y_o)+r_{33}(Z-Z_o)} \\[2ex] y-y_0=-c\dfrac{r_{21}(X-X_o)+r_{22}(Y-Y_o)+r_{23}(Z-Z_o)}{r_{31}(X-X_o)+r_{32}(Y-Y_o)+r_{33}(Z-Z_o)} \end{cases}$$

Projection center

Rotation of camera

3D object point

Exterior orientation

- Assume object points to be constants (extremely accurate points) => an explicit non-linear model. Alternatives:
  - LS adjustment with observation equations. Object points are constants.
  - Weighted LS adjustment with observation equations. Both exterior orientation parameters and object points are unknowns. For object points, we write additional constraints equations that are given very high (∞) weights in the adjustment.
- Object coordinates are also observations (less accurate ground measurements) => a non-linear mixed model. Alternatives:
  - Weighted least squares (LS) adjustment with observation equations. We add additional constraints equations of object coordinates. Weights are proportional to accuracy!
  - General LS adjustment

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Exterior orientation, Object points are constants

Design matrix (Jacobian matrix)

Correction vector

- After the linearization,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\partial f_x}{\partial \omega} & \frac{\partial f_x}{\partial \varphi} & \frac{\partial f_x}{\partial \kappa} & \frac{\partial f_x}{\partial X_0} & \frac{\partial f_x}{\partial Y_0} & \frac{\partial f_x}{\partial Z_0} \\ \frac{\partial f_y}{\partial \omega} & \frac{\partial f_y}{\partial \varphi} & \frac{\partial f_y}{\partial \kappa} & \frac{\partial f_y}{\partial X_0} & \frac{\partial f_y}{\partial Y_0} & \frac{\partial f_y}{\partial Z_0} \end{bmatrix} \begin{bmatrix} d\omega \\ d\varphi \\ d\kappa \\ dX_0 \\ dY_0 \\ dZ_0 \end{bmatrix} + \begin{bmatrix} f_x^0 \\ f_y^0 \end{bmatrix}$$

in which $\begin{bmatrix} f_x^0 \\ f_y^0 \end{bmatrix} = \begin{bmatrix} f_x(\omega^0, \varphi^0, \kappa^0, X_0^0, Y_0^0, Z_0^0, X, Y, Z) \\ f_y(\omega^0, \varphi^0, \kappa^0, X_0^0, Y_0^0, Z_0^0, X, Y, Z) \end{bmatrix}$ i.e. the values of collinearity equations calculated using the current approximate values of parameters

Exterior orientation, Object points are constants

- The error equation becomes

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \frac{\partial f_x}{\partial \omega} & \frac{\partial f_x}{\partial \varphi} & \frac{\partial f_x}{\partial \kappa} & \frac{\partial f_x}{\partial X_0} & \frac{\partial f_x}{\partial Y_0} & \frac{\partial f_x}{\partial Z_0} \\ \frac{\partial f_y}{\partial \omega} & \frac{\partial f_y}{\partial \varphi} & \frac{\partial f_y}{\partial \kappa} & \frac{\partial f_y}{\partial X_0} & \frac{\partial f_y}{\partial Y_0} & \frac{\partial f_y}{\partial Z_0} \end{bmatrix} \begin{bmatrix} d\omega \\ d\varphi \\ d\kappa \\ dX_0 \\ dY_0 \\ dZ_0 \end{bmatrix} - \begin{bmatrix} x - f_x^0 \\ y - f_y^0 \end{bmatrix} \qquad v = Ax - y$$

We get residuals of image observations v_x and v_y

- In this case, the redundancy of equations (the number of observations – the number of parameters) is (N = the number of observations between an image and object points)

$$r = 2N - 6$$

Exterior orientation, Partial derivatives of rotation matrix

- When we linearize the collinear equations, we have to make partial derivations of a 3D rotation matrix
- To assist this task, we examine what happens when a rotation matrix is partially derivated with respect to omega, phi, kappa (rotations)
- A 3D rotation matrix is (rotations happen around moving axes)

$$R = R_\kappa R_\varphi R_\omega = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix}$$

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Exterior orientation, Partial derivatives of rotation matrix

- The first example: partial derivative with respect to an omega rotation

$$\frac{\partial R}{\partial\omega} = R_\kappa R_\varphi \frac{\partial R_\omega}{\partial\omega} = R_\kappa R_\varphi \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\omega & \cos\omega \\ 0 & -\cos\omega & -\sin\omega \end{bmatrix} = R_\kappa R_\varphi R_\omega \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = R \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

- We have to make derivation to only one of the sub-matrices R_ω . In addition, we detect that the result of the derivation can, actually, be expressed by using the original rotation matrix R_ω and an assisting matrix

$$\frac{\partial R_\omega}{\partial\omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\omega & \cos\omega \\ 0 & -\cos\omega & -\sin\omega \end{bmatrix} = R_\omega \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

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Exterior orientation, Partial derivatives of rotation matrix

- Corresponding partial derivatives with respect to phi and kappa rotations are:

$$\frac{\partial R}{\partial \varphi} = R_\kappa \frac{\partial R_\varphi}{\partial \varphi} R_\omega = R_\kappa \begin{bmatrix} -\sin\varphi & 0 & -\cos\varphi \\ 0 & 0 & 0 \\ \cos\varphi & 0 & -\sin\varphi \end{bmatrix} R_\omega = R_\kappa R_\varphi \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} R_\omega = R \begin{bmatrix} 0 & \sin\omega & -\cos\omega \\ -\sin\omega & 0 & 0 \\ \cos\omega & 0 & 0 \end{bmatrix}$$

$$\frac{\partial R}{\partial \kappa} = \frac{\partial R_\kappa}{\partial \kappa} R_\varphi R_\omega = \begin{bmatrix} -\sin\kappa & \cos\kappa & 0 \\ -\cos\kappa & -\sin\kappa & 0 \\ 0 & 0 & 0 \end{bmatrix} R_\varphi R_\omega = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_\kappa R_\varphi R_\omega = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R$$

- As a conclusion, we can express all three partial derivatives by using the original 3D rotation matrix and a proper assisting matrix

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Exterior orientation, Linearization of collinearity equations

- For a while, we shorten the collinearity equations

$$\begin{cases} x = -c\dfrac{r_{11}(X - X_0) + r_{12}(Y - Y_0) + r_{13}(Z - Z_0)}{r_{31}(X - X_0) + r_{32}(Y - Y_0) + r_{33}(Z - Z_0)} = f_x \\ y = -c\dfrac{r_{21}(X - X_0) + r_{22}(Y - Y_0) + r_{23}(Z - Z_0)}{r_{31}(X - X_0) + r_{32}(Y - Y_0) + r_{33}(Z - Z_0)} = f_y \end{cases}$$

by introducing new variables

$$x = f_x = -c\frac{U}{W} \quad and \quad y = f_y = -c\frac{V}{W}$$

- Therefore, variables *U* (numerator), *V* (numerator) and *W* (denominator) are (temporarily, we take these out from the context for partial derivations)

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = R \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix}$$

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Exterior orientation, Linearization of collinearity equations

- Partial derivation of collinearity equations can be done by using following rules of partial derivatives of quotients

$$\begin{cases} \dfrac{\partial f_x}{\partial p} = \dfrac{-c}{W^2}\left(\dfrac{\partial U}{\partial p}W - U\dfrac{\partial W}{\partial p}\right) = \dfrac{-c}{W}\left(\dfrac{\partial U}{\partial p} - \dfrac{U}{W}\dfrac{\partial W}{\partial p}\right) \\ \dfrac{\partial f_y}{\partial p} = \dfrac{-c}{W^2}\left(\dfrac{\partial V}{\partial p}W - V\dfrac{\partial W}{\partial p}\right) = \dfrac{-c}{W}\left(\dfrac{\partial V}{\partial p} - \dfrac{V}{W}\dfrac{\partial W}{\partial p}\right) \end{cases}$$

In these equations, *p* symbolizes any parameter with respect to we make partial derivation (i.e. omega, phi, kappa, X_0, Y_0, Z_0, X, Y, Z)

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Exterior orientation, Linearization of collinearity equations

- Case X_0: the partial derivatives of U, V and W (with respect to X_0

$$\frac{\partial}{\partial X_0}\begin{bmatrix} U \\ V \\ W \end{bmatrix} = R\frac{\partial}{\partial X_0}\begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix} = R\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -r_{11} \\ -r_{21} \\ -r_{31} \end{bmatrix}$$

- The result after substituting partial derivatives is

$$\begin{cases} \dfrac{\partial f_x}{\partial X_0} = \dfrac{-c}{W}\left(\dfrac{\partial U}{\partial X_0} - \dfrac{U}{W}\dfrac{\partial W}{\partial X_0}\right) = \dfrac{-c}{W}\left(-r_{11} - \dfrac{U}{W}(-r_{31})\right) = \dfrac{c}{W}\left(r_{11} - \dfrac{U}{W}(r_{31})\right) \\ \dfrac{\partial f_y}{\partial X_0} = \dfrac{-c}{W}\left(\dfrac{\partial V}{\partial X_0} - \dfrac{V}{W}\dfrac{\partial W}{\partial X_0}\right) = \dfrac{-c}{W}\left(-r_{21} - \dfrac{V}{W}(-r_{31})\right) = \dfrac{c}{W}\left(r_{21} - \dfrac{V}{W}(r_{31})\right) \end{cases}$$

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Exterior orientation, Linearization of collinearity equations

- Case X:

$$\frac{\partial}{\partial X}\begin{bmatrix}U\\V\\W\end{bmatrix}=R\frac{\partial}{\partial X}\begin{bmatrix}X-X_0\\Y-Y_0\\Z-Z_0\end{bmatrix}=R\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}r_{11}&r_{12}&r_{13}\\r_{21}&r_{22}&r_{23}\\r_{31}&r_{32}&r_{33}\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}r_{11}\\r_{21}\\r_{31}\end{bmatrix}$$

- The result is

$$\begin{cases}\dfrac{\partial f_x}{\partial X}=\dfrac{-c}{W}\left(\dfrac{\partial U}{\partial X}-\dfrac{U}{W}\dfrac{\partial W}{\partial X}\right)=\dfrac{-c}{W}\left(r_{11}-\dfrac{U}{W}(r_{31})\right)\\[2ex]\dfrac{\partial f_y}{\partial X_0}=\dfrac{-c}{W}\left(\dfrac{\partial V}{\partial X}-\dfrac{V}{W}\dfrac{\partial W}{\partial X}\right)=\dfrac{-c}{W}\left(r_{21}-\dfrac{V}{W}(r_{31})\right)\end{cases}$$

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Exterior orientation, Linearization of collinearity equations

- Case omega: At first, we make partial derivation to *U*, *V* and *W* *(with respect to omega)*

$$\frac{\partial}{\partial\omega}\begin{bmatrix}U\\V\\W\end{bmatrix}=\begin{bmatrix}\partial U/\partial\omega\\\partial V/\partial\omega\\\partial W/\partial\omega\end{bmatrix}=\frac{\partial R}{\partial\omega}\begin{bmatrix}X-X_0\\Y-Y_0\\Z-Z_0\end{bmatrix}=R\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}\begin{bmatrix}X-X_0\\Y-Y_0\\Z-Z_0\end{bmatrix}=R\begin{bmatrix}0\\Z-Z_0\\-(Y-Y_0)\end{bmatrix}$$

- These results are then placed in the equation

$$\begin{cases}\dfrac{\partial f_x}{\partial\omega}=\dfrac{-c}{W^2}\left(\dfrac{\partial U}{\partial\omega}W-U\dfrac{\partial W}{\partial\omega}\right)=\dfrac{-c}{W}\left(\dfrac{\partial U}{\partial\omega}-\dfrac{U}{W}\dfrac{\partial W}{\partial\omega}\right)\\[2ex]\dfrac{\partial f_y}{\partial\omega}=\dfrac{-c}{W^2}\left(\dfrac{\partial V}{\partial\omega}W-V\dfrac{\partial W}{\partial\omega}\right)=\dfrac{-c}{W}\left(\dfrac{\partial V}{\partial\omega}-\dfrac{V}{W}\dfrac{\partial W}{\partial\omega}\right)\end{cases}$$

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Exterior orientation, Linearization of collinearity equations

- The result is:

$$\begin{cases} \dfrac{\partial f_x}{\partial \omega} = \dfrac{-c}{W}\left(\dfrac{\partial U}{\partial \omega} - \dfrac{U}{W}\dfrac{\partial W}{\partial \omega}\right) = \dfrac{-c}{W}\left[\{r_{12}(Z-Z_0) - r_{13}(Y-Y_0)\} - \dfrac{U}{W}\{r_{32}(Z-Z_0) - r_{33}(Y-Y_0)\}\right] \\ \dfrac{\partial f_y}{\partial \omega} = \dfrac{-c}{W}\left(\dfrac{\partial V}{\partial \omega} - \dfrac{V}{W}\dfrac{\partial W}{\partial \omega}\right) = \dfrac{-c}{W}\left[\{r_{22}(Z-Z_0) - r_{23}(Y-Y_0)\} - \dfrac{V}{W}\{r_{32}(Z-Z_0) - r_{33}(Y-Y_0)\}\right] \end{cases}$$

$$\frac{\partial}{\partial \omega}\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \partial U/\partial \omega \\ \partial V/\partial \omega \\ \partial W/\partial \omega \end{bmatrix} = R\begin{bmatrix} 0 \\ Z-Z_0 \\ -(Y-Y_0) \end{bmatrix}$$

Correspondingly, we should solve partial derivatives with respect to phi, kappa, Y_0, Z_0, Y and Z.

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Exterior orientation, Object points are constants

- This method is obviously iterative (non-linear). This means that we have to give some initial values (relatively good ones) to each parameter, and after several iteration rounds, we approach close to the correct solution of unknown parameters (until the corrections become very small ones)
- The correction vector $\hat{x}$ is computed by using LS adjustment with observation equations, and these corrections are added to current approximate values of parameters (l=image observations. $f^{\,0}$=estimated image observations)

$$\hat{x} = (A^T A)^{-1} A^T (l - f^{\,0}) \qquad \hat{x} = \left(A^T A\right)^{-1} A^T y$$

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Exterior orientation, Object coordinates are also observations

- In following, a method (also iterative) in which, in addition to image observations (x,y), also object space coordinates (X,Y,Z) are considered to be observations (e.g. GPS measurement). This requires so called combined LS adjustment
- Linearized error equations become now as

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \frac{\partial f_x}{\partial \omega} & \frac{\partial f_x}{\partial \varphi} & \frac{\partial f_x}{\partial \kappa} & \frac{\partial f_x}{\partial X_0} & \frac{\partial f_x}{\partial Y_0} & \frac{\partial f_x}{\partial Z_0} & \frac{\partial f_x}{\partial X} & \frac{\partial f_x}{\partial Y} & \frac{\partial f_x}{\partial Z} \\ \frac{\partial f_y}{\partial \omega} & \frac{\partial f_y}{\partial \varphi} & \frac{\partial f_y}{\partial \kappa} & \frac{\partial f_y}{\partial X_0} & \frac{\partial f_y}{\partial Y_0} & \frac{\partial f_y}{\partial Z_0} & \frac{\partial f_y}{\partial X} & \frac{\partial f_y}{\partial Y} & \frac{\partial f_y}{\partial Z} \end{bmatrix} \begin{bmatrix} d\omega \\ d\varphi \\ d\kappa \\ dX_0 \\ dY_0 \\ dZ_0 \\ dX \\ dY \\ dZ \end{bmatrix} - \begin{bmatrix} x - f_x^0 \\ y - f_y^0 \end{bmatrix}$$

$$v = Ax - y$$

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Exterior orientation, Object coordinates are also observations

- Error equation

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \frac{\partial f_x}{\partial \omega} & \frac{\partial f_x}{\partial \varphi} & \frac{\partial f_x}{\partial \kappa} & \frac{\partial f_x}{\partial X_0} & \frac{\partial f_x}{\partial Y_0} & \frac{\partial f_x}{\partial Z_0} & \frac{\partial f_x}{\partial X} & \frac{\partial f_x}{\partial Y} & \frac{\partial f_x}{\partial Z} \\ \frac{\partial f_y}{\partial \omega} & \frac{\partial f_y}{\partial \varphi} & \frac{\partial f_y}{\partial \kappa} & \frac{\partial f_y}{\partial X_0} & \frac{\partial f_y}{\partial Y_0} & \frac{\partial f_y}{\partial Z_0} & \frac{\partial f_y}{\partial X} & \frac{\partial f_y}{\partial Y} & \frac{\partial f_y}{\partial Z} \end{bmatrix} \begin{bmatrix} d\omega \\ d\varphi \\ d\kappa \\ dX_0 \\ dY_0 \\ dZ_0 \\ dX \\ dY \\ dZ \end{bmatrix} - \begin{bmatrix} x - f_x^0 \\ y - f_y^0 \end{bmatrix}$$

$$\begin{bmatrix} f_x^0 \\ f_y^0 \end{bmatrix} = \begin{bmatrix} f_x(\omega^0, \varphi^0, \kappa^0, X_0^0, Y_0^0, Z_0^0, X^0, Y^0, Z^0) \\ f_y(\omega^0, \varphi^0, \kappa^0, X_0^0, Y_0^0, Z_0^0, X^0, Y^0, Z^0) \end{bmatrix}$$

i.e. values of equations calculated using approximate values of parameters

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Exterior orientation, Object coordinates are also observations

- In this case, we need additional error equations of object coordinate observations for the combined LS adjustment

$$\begin{bmatrix} v_X \\ v_Y \\ v_Z \end{bmatrix} = \begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix} + \begin{bmatrix} X^0 - X \\ Y^0 - Y \\ Z^0 - Z \end{bmatrix}$$

- The combined LS adjustment can be solved (*P* = a weight matrix) $x = \left(A^T PA + P_x\right)^{-1}\left(A^T Pl + P_x l_x\right)$

$$\begin{bmatrix} A^T & I \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & P_x \end{bmatrix} \begin{bmatrix} A \\ I \end{bmatrix} x = \begin{bmatrix} A^T & I \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & P_x \end{bmatrix} \begin{bmatrix} l \\ l_x \end{bmatrix}$$

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Exterior orientation, Object coordinates are also observations

- Weights P_x (a diagonal matrix) of object coordinates are proportional to accuracy!
- Example: We assume that the standard deviation of image coordinate observations (unit: mm) is 0.005mm (5 μm), and the standard deviation of object coordinate observations (unit: m) is 0.01m
- If we select image coordinate observations as a unit of weight (weight=1), we get the weight of object coordinate observations as (unit mm^2 / m^2)

  $p = (0.005 / 0.01)^2 = 0.25$
- Redundancy is $r = 5N - (6 + 3N)$

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Exterior orientation, Object coordinates are also observations

- An alternative to the combined LS adjustment is to solve the problem by using an implicit model

$$\begin{cases} F_x(x, X, Y, Z, \omega, \phi, \kappa, X_0, Y_0, Z_0) = -x + f_x(X, Y, Z, \omega, \phi, \kappa, X_0, Y_0, Z_0) = 0 \\ F_y(y, X, Y, Z, \omega, \phi, \kappa, X_0, Y_0, Z_0) = -y + f_y(X, Y, Z, \omega, \phi, \kappa, X_0, Y_0, Z_0) = 0 \end{cases}$$

- However, this leads to the general LS adjustment (4 observations/equation), which is more complicated than the (combined) LS adjustment with observation equations

Exterior orientation, Direct solutions

- So far least-squares solutions were applied to linearized non-linear models
- Such solutions require good initial values. To get initial approximation we need direct methods, in which solution doesn't require iteration

- Direct methods are usually not as accurate than indirect (iterative) methods
- Examples of direct methods
  - A pyramid method (Presented in the course Photogrammetry, Laser Scanning and Remote Sensing, also available in Kraus, 2007, Photogrammetry, Vol. 2, pp. 48-58)
  - A method basing on vanishing lines (requires such objects that has perpendicular break-lines, such as buildings)

Change of exterior orientation when the object coordinate system is transformed

- Let's make a similarity transformation to the object coordinate system (scaling, rotation and translation)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mu S \begin{bmatrix} X' - X'_m \\ Y' - Y'_m \\ Z' - Z'_m \end{bmatrix}$$

- When this is placed in the collinearity equations

$$\begin{bmatrix} x \\ y \\ -c \end{bmatrix} = \lambda R \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix}$$, we get $$\begin{bmatrix} x \\ y \\ -c \end{bmatrix} = \lambda' R' \begin{bmatrix} X' - X'_0 \\ Y' - Y'_0 \\ Z' - Z'_0 \end{bmatrix}$$

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Change of exterior orientation when the object coordinate system is transformed

- In the equation $$\begin{bmatrix} x \\ y \\ -c \end{bmatrix} = \lambda' R' \begin{bmatrix} X' - X'_0 \\ Y' - Y'_0 \\ Z' - Z'_0 \end{bmatrix}$$
  - A new scale is $\lambda' = \mu\lambda$
  - A new rotation matrix is $R' = RS$
  - A new location of projection center is $\begin{bmatrix} X'_0 \\ Y'_0 \\ Z'_0 \end{bmatrix} = \frac{1}{\mu} S^T \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} + \begin{bmatrix} X'_m \\ Y'_m \\ Z'_m \end{bmatrix}$

- Using these equation, you can calculate the exterior orientation of images when you know the relative and absolute orientation of images

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Relative orientation

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Relative orientation

- In relative orientation, we try to solve relative location and attitude between two images (observation rays)
- We have several alternative methods to solve relative orientation
- First of all, we can select our mathematical model to be
  - *Coplanarity condition*
  - *Collinearity condition*

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Relative orientation

- In both cases, we can select parameters to be solved with several variations (we define the model coordinate system i.e. remove datum defect =7)
  - *Relative orientation of successive images:* we fix $\kappa_1, \varphi_1, \omega_1, X_{0_1}, Y_{0_1}, Z_{0_1}$ *and* X_{0_2} (i.e. left image+ X component of base) and solve $\kappa_2, \varphi_2, \omega_2, Y_{0_2}, Z_{0_2}$
  - *Relative orientation by rotations of the images:* we fix $\omega_1, X_{0_1}, Y_{0_1}, Z_{0_1}, X_{0_2}, Y_{0_2}, Z_{0_2}$ (base and rotation around the base) and solve $\kappa_1, \varphi_1, \kappa_2, \varphi_2, \omega_2$
  - *Minimum norm method:* a unique solution is ensured by using minimum norm condition $\|p\| = \min$, in which *p* is a vector that includes 12 exterior orientation parameters (and possibly also coordinates of object points).

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Datum defect

- In relative orientation, image observations give no information about a 7-parameter transformation of the object coordinate system, because we measure only corresponding points between images
  - Scale, three rotations (e.g. omega, phi, kappa) and three shifts (e.g. X_0, Y_0, Z_0)
  - If we make any 3D measurements from stereo images (relative oriented), we'll have an arbitrary scale and coordinate system

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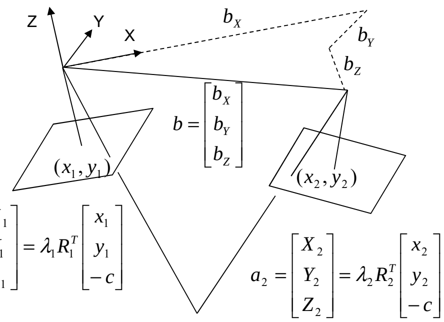
Relative orientation of successive images, Coplanarity condition

- Solution that is basing on the Coplanarity condition relies on the fact that corresponding observation rays lay on the same plane

$$G = \begin{vmatrix} b_X & b_Y & b_Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = b_X(Y_1Z_2 - Y_2Z_1) - b_Y(X_1Z_2 - X_2Z_1) + b_Z(X_1Y_2 - X_2Y_1) = 0$$



$$b = \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix}$$

$$a_1 = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \lambda_1 R_1^T \begin{bmatrix} x_1 \\ y_1 \\ -c \end{bmatrix}$$

$$a_2 = \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \lambda_2 R_2^T \begin{bmatrix} x_2 \\ y_2 \\ -c \end{bmatrix}$$

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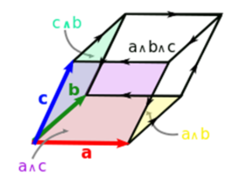
The Coplanarity Condition



Vectors *b*, U_1 and U_2 should lie on the same plane. If the volume of the parallelepiped (triple product) defined by these three vectors equals to zero, we know that the vectors lie at the same plane

$$G = b \cdot U_1 \times U_2 = 0$$

$$b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \quad U_1 = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} \quad U_2 = \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}$$

The triple product in the determinant form:

$$G = \begin{vmatrix} b_X & b_Y & b_Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = 0$$

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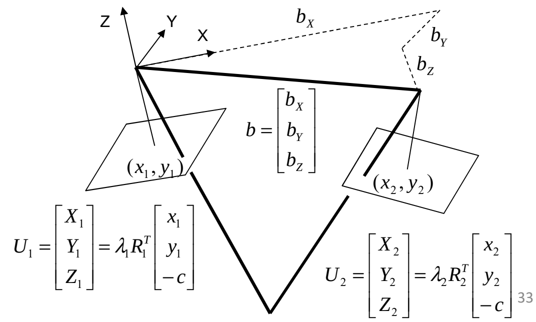
Relative orientation of successive images, coplanarity condition

- The observation vector of the left image (rotations is fixed to zero) $\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ -c \end{bmatrix}$
- The observation vector of the right image $\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = R_2^T \begin{bmatrix} x_2 \\ y_2 \\ -c \end{bmatrix}$
- Base vector

$$b = \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix} = \begin{bmatrix} X_{02} \\ Y_{02} \\ Z_{02} \end{bmatrix}$$



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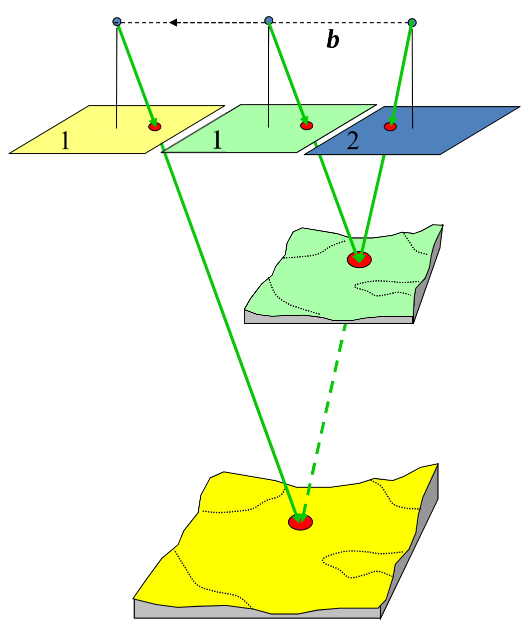
Changing the length of the image base (b_x) affects only to the scale of 3D model



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Relative orientation of successive images, coplanarity condition

- A non-linear mixed model => linearization and the general LS adjustment
- Coplanarity equation $G = \begin{vmatrix} b_X & b_Y & b_Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = b_X(Y_1Z_2 - Y_2Z_1) - b_Y(X_1Z_2 - X_2Z_1) + b_Z(X_1Y_2 - X_2Y_1) = 0$ becomes after linearization (for one corresponding point pair)

$$\begin{bmatrix} \frac{\partial G^0}{\partial x_1} & \frac{\partial G^0}{\partial y_1} & \frac{\partial G^0}{\partial x_2} & \frac{\partial G^0}{\partial y_2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dy_1 \\ dx_2 \\ dy_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial G^0}{\partial b_Y} & \frac{\partial G^0}{\partial b_Z} & \frac{\partial G^0}{\partial \omega_2} & \frac{\partial G^0}{\partial \varphi_2} & \frac{\partial G^0}{\partial \kappa_2} \end{bmatrix} \begin{bmatrix} db_Y \\ db_Z \\ d\omega_2 \\ d\varphi_2 \\ d\kappa_2 \end{bmatrix} + G^0 = 0$$

i.e. $C^T dl + Ddp + G^0 = 0$

- In which $G^0 = G(x_1^0, y_1^0, x_2^0, y_2^0, b_Y^0, b_Z^0, \omega_2^0, \varphi_2^0, \kappa_2^0)$ is the value of the coplanarity equation calculated using a current approximate values of parameters

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Relative orientation of successive images, coplanarity condition

- In the general LS adjustment, the corrections to parameters (*dp*) and observations (residuals, *v*) and Lagrange multipliers (*k*) are solved from

$$\begin{bmatrix} -P & C & 0 \\ C^T & 0 & D \\ 0 & D^T & 0 \end{bmatrix} \begin{bmatrix} \hat{v} \\ \hat{k} \\ d\hat{p} \end{bmatrix} = \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix}$$

in which $h = -G(l^0, x^0) - C^T(l - l^0)$ and a weight matrix P usually equals to I (identity matrix)

$$h_j = -G_j - \begin{bmatrix} \frac{\partial G_j}{\partial x_{1j}} & \frac{\partial G_j}{\partial y_{1j}} & \frac{\partial G_j}{\partial x_{2j}} & \frac{\partial G_j}{\partial y_{2j}} \end{bmatrix} \begin{bmatrix} x_{1j} - x_{1j}^0 \\ y_{1j} - y_{1j}^0 \\ x_{2j} - x_{2j}^0 \\ y_{2j} - y_{2j}^0 \end{bmatrix} \qquad h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \qquad \hat{v} = ld$$

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Relative orientation of successive images, coplanarity condition

$$\begin{bmatrix} -P & C & 0 \\ C^T & 0 & D \\ 0 & D^T & 0 \end{bmatrix} \begin{bmatrix} \hat{v} \\ \hat{k} \\ d\hat{p} \end{bmatrix} = \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix}$$

- In addition, design matrices are

The design matrix of observations

$$C^T = \begin{bmatrix} \frac{\partial G_1}{\partial x_{11}} & \frac{\partial G_1}{\partial y_{11}} & \frac{\partial G_1}{\partial x_{21}} & \frac{\partial G_1}{\partial y_{21}} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial G_2}{\partial x_{12}} & \frac{\partial G_2}{\partial y_{12}} & \frac{\partial G_2}{\partial x_{22}} & \frac{\partial G_2}{\partial y_{22}} & \dots & 0 & 0 & 0 & 0 \\ & & & & & & & & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \frac{\partial G_n}{\partial x_{1n}} & \frac{\partial G_n}{\partial y_{1n}} & \frac{\partial G_n}{\partial x_{2n}} & \frac{\partial G_n}{\partial y_{2n}} \end{bmatrix}$$

The design matrix of orientation parameters

$$D = \begin{bmatrix} \frac{\partial G_1}{\partial b_Y} & \frac{\partial G_1}{\partial b_Z} & \frac{\partial G_1}{\partial \omega_2} & \frac{\partial G_1}{\partial \varphi_2} & \frac{\partial G_1}{\partial \kappa_2} \\ \frac{\partial G_2}{\partial b_Y} & \frac{\partial G_2}{\partial b_Z} & \frac{\partial G_2}{\partial \omega_2} & \frac{\partial G_2}{\partial \varphi_2} & \frac{\partial G_2}{\partial \kappa_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial G_n}{\partial b_Y} & \frac{\partial G_n}{\partial b_Z} & \frac{\partial G_n}{\partial \omega_2} & \frac{\partial G_n}{\partial \varphi_2} & \frac{\partial G_n}{\partial \kappa_2} \end{bmatrix}$$

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Relative orientation of successive images, coplanarity condition

- Equations have to be linearized by computing partial derivatives with respect to all unknown parameters
- As an example, here we present only the partial derivation of the coplanarity equation with respect to a parameter b_y

$$\frac{\partial G}{\partial b_Y} = \begin{vmatrix} \partial b_X / \partial b_Y & \partial b_Y / \partial b_Y & \partial b_Z / \partial b_Y \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = -\begin{vmatrix} X_1 & Z_1 \\ X_2 & Z_2 \end{vmatrix} = -X_1 Z_2 + X_2 Z_1$$

- Because parameter b_y exists only in one row (in the determinant form), we are able to make partial derivation separately to that row, and after that to calculate the determinant
- In the case of rotations, we can use previously presented methods how to make partial derivatives of 3D rotation matrices with respect to rotation angles

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Relative orientation of successive images, coplanarity condition

- Step-by-step solution is

$$\begin{bmatrix} -P & C & 0 \\ C^T & 0 & D \\ 0 & D^T & 0 \end{bmatrix} \begin{bmatrix} \hat{v} \\ \hat{k} \\ d\hat{p} \end{bmatrix} = \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix}$$

Corrections to orientation parameters $d\hat{p} = \left(D^T W D\right)^{-1} D^T W h$

$$W = (C^T P^{-1} C)^{-1}$$

$$h = -G_0 - C^T l$$

Lagrange multipliers $\hat{k} = W(h - Dd\hat{p})$

Corrections to image observations $\hat{v} = C\hat{k}$

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Relative orientation of successive images, coplanarity condition

- Again, the method is iterative and therefore we need initial values to parameters
- One method to get approximate values is to use a linear (projective) method (corresponding points $\rightarrow$ epipolar matrix $\rightarrow$ physical parameters)
- After each iteration round, new approximations to parameters are calculated $p^{(k)} = p^{(k-1)} + dp^{(k)}$
  $l^{(k)} = l^{(k-1)} + v^{(k)}$
- If the number of measured corresponding points is *n*, redundancy is $r = n - 5$

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Relative orientation of successive images, collinearity condition

- The relative orientation of successive images can be established also by using the collinearity equations (instead of the coplanarity equations)
- This method is called as the relative orientation with the bundle block method, because it is only a special case of bundle block adjustment (the number of images is only two)

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Relative orientation of successive images, collinearity condition

- In this case, we have to linearize collinearity equations (with respect to unknown parameters)

$$\begin{cases} x = -c\dfrac{r_{11}(X - X_0) + r_{12}(Y - Y_0) + r_{13}(Z - Z_0)}{r_{31}(X - X_0) + r_{32}(Y - Y_0) + r_{33}(Z - Z_0)} = f_x \\ y = -c\dfrac{r_{21}(X - X_0) + r_{22}(Y - Y_0) + r_{23}(Z - Z_0)}{r_{31}(X - X_0) + r_{32}(Y - Y_0) + r_{33}(Z - Z_0)} = f_y \end{cases}$$

- The result of linearization is (for one corresponding point pair)

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\partial f_{x1}}{\partial X} & \frac{\partial f_{x1}}{\partial Y} & \frac{\partial f_{x1}}{\partial Z} \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial f_{x1}}{\partial X} & \frac{\partial f_{x1}}{\partial Y} & \frac{\partial f_{x1}}{\partial Z} \\ \frac{\partial f_{x2}}{\partial \omega_2} & \frac{\partial f_{x2}}{\partial \varphi_2} & \frac{\partial f_{x2}}{\partial \kappa_2} & \frac{\partial f_{x2}}{\partial Y_{02}} & \frac{\partial f_{x2}}{\partial Z_{02}} & \frac{\partial f_{x2}}{\partial X} & \frac{\partial f_{x2}}{\partial Y} & \frac{\partial f_{x2}}{\partial Z} \\ \frac{\partial f_{y2}}{\partial \omega_2} & \frac{\partial f_{y2}}{\partial \varphi_2} & \frac{\partial f_{y2}}{\partial \kappa_2} & \frac{\partial f_{y2}}{\partial Y_{02}} & \frac{\partial f_{y2}}{\partial Z_{02}} & \frac{\partial f_{y2}}{\partial X} & \frac{\partial f_{y2}}{\partial Y} & \frac{\partial f_{y2}}{\partial Z} \end{bmatrix} \begin{bmatrix} d\omega_2 \\ d\varphi_2 \\ d\kappa_2 \\ dY_{02} \\ dZ_{02} \\ dX \\ dY \\ dZ \end{bmatrix} + \begin{bmatrix} f_{x1}^0 \\ f_{y1}^0 \\ f_{x2}^0 \\ f_{y1}^0 \end{bmatrix}$$

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Relative orientation of successive images, collinearity condition

- In these equation, values of the collinearity equations are calculated using current approximations of parameters

$$\begin{bmatrix} f_{x1}^0 \\ f_{y1}^0 \\ f_{x2}^0 \\ f_{y2}^0 \end{bmatrix} = \begin{bmatrix} f_{x1}(\omega_1, \varphi_1, \kappa_1, X_{01}, Y_{01}, Z_{01}, X^0, Y^0, Z^0) \\ f_{y1}(\omega_1, \varphi_1, \kappa_1, X_{01}, Y_{01}, Z_{01}, X^0, Y^0, Z^0) \\ f_{x2}(\omega_2^0, \varphi_2^0, \kappa_2^0, X_{02}^0, Y_{02}^0, Z_{02}^0, X^0, Y^0, Z^0) \\ f_{y2}(\omega_2^0, \varphi_2^0, \kappa_2^0, X_{02}^0, Y_{02}^0, Z_{02}^0, X^0, Y^0, Z^0) \end{bmatrix}$$

- In a design matrix, two first rows are associated to one image (because the coordinate system is set to the camera coordinate system of this image, we don't have to solve its rotations or translations -> = 0)
- Next two rows are associated to another image (the adjacent image to the first one)

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Relative orientation of successive images, collinearity condition

- A solution can be calculated using LS adjustment with observation equations
- Notice that object coordinates are now included in adjustment. Therefore, we need approximate values also for them
- However, there is a possibility to eliminate object points from the normal equation
- As a result, we get a reduced normal equation, from which we can solve five orientation parameters
- If the number of corresponding points is N => we get $u = 5 + 3N$ equations => $n = 4N$ unknowns => should be $N \geq 5$ in order to get redundancy

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Space intersection

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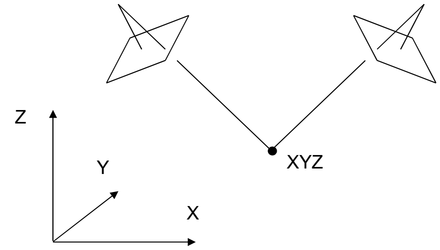
Space intersection

- We know (interior and) exterior orientation of images
- We measure image observations (image coordinates) of a common object point from two or more images
- We solve object coordinates (X,Y,Z)



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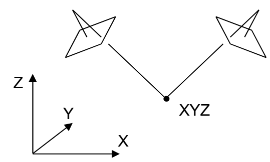
Space intersection

- The collinearity equations are used as the mathematical model



$$\begin{cases} x = -c\dfrac{r_{11}(X-X_0)+r_{12}(Y-Y_0)+r_{13}(Z-Z_0)}{r_{31}(X-X_0)+r_{32}(Y-Y_0)+r_{33}(Z-Z_0)} = f_x \\ y = -c\dfrac{r_{21}(X-X_0)+r_{22}(Y-Y_0)+r_{23}(Z-Z_0)}{r_{31}(X-X_0)+r_{32}(Y-Y_0)+r_{33}(Z-Z_0)} = f_y \end{cases}$$

- After the linearization (*X*, *Y* and *Z* are unknown parameters), we get normal equations (indices: image *i*, object point *j*)

$$\begin{bmatrix} v_{xij} \\ v_{yij} \end{bmatrix} = \begin{bmatrix} \partial f_x^0/\partial X & \partial f_x^0/\partial Y & \partial f_x^0/\partial Z \\ \partial f_y^0/\partial X & \partial f_y^0/\partial Y & \partial f_y^0/\partial Z \end{bmatrix} \begin{bmatrix} dX_j \\ dY_j \\ dZ_j \end{bmatrix} - \begin{bmatrix} x_{ij} - f_{xij}^0 \\ y_{ij} - f_{yij}^0 \end{bmatrix}$$

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Space intersection

- In error equations, values of the collinearity equation are computed by using approximate values of parameters

$$\begin{bmatrix} f_{xij}^0 \\ f_{yij}^0 \end{bmatrix} = \begin{bmatrix} f_x\,(\omega_i, \varphi_i, \kappa_i, X_{0i}, Y_{0i}, Z_{0i}, X_j^0, Y_j^0, Z_j^0) \\ f_y\,(\omega_i, \varphi_i, \kappa_i, X_{0i}, Y_{0i}, Z_{0i}, X_j^0, Y_j^0, Z_j^0) \end{bmatrix}$$

- 3 unknowns
- 2 equations/images
- overdetermined, if we have at least two images
- Iterative LS solution

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Space intersection, alternative solution

- We modify the collinearity equations to the form

$$\begin{cases} x(r_{31}(X - X_0) + r_{32}(Y - Y_0) + r_{33}(Z - Z_0)) = -c(r_{11}(X - X_0) + r_{12}(Y - Y_0) + r_{13}(Z - Z_0)) \\ y(r_{31}(X - X_0) + r_{32}(Y - Y_0) + r_{33}(Z - Z_0)) = -c(r_{21}(X - X_0) + r_{22}(Y - Y_0) + r_{23}(Z - Z_0)) \end{cases}$$

and furthermore

$$\begin{cases} (xr_{31} + cr_{11})X + (xr_{32} + cr_{12})Y + (xr_{33} + cr_{13})Z = (xr_{31} + cr_{11})X_0 + (xr_{32} + cr_{12})Y_0 + (xr_{33} + cr_{13})Z_0 \\ (yr_{31} + cr_{21})X + (yr_{32} + cr_{22})Y + (yr_{33} + cr_{23})Z = (yr_{31} + cr_{21})X_0 + (yr_{32} + cr_{22})Y_0 + (yr_{33} + cr_{23})Z_0 \end{cases}$$

- When image coordinates *(x,y)* of an unknown object point are measured from $k \geq 2$ images, we can solve object coordinates directly without iterations or initial values (we get *2N* linear equations)

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Space intersection, accuracy

- The accuracy of object points is dependent on
  - Measuring accuracy
  - The number of intersecting rays
  - Scale (c/Z)
  - Geometry of intersecting rays (imaging geometry)

- The effect of geometry :
  - Planimetric accuracy is linearly dependent on *Z*. The best geometry is achieved when *X*=*Y*=0.
  - Height accuracy is proportional to Z^3

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