GIS-E3010 Least-Squares Methods in Geoscience

Lecture 6

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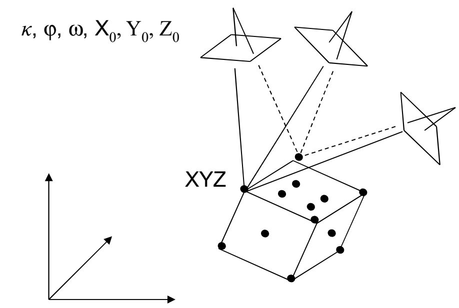
Learning objectives

- To understand
  - Bundle block adjustment

Bundle block adjustment /triangulation

- In bundle adjustment, we solve simultaneously
  - Exterior orientations of images (observation ray bundle) (6m parameters, m = the number of images) and
  - 3D coordinates of object points i.e. triangulation points (3n parameters, n = the number of points)



Bundle block adjustment /triangulation

- The shape of an object become known (usually not a very dense point cloud) by tie points
- In order to define size and location, we need additional information, such as geodetically measured ground control points or directly measured exterior orientation parameters (GPS/INS)
- When triangulation is based on aerial images, we call it as aerial triangulation
- Aerial triangulation can ”cross” large areas without ground control points and, therefore, also can densify a sparse geodetic network to the requirements of , e.g., stereo mapping (point densification)

Bundle block adjustment /triangulation

- Usually, the mathematical model is the collinear equations (indexing: image *i*, object point *j*)

$$\begin{cases} x_{ij} = -c\dfrac{r_{11}^{(i)}(X_j - X_{0i}) + r_{12}^{(i)}(Y_j - Y_{0i}) + r_{13}^{(i)}(Z_j - Z_{0i})}{r_{31}^{(i)}(X_j - X_{0i}) + r_{32}^{(i)}(Y_j - Y_{0i}) + r_{33}^{(i)}(Z_j - Z_{0i})} \\ y_{ij} = -c\dfrac{r_{21}^{(i)}(X_j - X_{0i}) + r_{22}^{(i)}(Y_j - Y_{0i}) + r_{23}^{(i)}(Z_j - Z_{0i})}{r_{31}^{(i)}(X_j - X_{0i}) + r_{32}^{(i)}(Y_j - Y_{0i}) + r_{33}^{(i)}(Z_j - Z_{0i})} \end{cases}$$

- These equations can also be expressed as

$$\begin{cases} x_{ij} = x(\omega_i, \varphi_i, \kappa_i, X_{0i}, Y_{0i}, Z_{0i}, X_j, Y_j, Z_j) \\ y_{ij} = y(\omega_i, \varphi_i, \kappa_i, X_{0i}, Y_{0i}, Z_{0i}, X_j, Y_j, Z_j) \end{cases}$$

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Bundle block adjustment /triangulation

- The collinearity equations are non-linear, and therefore, we have to linearize them (Taylor series)

$$\begin{cases} x_{ij} \approx x_{ij}^0 + \dfrac{\partial x_{ij}^0}{\partial \omega_i} d\omega_i + \ldots + \dfrac{\partial x_{ij}^0}{\partial X_{0i}} dX_{0i} + \ldots + \dfrac{\partial x_{ij}^0}{\partial X_j} dX_j + \ldots + \dfrac{\partial x_{ij}^0}{\partial Z_j} dZ_j \\ y_{ij} \approx y_{ij}^0 + \end{cases}$$

- How to find initial values of parameters?
  - Not a big problem in aerial triangulation!
  - Might be problematic in the case of terrestrial applications

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Bundle block adjustment /triangulation

- Each measured image point (observation ray) establishes two error equations

$$\begin{bmatrix} v_{xij} \\ v_{yij} \end{bmatrix} = \begin{bmatrix} \partial x_{ij}^0 / \partial \omega_i & . & . & . & . & . \\ . & . & . & . & . & \partial y_{ij}^0 / \partial Z_{0i} \end{bmatrix} \begin{bmatrix} d\omega_i \\ d\varphi_i \\ d\kappa_i \\ dX_{0i} \\ dY_{0i} \\ dZ_{0i} \end{bmatrix} + \begin{bmatrix} . & . & . \\ . & . & . \end{bmatrix} \begin{bmatrix} dX_j \\ dY_j \\ dZ_j \end{bmatrix} - \begin{bmatrix} x_{ij} - x_{ij}^0 \\ y_{ij} - y_{ij}^0 \end{bmatrix}$$

i.e. $v_{ij} = \dot{A}_{ij} \dot{\Delta}_i + \ddot{A}_{ij} \ddot{\Delta}_j - f_{ij}$

Bundle block adjustment /triangulation

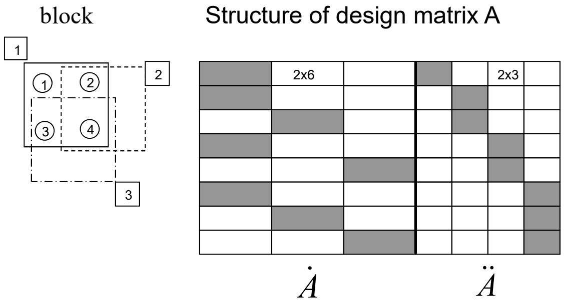
- If all points are included, we can write the error equations as

$$v = A\Delta - f = \begin{bmatrix} \dot{A} & \ddot{A} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} - f$$

- $\dot{A}$ includes information about image orientation parameters and $\ddot{A}$ includes information about object points

Bundle block adjustment /triangulation

- The structure of a block is following (in this example, we have 3 images and 4 points; the point 1 is visible from one image, points 3 and 4 from two images and the point 4 from three images



Alternatively, the structure can be established in a way that at first we deal with all points that are visible from image 1 and then we move on with points that are visible from image 2.

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Bundle block adjustment /triangulation

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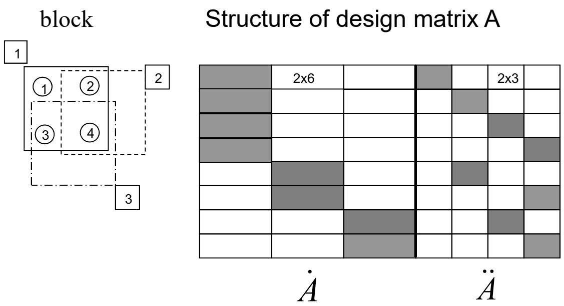

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Bundle block adjustment /triangulation

- If we examine only image observations, the normal equations are (LS adjustment with observation equations)

$$A^T PA\Delta = A^T Pf \quad \text{i.e.} \quad N\Delta = u$$

- If we remember partitioning $A = \begin{bmatrix} \dot{A} & \ddot{A} \end{bmatrix}$, normal equations become as

$$\begin{bmatrix} \dot{A}^T P\dot{A} & \dot{A}^T P\ddot{A} \\ \ddot{A}^T P\dot{A} & \ddot{A}^T P\ddot{A} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \dot{A}^T Pf \\ \ddot{A}^T Pf \end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix} \dot{N} & \overline{N} \\ \overline{N}^T & \ddot{N} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \ddot{u} \end{bmatrix}$$

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Bundle block adjustment /triangulation

- The size of a normal matrix *N* is (6m+3n)x(6m+3n), in which *m* is the number of images and *n* the number of object points
- Therefore in bundle block adjustment, we usually have to solve an extremely large group of equations
- On the other side, such groups of equations are typically sparse, i.e., most of the elements are zeroes
- The reason for this is that each observation equation includes only a small amount of all unknowns, and therefore *A* and also $N = A^T A$ are sparse

$$\begin{bmatrix} \dot{N} & \overline{N} \\ \overline{N}^T & \ddot{N} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \ddot{u} \end{bmatrix}$$

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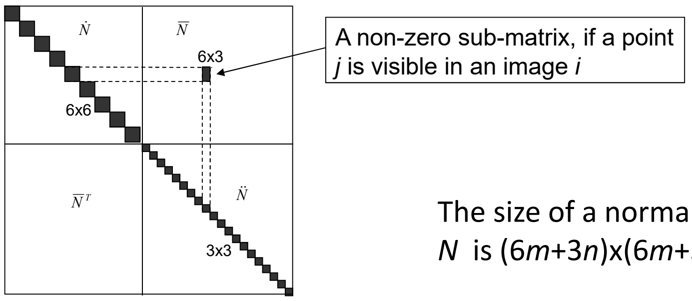
$$\begin{bmatrix} \dot{N} & \overline{N} \\ \overline{N}^T & \ddot{N} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \ddot{u} \end{bmatrix}$$

Bundle block adjustment /triangulation

- It's also important to detect that a matrix *N* has a special structure



The size of a normal matrix *N* is (6*m*+3*n*)x(6*m*+3*n*)

E.g. A block diagonal matrix (such as $\dot{N}$ and $\ddot{N}$) can be inverted by inverting separately small sub-matrices along the diagonal.

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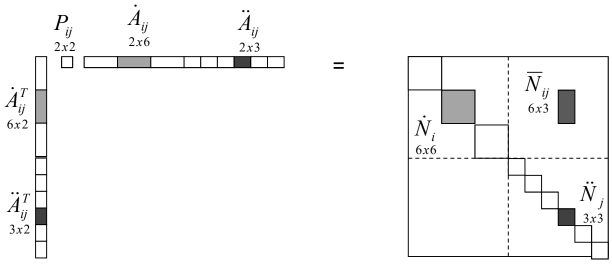
$$\begin{bmatrix} \dot{N} & \overline{N} \\ \overline{N}^T & \ddot{N} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \ddot{u} \end{bmatrix}$$

Bundle block adjustment /triangulation

- By partitioning a matrix *A* row by row, we can conclude that it is possible to establish a matrix *N* in a cumulative way

$$N = A^T P A = \sum_k A_k^T P_k A_k$$



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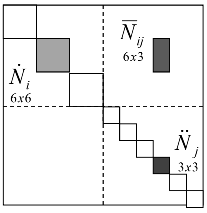
Bundle block adjustment /triangulation

- The algorithm of building a matrix *N* in a cumulative way is:

$$\begin{bmatrix}\dot{N} & \bar{N}\\ \bar{N}^T & \ddot{N}\end{bmatrix}\begin{bmatrix}\dot{\Delta}\\ \ddot{\Delta}\end{bmatrix}=\begin{bmatrix}\dot{u}\\ \ddot{u}\end{bmatrix}$$

$\bar{N}_{ij}=0$, if a point j is not visible in an image i.



$i=1,\ldots,m$

$\quad j=1,\ldots,n$

$\quad\quad$ if a point j is visible in an image i, compute

$$\dot{A}_{ij}=\ldots;\quad \ddot{A}_{ij}=\ldots;\quad f_{ij}=\ldots$$

$$\dot{N}_i=\dot{N}_i+\dot{A}_{ij}^T P_{ij}\dot{A}_{ij};\quad \dot{u}_i=\dot{u}_i+\dot{A}_{ij}^T P_{ij} f_{ij}$$

$$\ddot{N}_j=\ddot{N}_j+\ddot{A}_{ij}^T P_{ij}\ddot{A}_{ij};\quad \ddot{u}_j=\ddot{u}_j+\ddot{A}_{ij}^T P_{ij} f_{ij}$$

$$\bar{N}_{ij}=\dot{A}_{ij}^T P_{ij}\ddot{A}_{ij}$$

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Bundle block adjustment /triangulation

- In order to solve normal equations $\begin{bmatrix}\dot{N} & \bar{N}\\ \bar{N}^T & \ddot{N}\end{bmatrix}\begin{bmatrix}\dot{\Delta}\\ \ddot{\Delta}\end{bmatrix}=\begin{bmatrix}\dot{u}\\ \ddot{u}\end{bmatrix}$

  we multiply both sides (from left) with a matrix

$$\begin{bmatrix}I & -\bar{N}\ddot{N}^{-1}\\ 0 & I\end{bmatrix}$$

- The result is $\begin{bmatrix}\dot{N}-\bar{N}\ddot{N}^{-1}\bar{N}^T & 0\\ \bar{N}^T & \ddot{N}\end{bmatrix}\begin{bmatrix}\dot{\Delta}\\ \ddot{\Delta}\end{bmatrix}=\begin{bmatrix}\dot{u}-\bar{N}\ddot{N}^{-1}\ddot{u}\\ \ddot{u}\end{bmatrix}$

- Now we are able to solve equations in two parts:

$$\dot{\Delta}=\left(\dot{N}-\bar{N}\ddot{N}^{-1}\bar{N}^T\right)^{-1}\left(\dot{u}-\bar{N}\ddot{N}^{-1}\ddot{u}\right)$$

So called ***Reduced normal equations***

$$\ddot{\Delta}=\ddot{N}^{-1}\left(\ddot{u}-\bar{N}^T\dot{\Delta}\right)$$

The same result can be found if we solve $\ddot{\Delta}$ from the lower equation and place that in the upper equation

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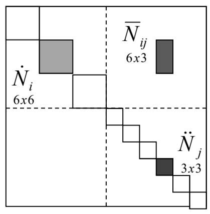
Bundle block adjustment /triangulation

- The zeroes of a block diagonal matrix $\ddot{N}$ can be bypassed using elimination, because the inverse of a block diagonal matrix is also a block diagonal matrix

$$(diag(N_1,...,N_k))^{-1} = diag(N_1^{-1},...,N_k^{-1})$$

- Elimination is computationally fast because sub matrices $\ddot{N}_j$ are 3x3 matrices



Bundle block adjustment /triangulation

- The method is iterative i.e. we need initial values for all unknown parameters, and the adjustment provides us corrections to the approximate values of parameters. These corrections are added to current approximate values.
- However, if the block adjustment includes only image observations, we have a problem
  - LS condition ensures solution, but not a unique solution

Bundle block adjustment /triangulation

- The problem is datum defect, and the reason of it is that photogrammetric observations do not include datum information (image observations are invariants to 7-parametric similarity transformation)
- Such block is called as *a free network* (observations define shape, but not location, attitude or size/scale)
- For example following additional data include needed datum information:
  - Coordinates of points (object points or projection centers of images)
  - Coordinate differences or distances between points

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Bundle block adjustment /triangulation

- Datum defect can be removed
  - With additional observations (in following we denote those with a sub matrix *B*) such as GPS/IMU or geodetic object point measurements
  - Additional constraints (in following we denote them with a sub matrix *C*) such as forcing a point to lay on a known line or curve
- Following conditions should be fulfilled
  - The number of linearly independent rows of matrices B/C should be ≥ 7
  - Rows of matrices B/C are linearly independent from the rows of matrix A

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Bundle block adjustment, additional observations

- In addition to image observations, we can include measured values of any function or parameters
- Those establish (additional) error equations

$$v' = B\Delta - f' \quad \text{i.e.} \quad v' = \begin{bmatrix} \dot{B} & \ddot{B} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} - f'$$

- For example, measuring an object point *j* directly with GPS establishes an additional error equation

$$\ddot{v}_j = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} dX_j \\ dY_j \\ dZ_j \end{bmatrix} + \begin{bmatrix} X_j^0 \\ Y_j^0 \\ Z_j^0 \end{bmatrix} - \begin{bmatrix} X_j \\ Y_j \\ Z_j \end{bmatrix} \quad \text{i.e.} \quad \ddot{v}_j = I\ddot{\Delta}_j - \ddot{f}_j$$

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Bundle block adjustment, additional observations

- Corresponding example is the measurement of exterior orientations of images using direct orientation sensors (e.g. GPS/INS). This gives us additional error equations

$$\dot{v}_i = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} d\omega_i \\ d\varphi_i \\ d\kappa_i \\ dX_{0i} \\ dY_{0i} \\ dZ_{0i} \end{bmatrix} + \begin{bmatrix} \omega_i^0 \\ \varphi_i^0 \\ \kappa_i^0 \\ X_{0i}^0 \\ Y_{0i}^0 \\ Z_{0i}^0 \end{bmatrix} - \begin{bmatrix} \omega_i \\ \varphi_i \\ \kappa_i \\ X_{0i} \\ Y_{0i} \\ Z_{0i} \end{bmatrix} \quad \text{i.e.} \quad \dot{v}_i = I\dot{\Delta}_i - \dot{f}_i$$

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Bundle block adjustment, additional observations

- *A distance observation* $s_{ij} = \sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2}$

  gives an additional error equation

$$v_{jk} = a^0_{jk} dX_j + b^0_{jk} dY_j + c^0_{jk} dZ_j - a^0_{jk} dX_k - b^0_{jk} dY_k - c^0_{jk} dZ_k + s^0_{jk} - s_{jk}$$

  in which

$$a_{jk} = \frac{X^0_j - X^0_k}{s^0_{jk}} \qquad b_{jk} = \frac{Y^0_j - Y^0_k}{s^0_{jk}} \qquad c_{jk} = \frac{Z^0_j - Z^0_k}{s^0_{jk}}$$

- A height distance observation $h_{ij} = Z_i - Z_j$ gives an additional error equation

$$v_{ij} = dZ_i - dZ_j + (Z^0_i - Z^0_j) - h_{ij}$$

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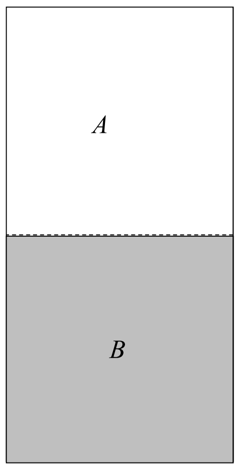
Bundle block adjustment, additional observations

- Together image and additional observations establish an error equation

$$\begin{bmatrix} v \\ v' \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} \Delta - \begin{bmatrix} f \\ f' \end{bmatrix}$$



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Bundle block adjustment, additional observations

- An error equation, in which we have image observations and additional observations, gives us normal equations

$$(A^T PA + B^T P'B)\Delta = A^T Pf + B^T P'f'$$

i.e.

$$(N + B^T P'B)\Delta = u + B^T P'f'$$

The effect of additional observations can be established by simple adding!

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Bundle block adjustment, additional observations

- The solution of a normal equation can be calculated using the Cholesky decomposition
- Let's assume that the rank of a coefficient matrix *M* in the normal equation is full (image observations and additional observations)

$$(A^T PA + B^T P'B)\Delta = A^T Pf + B^T P'f' \quad \text{i.e.} \quad M\Delta = t$$

- Simple examples of additional observations that can remove datum defect are, e.g., following:
  - We known *XYZ* coordinates of two points and *Z* coordinate of a third point
  - We know all six exterior orientation parameters of one image and additionally one distance or coordinate difference between projection centers

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Bundle block adjustment, additional observations

- Because weight matrices *P* and *P´* are positively definite matrices, a coefficient matrix *M* is also positively definite
- Therefore, a normal equation $M\Delta = t$ can be solved using e.g. Cholesky method, which is an efficient direct solving method

Bundle block adjustment, additional observations

- An algorithm for solving a normal equation $M\Delta = t$ includes following steps:

1. Compute Cholesky decomposition $LL^T = M$
2. Solve lower triangle system $Ly = t$
3. Solve upper triangle system $L^T\Delta = y$

- Repeat computation until $\Delta^{(k)} \approx 0$

Bundle block adjustment, additional observations

- After the iteration, we get residuals of observations with the equation

$$\begin{bmatrix} v \\ v' \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} \Delta - \begin{bmatrix} f \\ f' \end{bmatrix} \approx \begin{bmatrix} f \\ f' \end{bmatrix}$$

- The correctness of the computation can be ensured with the equation

$$\begin{bmatrix} A \\ B \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & P' \end{bmatrix} \begin{bmatrix} v \\ v' \end{bmatrix} = 0 \quad \text{i.e.} \quad A^T P v + B^T P' v' = 0$$

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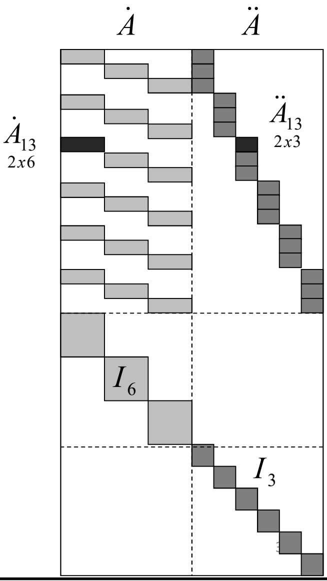
Bundle block adjustment, additional observations

- In a special case, in which additional observations are measurements of orientation parameters (e.g. GPS/INS) and object coordinates (e.g. Geodetic measurements), we get simpler equations



$$\begin{bmatrix} v \\ \dot{v} \\ \ddot{v} \end{bmatrix} = \begin{bmatrix} \dot{A} & \ddot{A} \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} - \begin{bmatrix} f \\ \dot{f} \\ \ddot{f} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} P & 0 & 0 \\ 0 & \dot{P} & 0 \\ 0 & 0 & \ddot{P} \end{bmatrix}$$

Bundle block adjustment, additional observations

- Weight matrix

$$P = \begin{bmatrix} P & 0 & 0 \\ 0 & \dot{P} & 0 \\ 0 & 0 & \ddot{P} \end{bmatrix}$$

Weights of image observations (P_{ij} is a 2x2 matrix)

$$P = diag(P_{11}, P_{21}, ...P_{m1}, \ldots, P_{1n}, P_{2n}, ...P_{mn})$$

Weights of GPS/INS observations ($\dot{P}_i$ is a 6x6 matrix)

$$\dot{P} = diag(\dot{P}_1, \dot{P}_2, \ldots, \dot{P}_m)$$

Weights of ground control observations ($\ddot{P}_j$ is a 3x3 matrix)

$$\ddot{P} = diag(\ddot{P}_1, \ddot{P}_2, \ldots, \ddot{P}_n)$$

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Bundle block adjustment, additional observations

- The error equation

$$\begin{bmatrix} v \\ \dot{v} \\ \ddot{v} \end{bmatrix} = \begin{bmatrix} \dot{A} & \ddot{A} \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} - \begin{bmatrix} f \\ \dot{f} \\ \ddot{f} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} P & 0 & 0 \\ 0 & \dot{P} & 0 \\ 0 & 0 & \ddot{P} \end{bmatrix}$$

gives us a normal equation

$$\widetilde{A}^T \widetilde{P} \widetilde{A} \Delta = \widetilde{A}^T \widetilde{P} \widetilde{f}$$

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Bundle block adjustment, additional observations

- $\widetilde{A}^T \widetilde{P}\widetilde{A}\Delta = \widetilde{A}^T \widetilde{P}\widetilde{f}$ includes

$$\begin{bmatrix} \dot{A}^T P \dot{A} + \dot{P} & \dot{A}^T P \ddot{A} \\ \ddot{A}^T P \dot{A} & \ddot{A}^T P \ddot{A} + \ddot{P} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \dot{A}^T Pf + \dot{P}\dot{f} \\ \ddot{A}^T Pf + \ddot{P}\ddot{f} \end{bmatrix}$$

i.e.

$$\begin{bmatrix} \dot{N} + \dot{P} & \overline{N} \\ \overline{N}^T & \ddot{N} + \ddot{P} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} \dot{u} + \dot{P}\dot{f} \\ \ddot{u} + \ddot{P}\ddot{f} \end{bmatrix}$$

i.e.

$$\widetilde{N}\Delta = \widetilde{u}$$

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Bundle block adjustment, additional observations

- In this special case, the structure of normal matrix remains unchanged (zeroes/non-zeroes) i.e. additional equations do not have effect on utilization of sparseness of matrices.

- A solution can be computed from reduced normal equations

$$\{(\dot{N} + \dot{P}) - \overline{N}(\ddot{N} + \ddot{P})^{-1}\overline{N}^T\}\dot{\Delta} = (\dot{u} + \dot{P}\dot{f}) - \overline{N}(\ddot{N} + \ddot{P})^{-1}(\ddot{u} + \ddot{P}\ddot{f})$$

$$(\ddot{N} + \ddot{P})\ddot{\Delta} = \ddot{u} + \ddot{P}\ddot{f} - \overline{N}^T\dot{\Delta}$$

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