MEC-E8001

FINITE ELEMENT ANALYSIS

2019

WHY FINITE ELEMENTS AND ITS THEORY?

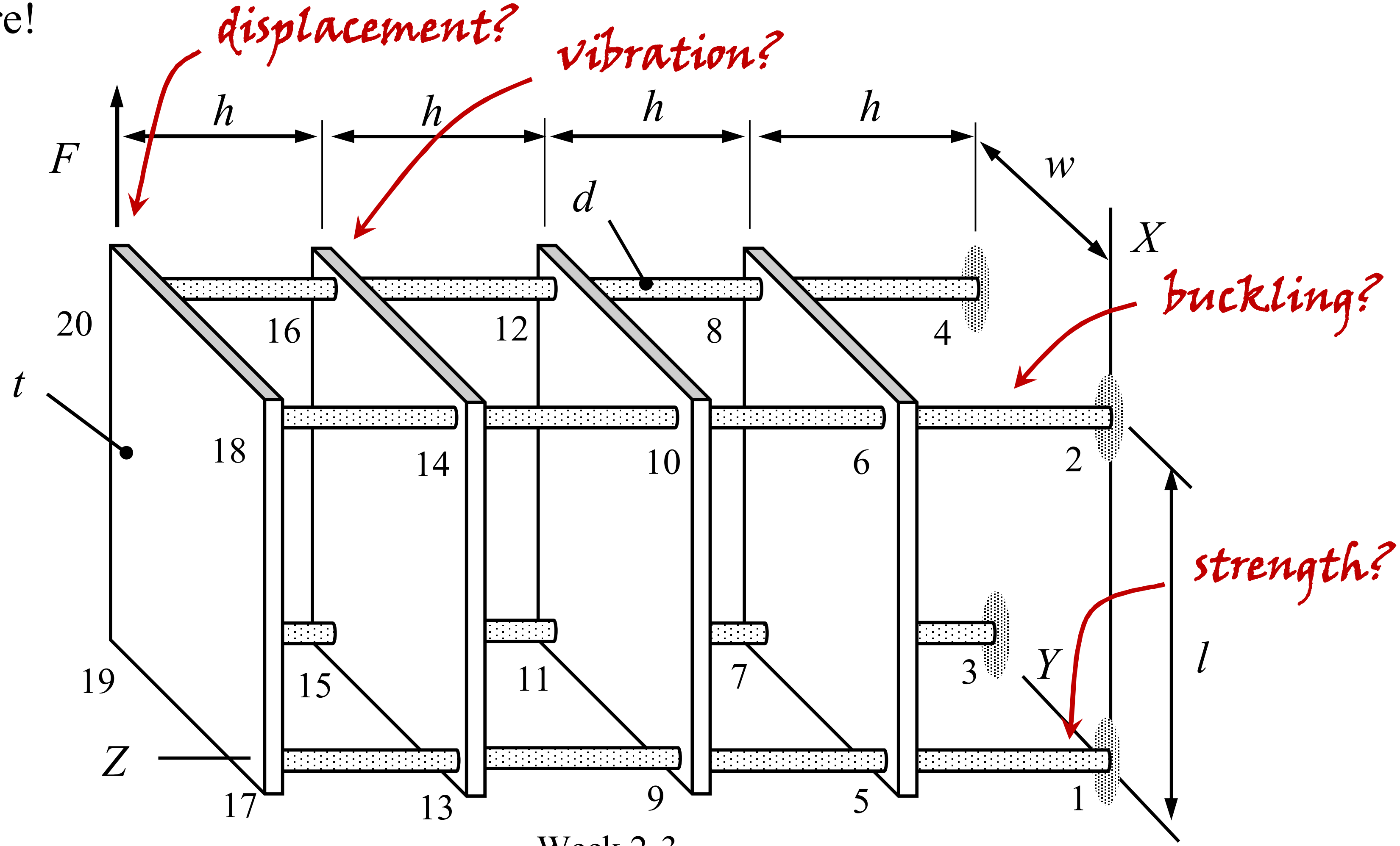
Design of machines and structures: Solution to stress or displacement by analytical method is often impossible due to complex geometry, heterogeneous material etc. Lack of the “exact solution” to an “approximate problem” is not an issue in engineering work.

Finite element method is the standard of solid mechanics: Commercial codes in common use are based on the finite element method. A graphical user interface may make living easier, but a user should always understand what the problem is and in what sense it is solved!

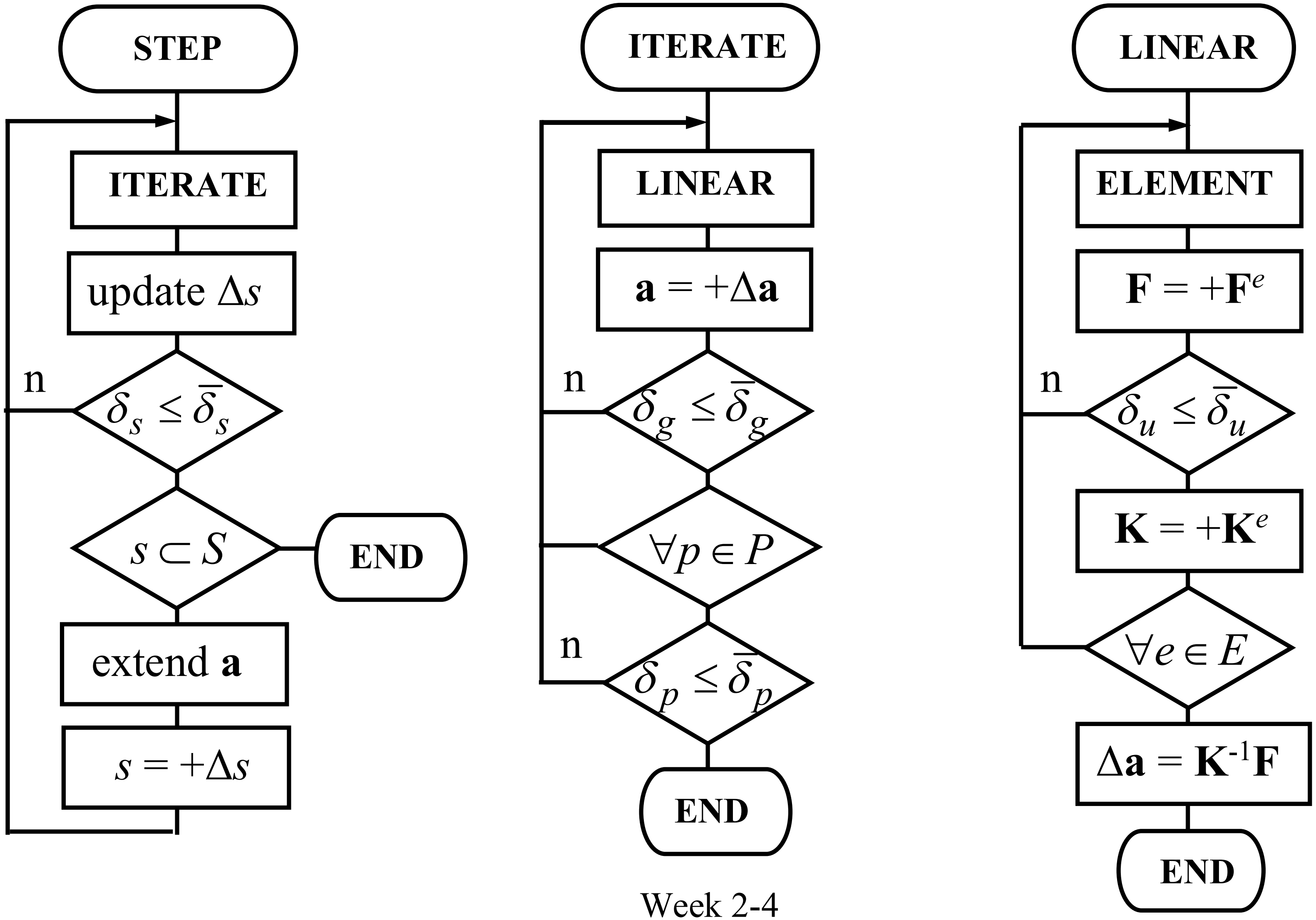
Finite element method has a strong theory: Although approximate solution is acceptable, knowing nothing about the error is not acceptable.

STRUCTURE ANALYSIS

Stress analysis according to the linear elasticity theory may not entirely explain behavior of a structure!



PROGRAMMER'S VIEWPOINT



LINEAR ELASTICITY

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. ←

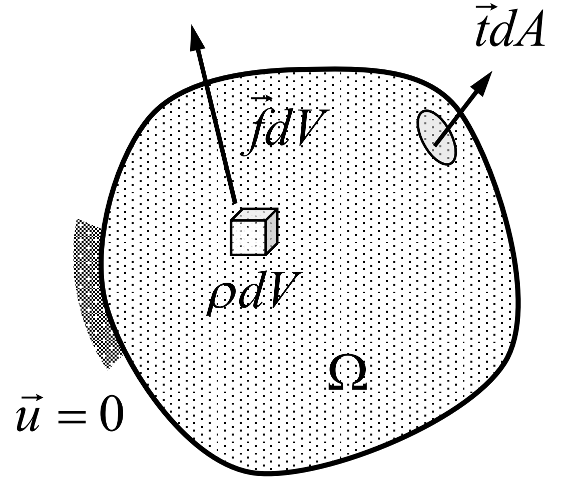
Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. ←

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

BOUNDARY VALUE PROBLEM

Assuming an equilibrium of a solid body (a set of particles) inside domain Ω, the aim is to find displacement $\vec{u}$ of the particles, when external forces or boundary conditions are changed in some manner:



Equilibrium equations $\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0$ in Ω,

Hooke's law $\vec{\vec{\sigma}}=\dfrac{E}{1+\nu}(\dfrac{\nu}{1-2\nu}\vec{\vec{I}}\nabla\cdot\vec{u}+\vec{\vec{\varepsilon}})$ in Ω,

Boundary conditions $\vec{n}\cdot\vec{\vec{\sigma}}=\vec{t}$ or $\vec{u}=\vec{g}$ on $\partial\Omega$.

The balance of angular momentum is satisfied 'a priori' by the symmetric form of the Hooke's law.

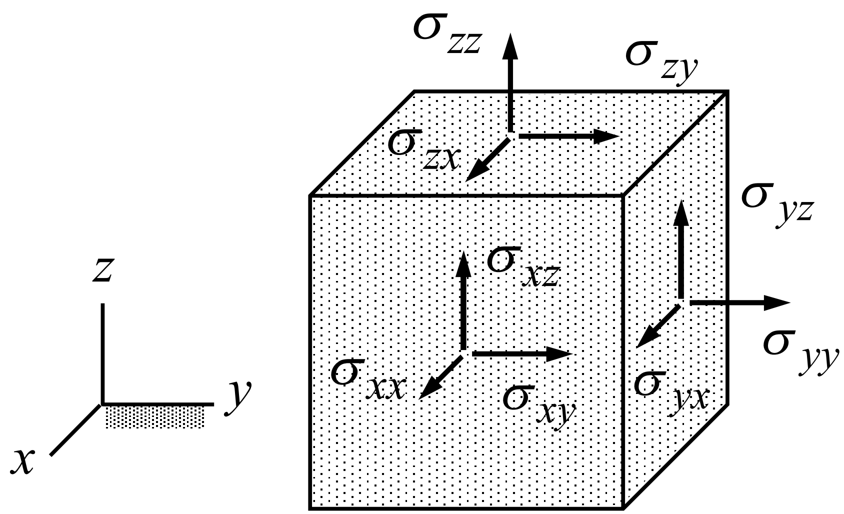
EQUILIBRIUM EQUATIONS

The left-hand side of the equilibrium equation is the sum of the volume and surface forces acting on a material element of the body. The component forms are

$$x: \quad \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x = 0,$$

$$y: \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + f_y = 0,$$

$$z: \quad \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0,$$



The first index of a stress component refers to the direction of the surface normal and the second that of the force component.

HOOKE'S LAW

The generalized Hooke's law of an isotropic homogeneous material and be expressed in the component forms

Strain-stress: $$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} = \frac{1}{G} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$$

Strain-displacement: $$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial y + \partial u_y / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \\ \partial u_z / \partial x + \partial u_x / \partial z \end{Bmatrix}$$

in which E is the Young's modulus, ν the Poisson's ratio, and $G = E/(2+2\nu)$ the shear modulus.

MATERIAL PARAMETERS

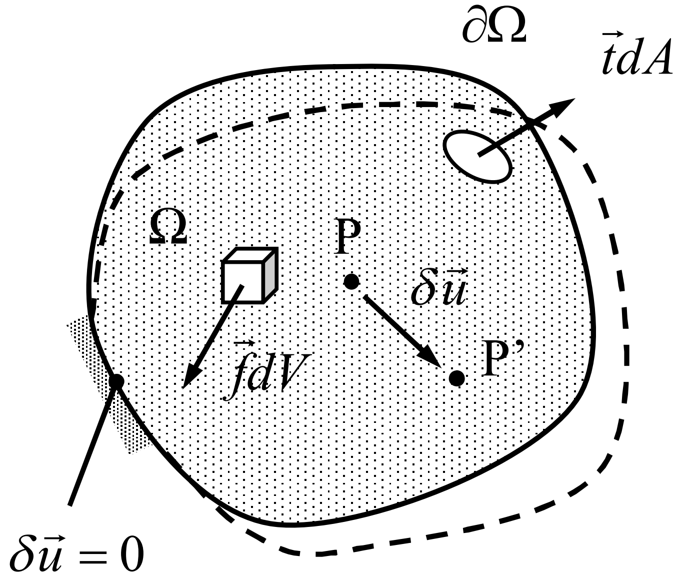
Material	ρ [kg / m^3]	E [GN / m^2]	v [1]
Steel	7800	210	0.3
Aluminum	2700	70	0.33
Copper	8900	120	0.34
Glass	2500	60	0.23
Granite	2700	65	0.23
Birch	600	16	-
Rubber	900	10^{-2}	0.5
Concrete	2300	25	0.1

PRINCIPLE OF VIRTUAL WORK

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \;\; \forall \delta \vec{u}$ is just one form of the equilibrium equations.

$$\delta W^{\text{int}} = -\int_{\Omega} \left(\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix} + \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix} \right) dV$$

$$\delta W^{\text{ext}} = \int_{\Omega} \left(\begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} \right) dV + \int_{\partial\Omega} \left(\begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} \right) dA$$



The details of the expressions vary case by case, but the principle itself does not!

VIRTUAL WORK DENSITIES

Virtual work densities of the internal forces, external volume forces, and external surface forces are (subscripts Ω and $\partial\Omega$ denote virtual work per unit volume and area, respectively)

Internal forces: $$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix} - \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}$$

External forces: $$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} \text{ and } \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}.$$

Virtual work densities consist of terms containing kinematic quantities and their "work conjugates" !

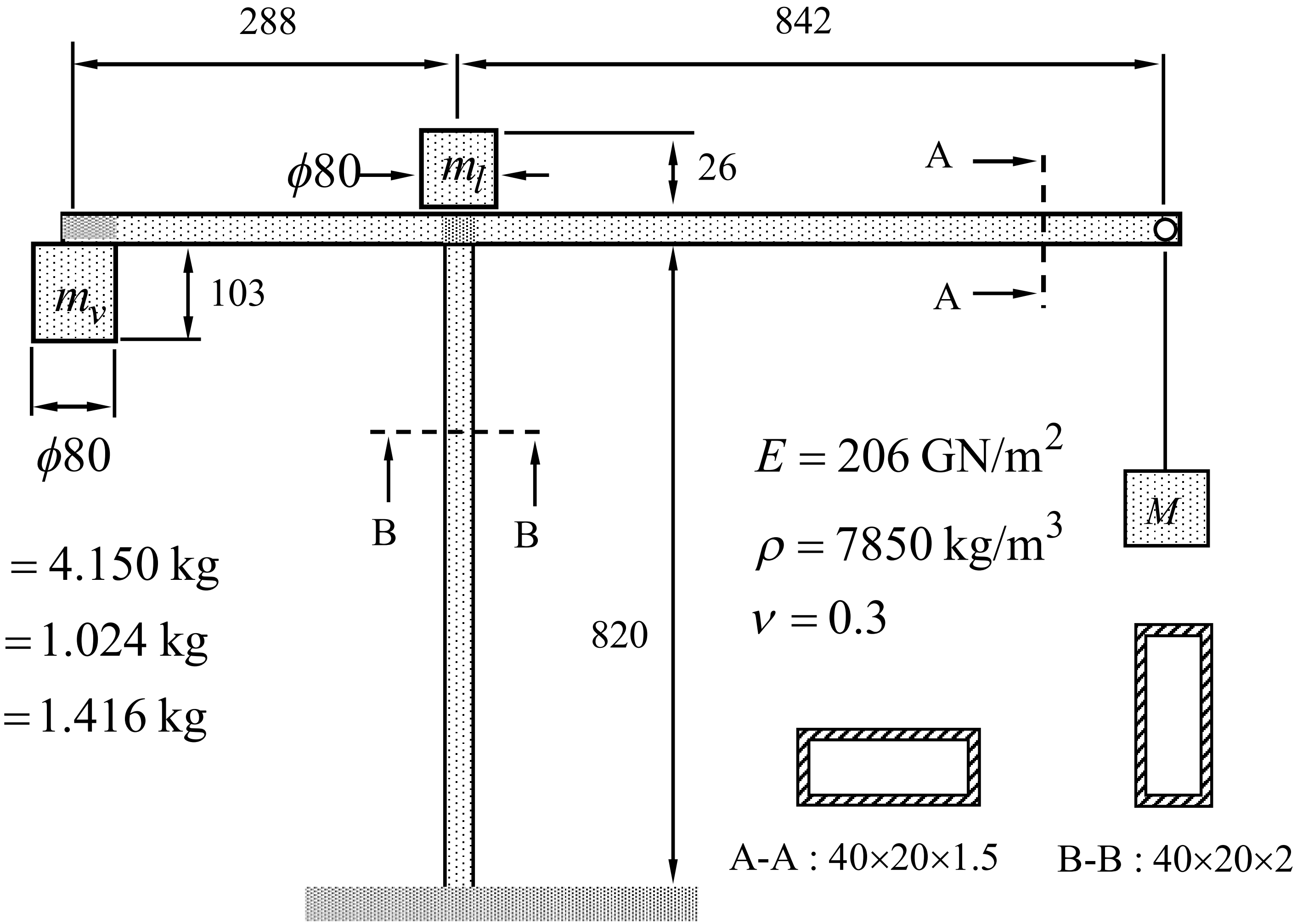
1 LINEAR DISPLACEMENT ANALYSIS

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on the topics of week 2:

- ☐ Engineer paradigm in FEM, elements and nodes, nodal quantities and sign conventions.
- ☐ Displacement analysis of simple structures by using the virtual work expressions of the elements.
- ☐ Calculations of the element constributions of force, solid, beam, and plate elements out of virtual work density of the model and element approximation.

1.1 NOTATIONS AND CONVENTIONS



$m_v = 4.150$ kg

$m_l = 1.024$ kg

$M = 1.416$ kg

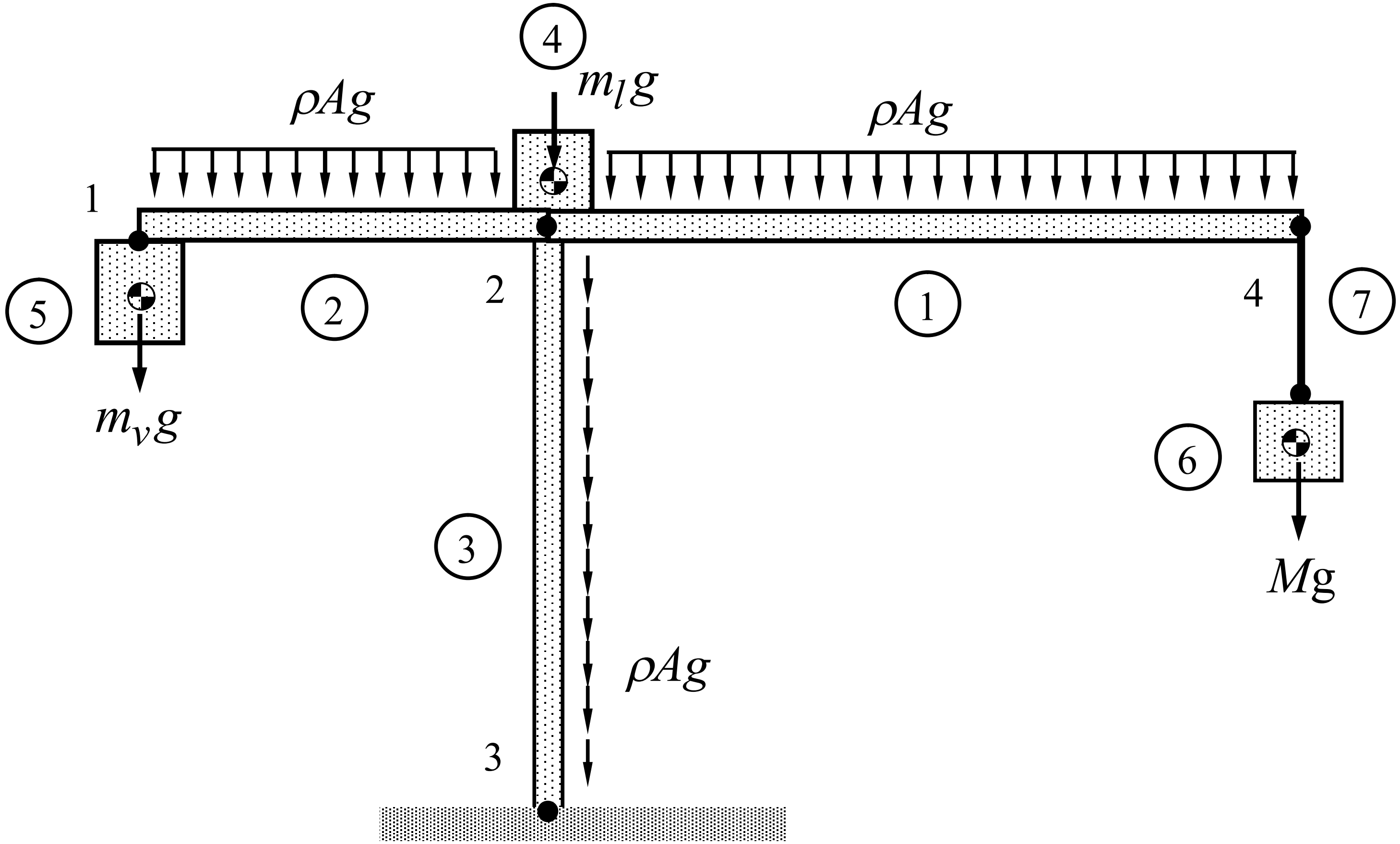
$E = 206$ GN/m^2

$\rho = 7850$ kg/m^3

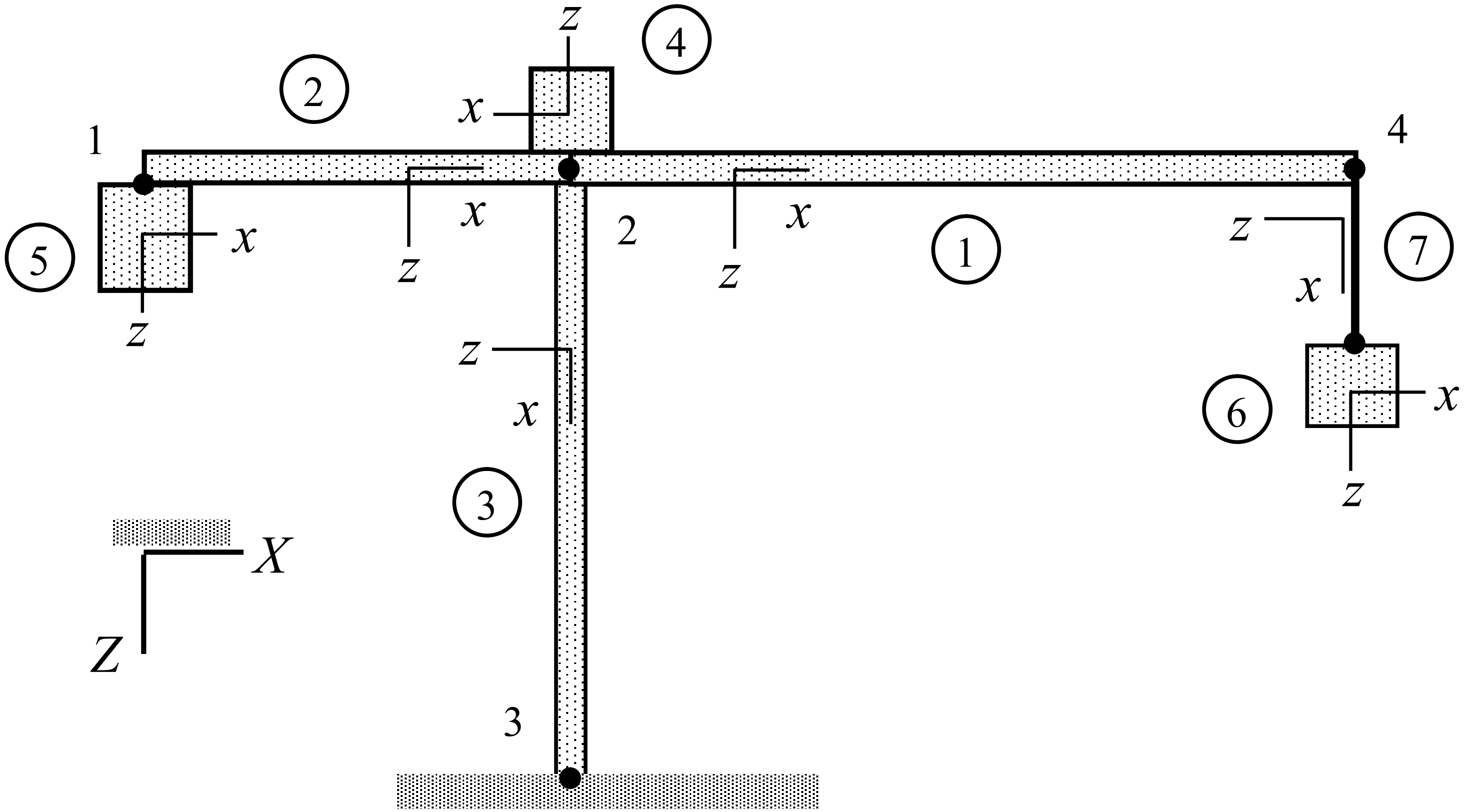
$\nu = 0.3$

ENGINEERING PARADIGM

A complex structure is represented as a collection of structural parts (or elements) which can be modelled as beams, plates etc.



Element contributions are represented in elementwise $(x, y, z)-$coordinate systems. Transformation into $(X, Y, Z)-$structural system is required for the structural integrity.



SIGN CONVENTIONS AND NOTATIONS

Displacements, rotations, forces and moments are vector quantities whose components are positive in the directions of the chosen coordinate axes. The convention may differ from that used in mechanics of materials courses (be careful with that).

	Displacement	**Force**	**Rotation**	**Moment**
Material	u_x, u_y, u_z	F_x, F_y, F_z	$\theta_x, \theta_y, \theta_z$	M_x, M_y, M_z
Structural	u_X, u_Y, u_Z	F_X, F_Y, F_Z	$\theta_X, \theta_Y, \theta_Z$	M_X, M_Y, M_Z

The basis vectors of the material and structural systems are $(\vec{i}, \vec{j}, \vec{k})$ and $(\vec{I}, \vec{J}, \vec{K})$, respectively!

1.2 DISPLACEMENT ANALYSIS

- Model the structure as a collection of beam, plate, etc. elements.

- Derive the element contributions δW^e and express the nodal displacement and rotation components of the material coordinate system in terms of those in the structural coordinate system.

- Sum up the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get the standard form $\delta W = -\delta \mathbf{a}^{\mathrm{T}}(\mathbf{Ka} - \mathbf{F})$.

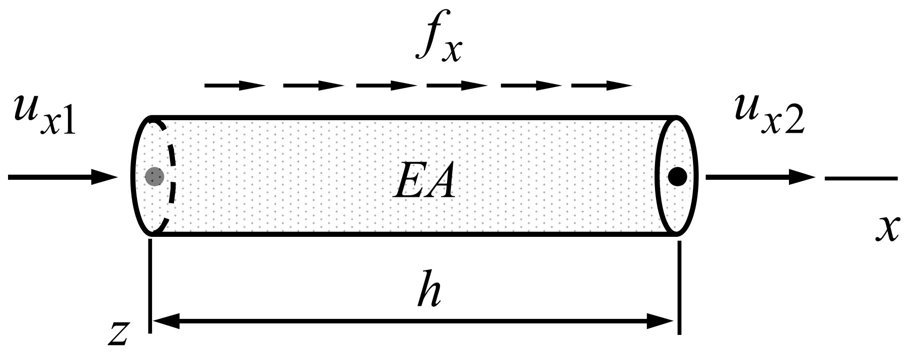
- Use the principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$, fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$, and solve for the unknown nodal displacement and rotation components from the system equations $\mathbf{Ka} - \mathbf{F} = 0$.

BAR ELEMENT

Assuming a linear interpolation for the axial displacement at the endpoints for $u_x \equiv u(x)$, virtual work expressions of the internal and external forces take the forms

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$



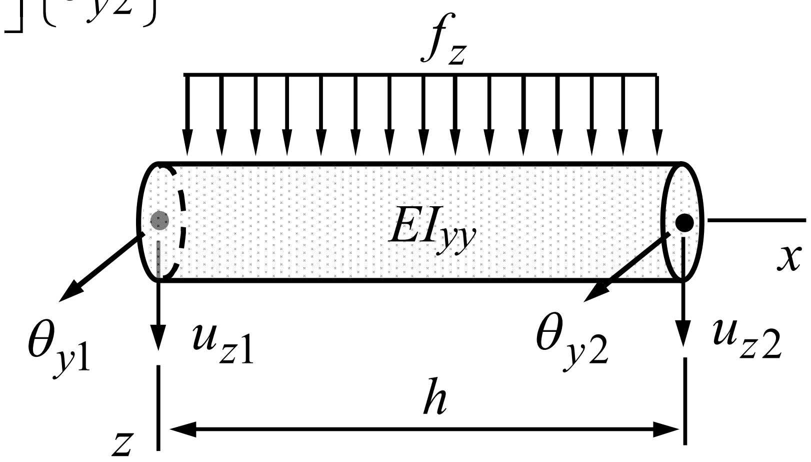
Above, f_x, E, and A are assumed to be constants. The relationship between the axial displacement component and the displacement components in the structural coordinate system is $u_x = \vec{i} \cdot \vec{u} = i_X u_X + i_Y u_Y + i_Z u_Z$ and $\delta u_x = \vec{i} \cdot \delta\vec{u} = i_X \delta u_X + i_Y \delta u_Y + i_Z \delta u_Z$.

BEAM BENDING ELEMENT

Assuming a cubic interpolation for $u_z \equiv w(x)$ in terms of the nodal displacements u_{z1}, u_{z2} and rotations θ_{y1}, θ_{y2}, virtual work expressions take the forms

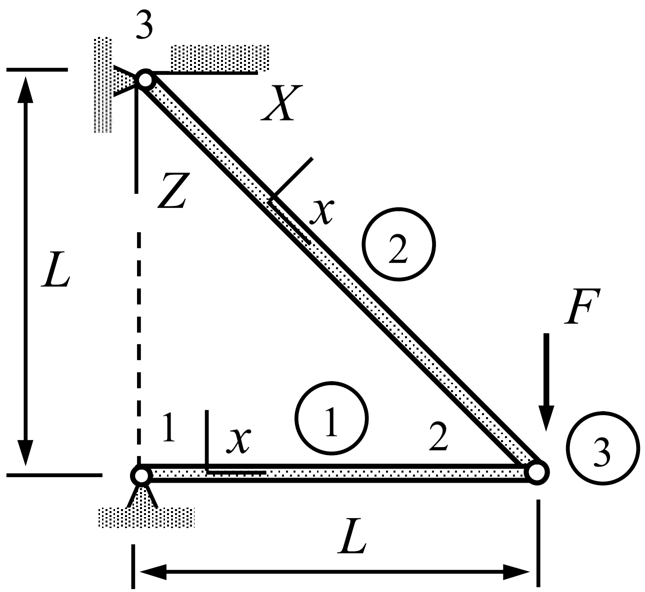
$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \hdashline \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \left[\begin{array}{cc:cc} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ \hdashline -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{array}\right] \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ \hdashline u_{z2} \\ \theta_{y2} \end{Bmatrix}$$

$$\delta W^{\mathrm{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \hdashline \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ \hdashline 6 \\ h \end{Bmatrix}$$



Above, f_z , I_{yy} and E are assumed to be constants.

EXAMPLE 1.1. A bar truss is loaded by a point force having magnitude F as shown in the figure. Determine the nodal displacements. Cross-sectional area of bar 1-2 is A and that for bar 3-2 $\sqrt{8}A$. Young's modulus is E and weight is omitted.



Answer $\begin{Bmatrix} u_{X1} \\ u_{Z1} \end{Bmatrix} = \dfrac{LF}{EA} \begin{Bmatrix} -1 \\ 2 \end{Bmatrix}$

- For element 1, the relationships between the nodal displacement components in the material and structural systems are $u_{x1}=0$ and $u_{x2}=u_{X2}$. Element contribution δW^1 to the virtual work expression of the structure is

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}) = -\frac{EA}{L} u_{X2}\delta u_{X2}.$$

- For element 2, $u_{x3}=0$ and $u_{x2}=(u_{X2}+u_{Z2})/\sqrt{2}$. Element contribution takes the form

$$\delta W^2 = -\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ \delta u_{X2}+\delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} (\frac{E\sqrt{8}A}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ u_{X2}+u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}) \quad \Leftrightarrow$$

$$\delta W^2 = -\frac{EA}{L}(\delta u_{X2}+\delta u_{Z2})(u_{X2}+u_{Z2}).$$

- Virtual work expression of the point force follows from the definition of work

$$\delta W^3 = \delta u_{Z2} F\,.$$

- Virtual work expression of the structure is obtained as the sum of the element contributions

$$\delta W = -\frac{EA}{L}\delta u_{X2} u_{X2} - \frac{EA}{L}(\delta u_{X2} + \delta u_{Z2})(u_{X2} + u_{Z2}) + \delta u_{Z2} F \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix}\right).$$ "standard" form

- Using the principle of virtual work $\delta W = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus

$$\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{LF}{EA}\begin{Bmatrix} -1 \\ 2 \end{Bmatrix}.$$

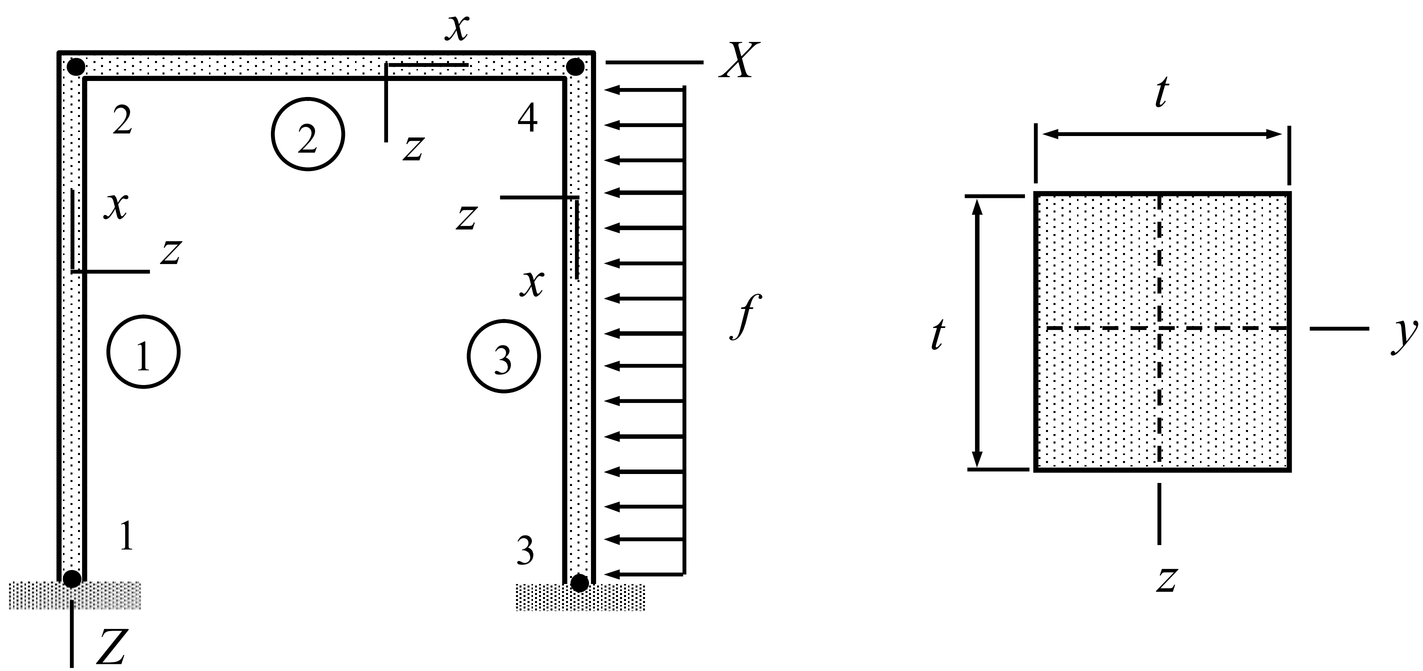
- The Mathematica description of the problem is given by

	type	properties	geometry
1	BAR	{{E}, {A}, {0}}	Line[{1, 2}]
2	BAR	{{E}, {$2\sqrt{2}$ A}, {0}}	Line[{3, 2}]
3	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, L}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}

$$\left\{ uX[2] \to -\frac{F\,L}{A\,E},\ uZ[2] \to \frac{2\,F\,L}{A\,E} \right\}$$

EXAMPLE 1.2. Consider the beam truss shown. Determine the displacements and rotations of nodes 2 and 4. Assume that the beams are rigid in the axial directions so that the axial *strain* vanishes. Cross-sections and lengths are the same and Young's modulus E is constant.



Answer $u_{X2} = u_{X4} = -\dfrac{3}{112}\dfrac{fL^4}{EI}$, $\theta_{Y2} = \dfrac{19}{1008}\dfrac{fL^3}{EI}$, and $\theta_{Y4} = \dfrac{5}{1008}\dfrac{fL^3}{EI}$

- Only the bending in *XZ*-plane needs to be accounted for. The displacement and rotation components of the structure are u_{X2}, θ_{Y2}, and θ_{Y4}. As the axial strain of beam 2 vanishes, axial displacements satisfy $u_{X4} = u_{X2}$.

$$\delta W^1 = -\left\{\begin{array}{c} 0 \\ 0 \\ \hline \delta u_{X2} \\ \delta\theta_{Y2} \end{array}\right\}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc|cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ \hline u_{X2} \\ \theta_{Y2} \end{array}\right\}\right) \quad (u_{z2} = u_{X2}, \theta_{y2} = \theta_{Y2})$$

$$\delta W^2 = -\left\{\begin{array}{c} 0 \\ \delta\theta_{Y2} \\ \hline 0 \\ \delta\theta_{Y4} \end{array}\right\}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc|cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\left\{\begin{array}{c} 0 \\ \theta_{Y2} \\ \hline 0 \\ \theta_{Y4} \end{array}\right\}\right) \quad (\theta_{y2} = \theta_{Y2}, \theta_{y4} = \theta_{Y4})$$

$$\delta W^3 = -\begin{Bmatrix} -\delta u_{X2} \\ \delta\theta_{Y4} \\ \hdashline 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc:cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hdashline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\begin{Bmatrix} -u_{X2} \\ \theta_{Y4} \\ \hdashline 0 \\ 0 \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ \hdashline 6 \\ L \end{Bmatrix}\right) \quad (u_{z4} = -u_{X2})$$

- Virtual work expression of the structure is

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \\ \delta\theta_{Y4} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} -6 \\ 0 \\ -L \end{Bmatrix}\right)$$

- Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} -6 \\ 0 \\ -L \end{Bmatrix} = 0 \iff \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} = \frac{fL^3}{1008EI}\begin{Bmatrix} -27L \\ 19 \\ 5 \end{Bmatrix}$$

- In the Mathematica code calculation, horizontal displacements of nodes 2 and 4 are forced to be same ($u_{X4} = u_{X2}$)

	type	properties	geometry
1	BEAM	{{E, G}, {A, I, I}, {0, 0, 0}}	Line[{1, 2}]
2	BEAM	{{E, G}, {A, I, I}, {0, 0, 0}}	Line[{2, 4}]
3	BEAM	{{E, G}, {A, I, I}, {0, 0, f}}	Line[{4, 3}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{0, 0, 0}	{uX[2], 0, 0}	{0, θY[2], 0}
3	{L, 0, L}	{0, 0, 0}	{0, 0, 0}
4	{L, 0, 0}	{uX[2], 0, 0}	{0, θY[4], 0}

$$\left\{uX[2] \to -\frac{3 f L^4}{112 E I},\ \theta Y[2] \to \frac{19 f L^3}{1008 E I},\ \theta Y[4] \to \frac{5 f L^3}{1008 E I}\right\}$$

1.3 ELEMENT CONTRIBUTIONS

Virtual work expressions for the solid, beam, plate elements combine virtual work densities representing the model and a case dependent approximation. To derive the expression for an element:

- ☐ Start with the virtual work densities $\delta w_{\Omega}^{\text{int}}$ and $\delta w_{\Omega}^{\text{ext}}$ of the formulae collection (if not available there, derive the expression in the manner discussed in MEC-E1050).

- ☐ Represent the unknown functions by interpolation of the nodal displacement and rotations (see formulae collection). Substitute the approximations into the density expressions.

- ☐ Integrate the virtual work density over the domain occupied by the element to get δW.

ELEMENT APPROXIMATION

In MEC-E8001, element approximation is a polynomial interpolant of the nodal displacement and rotations in terms of shape functions. In displacement analysis, shape functions depend on (x,y,z) and the nodal values are parameters to be evaluated by FEM.

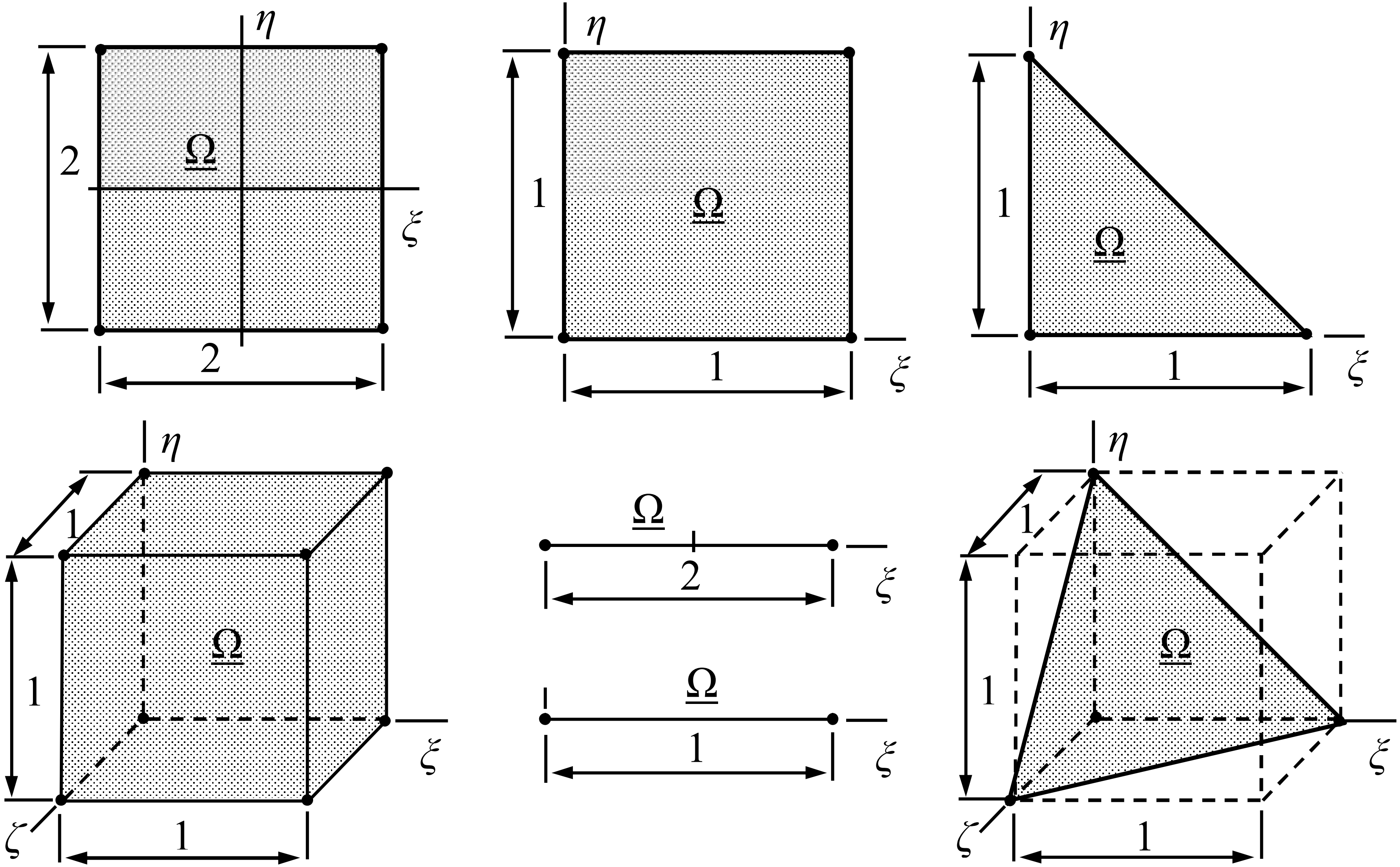
Approximation $\mathbf{u}=\mathbf{N}^{\mathrm{T}}\mathbf{a}$ *always of the same form!*

Shape functions $\mathbf{N}=\{N_1(x,y,z)\;\; N_2(x,y,z)\;\; \dots\;\; N_n(x,y,z)\}^{\mathrm{T}}$

Parameters $\mathbf{a}=\{\mathrm{a}_1\;\; \mathrm{a}_2\;\; \dots\;\; \mathrm{a}_n\}^{\mathrm{T}}$

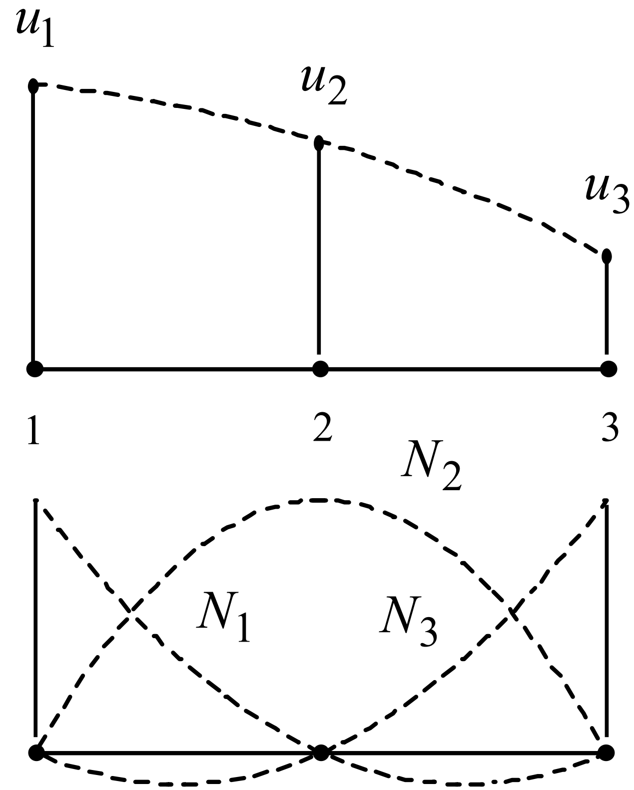
Nodal parameters $\mathrm{a}\in\{u_x,u_y,u_z,\theta_x,\theta_y,\theta_z\}$ may be just displacement or rotation components or a mixture of them (as with the Bernoulli beam model).

ELEMENT GEOMETRY



QUADRATIC SHAPE FUNCTIONS

Piecewise quadratic approximation is continuous in Ω and second order polynomial inside the elements. In a typical element Ω^e



Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \;\; u_2 \;\; u_3\}^{\mathrm{T}}$

Shape functions: $$\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1-3\xi+2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi-1) \end{Bmatrix}, \quad \xi = \frac{x}{h}$$

More nodes can be used to generate higher order approximations!

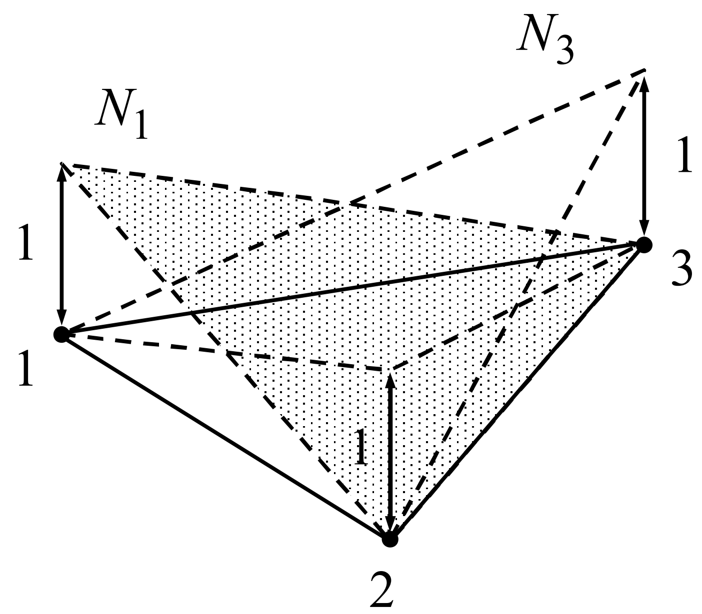
LINEAR SHAPE FUNCTIONS

A piecewise linear approximation is continuous in Ω and linear inside each element of triangle shape. In a typical element

Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \;\; u_2 \;\; u_3\}^{\mathrm{T}}$

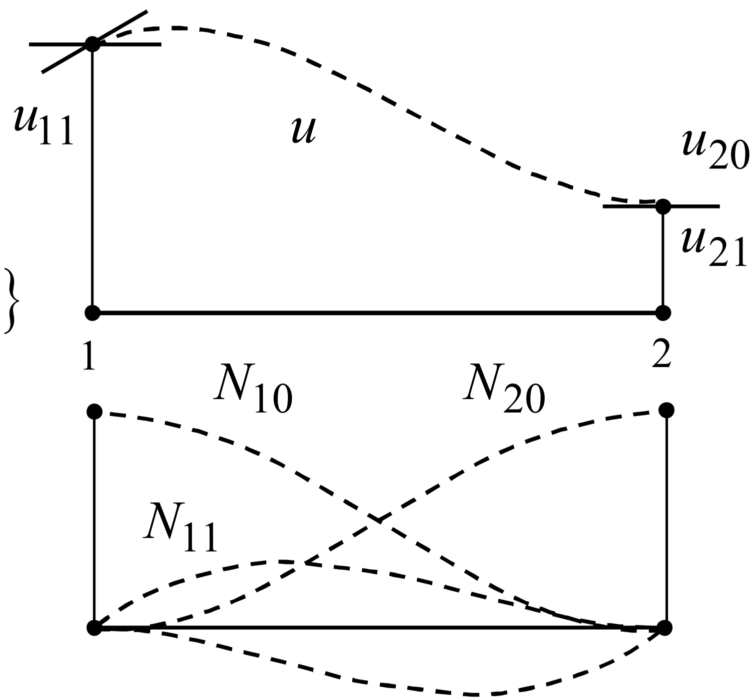
Shape functions: $\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$



Triangle element is the simplest element in two dimensions. Division of any 2D domain into triangles is always possible, which makes the element quite useful.

CUBIC SHAPE FUNCTIONS

Piecewise cubic approximation has continuous derivatives up to the first order in Ω and is a third order polynomial inside the elements.



Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \;\; (du/dx)_1 \,\vdots\, u_2 \;\; (du/dx)_2\}$

Shape functions:

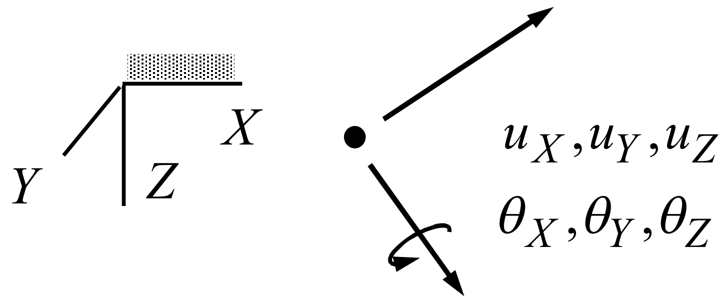
$$\mathbf{N} = \begin{Bmatrix} N_{10} \\ N_{11} \\ \hdashline N_{20} \\ N_{21} \end{Bmatrix} = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix}$$

In $xz-$plane bending $u = u_z$, $du/dx = -\theta_y$ and in $xy-$plane bending $u = u_y$, $du/dx = \theta_z$.

FORCE ELEMENT

External point forces and moments are assumed to act on the joints. They are treated as elements associated with one node only. Virtual work expression is usually simplest in the structural coordinate system:

$$\delta W = \begin{Bmatrix} \delta u_X \\ \delta u_Y \\ \delta u_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} + \begin{Bmatrix} \delta\theta_X \\ \delta\theta_Y \\ \delta\theta_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}$$



Above, F_X, F_Y, F_Z and M_X, M_Y, M_Z are the given components. A rigid body can be modeled as a particle at the center of mass connected to the other joints of the body by rigid links!

SOLID MODEL

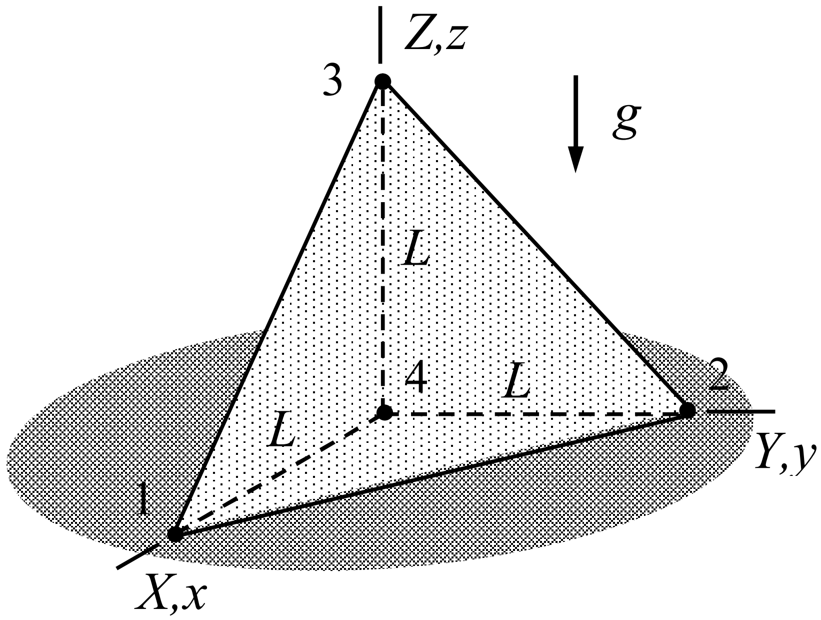
The model does not contain assumptions in addition to those of linear elasticity theory.

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta w / \partial z \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial w / \partial z \end{Bmatrix} - \begin{Bmatrix} \partial\delta u / \partial y + \partial\delta v / \partial x \\ \partial\delta v / \partial z + \partial\delta w / \partial y \\ \partial\delta w / \partial x + \partial\delta u / \partial z \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} \partial u / \partial y + \partial v / \partial x \\ \partial v / \partial z + \partial w / \partial y \\ \partial w / \partial x + \partial u / \partial z \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \text{ and } \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} \text{ in which } \quad [E] = E \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}^{-1}.$$

The solution domain can be represented, e.g, by tetrahedron elements with linear interpolation of the displacement components $u(x, y, z)$, $v(x, y, z)$, and $w(x, y, z)$

EXAMPLE 1.3. A tetrahedron of edge length L, density ρ, and elastic properties E and ν is subjected to its own weight on a horizontal floor. Calculate the displacement u_{Z3} of node 3 with one tetrahedron element and linear approximation. Assume that $u_{X3} = u_{Y3} = 0$, and that the bottom surface is fixed.



Answer: $u_{Z3} = -\frac{1}{4}\frac{\rho g L^2}{E}\frac{1-\nu-2\nu^2}{1-\nu}$

- Linear shape functions can be deduced directly from the figure $N_1 = x/L$, $N_2 = y/L$, $N_3 = z/L$, and $N_4 = 1 - x/L - y/L - z/L$. However, only the shape function of node 3 is needed as the other nodes are fixed. Approximations to the displacement components are

$$u = 0,\ \ v = 0,\ \text{ and }\ w = \frac{z}{L}u_{Z3},\ \text{ giving } \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0 \ \text{ and } \frac{\partial w}{\partial z} = \frac{1}{L}u_{Z3}.$$

- When the approximation is substituted there, the virtual work densities of the internal and external forces simplify to

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta_{Z3} \end{Bmatrix}^{\mathrm{T}} \frac{E}{L^2(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z3} \end{Bmatrix} = \frac{-E(1-\nu)}{(1+\nu)(1-2\nu)} \frac{u_{Z3}\delta u_{Z3}}{L^2}$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z3} \end{Bmatrix}^{\text{T}} \frac{z}{L} \begin{Bmatrix} 0 \\ 0 \\ -\rho g \end{Bmatrix} = -\frac{z}{L} \rho g \delta u_{Z3}.$$

- Virtual work expressions are obtained as integrals of densities over the volume:

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} dV = \delta w_\Omega^{\text{int}} \frac{L^3}{6} = -\frac{1}{6} \frac{1-\nu}{(1+\nu)(1-2\nu)} E L u_{Z3} \delta u_{Z3},$$

$$\delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} dV = -\frac{L^3}{24} \rho g \delta u_{Z3}.$$

- Finally, principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ with $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ and the fundamental lemma of variation calculus imply

$$u_{Z3} = -\frac{1}{4}\frac{\rho g L^2}{E}\frac{1-\nu-2\nu^2}{1-\nu}.$$

- In Mathematica code of the course, the problem description is given by

	type	properties	geometry
1	SOLID	{{E, ν}, {0, 0, -g ρ}}	Tetrahedron[{1, 2, 3, 4}]

	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{L, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{0, L, 0}	{0, 0, 0}	{0, 0, 0}
3	{0, 0, L}	{uX[3], uY[3], uZ[3]}	{0, 0, 0}
4	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}

$$\left\{uX[3] \to 0,\ uY[3] \to 0,\ uZ[3] \to -\frac{g L^2 \left(-1+\nu+2\nu^2\right)\rho}{4 E (-1+\nu)}\right\}$$

BEAM MODEL

The bar, torsion bar, and bending modes of a beam are connected unless the first and cross moments (off-diagonal terms of the matrix) of the cross-section vanish:

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d^2\delta v/dx^2 \\ d^2\delta w/dx^2 \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du/dx \\ d^2v/dx^2 \\ d^2w/dx^2 \end{Bmatrix} - \frac{d\delta\phi}{dx} GI_p \frac{d\phi}{dx}$$

traction on the end surfaces

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ -d\delta w/dx \\ d\delta v/dx \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix} \quad \left(\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_X \\ \delta u_Y \\ \delta u_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} + \begin{Bmatrix} \delta\theta_X \\ \delta\theta_Y \\ \delta\theta_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}\right)$$

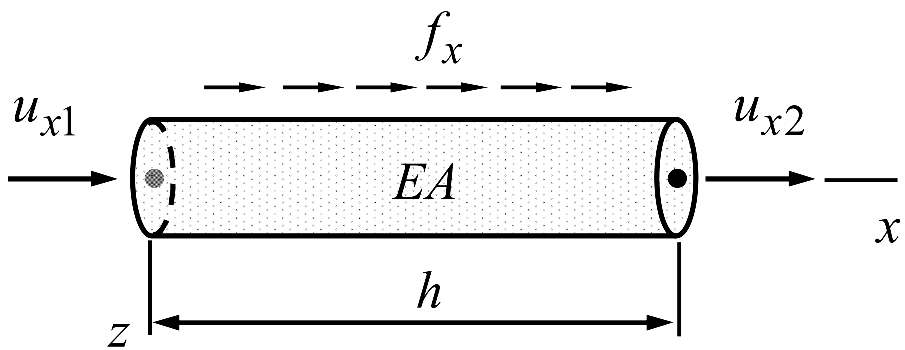
In FEM the solution domain (a line segment) is represented by line elements and the displacement and rotation components $u(x)$, $v(x)$, $w(x)$, and $\phi(x)$ by their interpolants.

BAR MODE

Assuming that $v=0$, $w=0$, $\phi=0$ and a linear approximation to $u(x)$ in terms of the end point displacements u_{x1}, u_{x2}, virtual work expressions of the internal and external forces take the forms

$$\delta W^{\text{int}} = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}} \frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}.$$



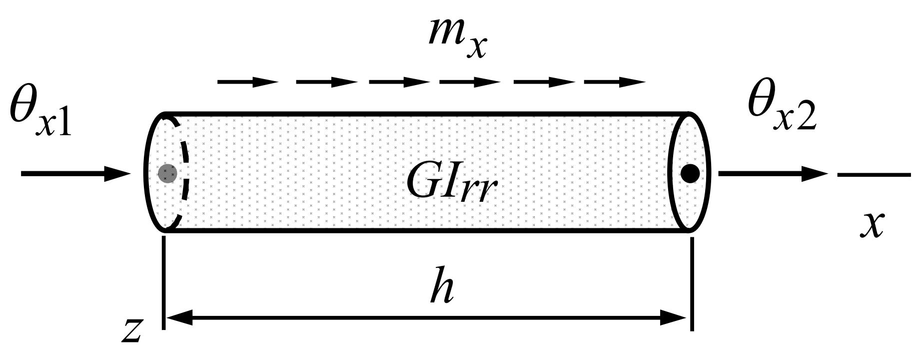
Above, f_x, E, and A are assumed to be constants. The relationship between the axial displacement component and the displacement components in the structural coordinate system is $u_x = \vec{i} \cdot \vec{u} = i_X u_X + i_Y u_Y + i_Z u_Z$.

TORSION MODE

Assuming that $u=0$, $v=0$, $w=0$, and a linear approximation to $\phi(x)$ in terms of the end point rotations θ_{x1},θ_{x2}, virtual work expressions of the internal and external forces take the forms

$$\delta W^{\text{int}} = -\begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\mathrm{T}} \frac{GI_{rr}}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\mathrm{T}} \frac{m_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}.$$



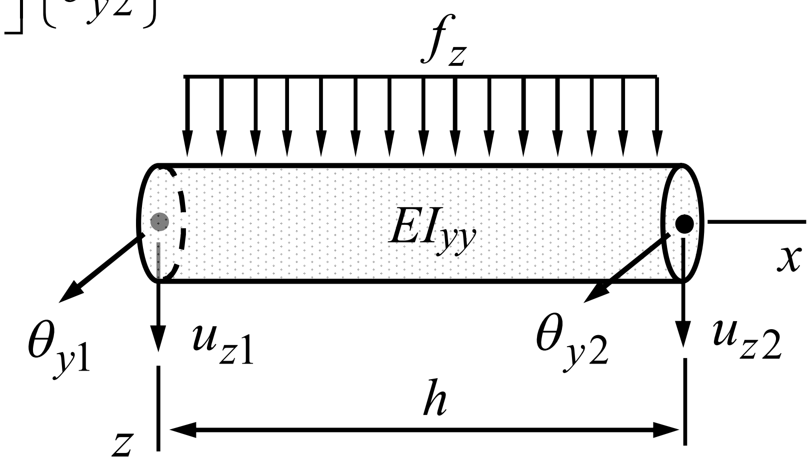
Above, m_x, E, and I_{rr} are assumed to be constants. The relationship between the axial rotation component and the rotation components in the structural coordinate system is $\theta_x = \vec{i}\cdot\vec{u} = i_X\theta_X + i_Y\theta_Y + i_Z\theta_Z$.

BENDING MODE (*xz*-plane)

Assuming that $u=0$, $v=0$, $\phi=0$, and a cubic approximation to $w(x)$ in terms of the end point displacements u_{z1}, u_{z2} and rotations θ_{y1},θ_{y2}:

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \hline \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \left[\begin{array}{cc|cc} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ \hline -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{array}\right] \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ \hline u_{z2} \\ \theta_{y2} \end{Bmatrix}$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \hline \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ \hline 6 \\ h \end{Bmatrix}$$



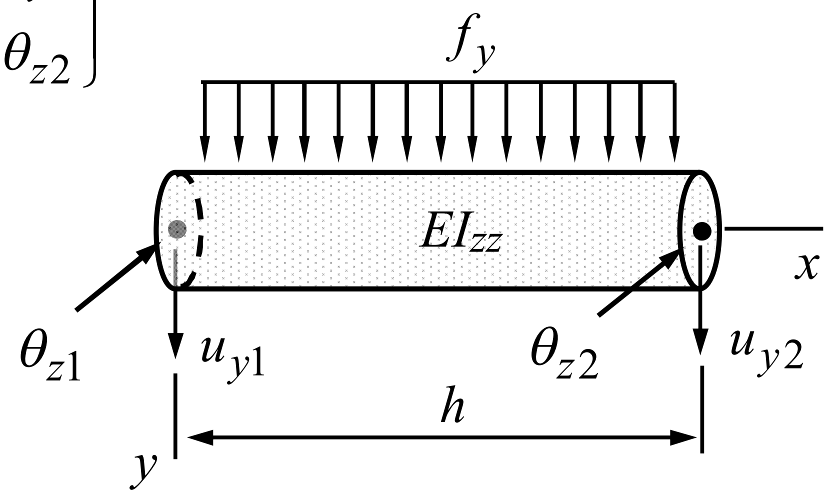
Above, f_z , I_{yy} and E are assumed to be constants.

BENDING MODE (*xy*-plane)

Assuming that $u = 0$, $w = 0$, $\phi = 0$, and a cubic approximation to $v(x)$ in terms of point displacements u_{y1}, u_{y2} and rotations θ_{z1}, θ_{z2}

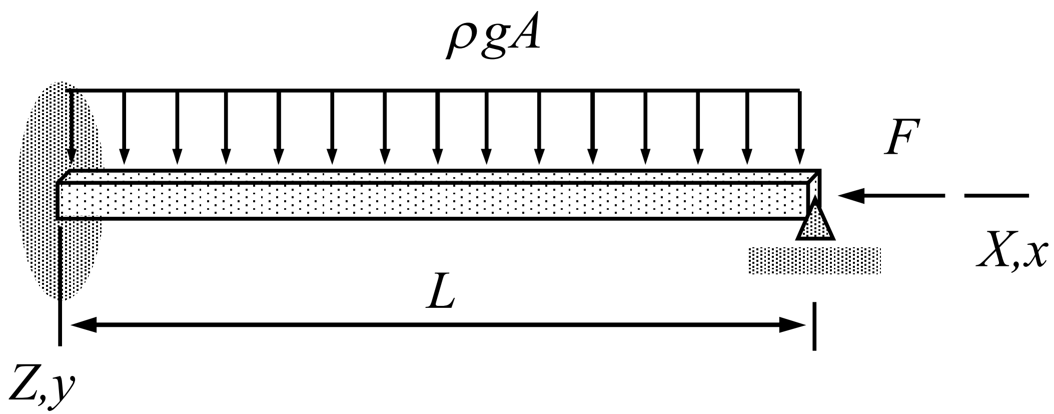
$$\delta W^{\text{int}} = \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \hline \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{zz}}{h^3} \left[\begin{array}{cc|cc} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ \hline -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{array}\right] \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ \hline u_{y2} \\ \theta_{z2} \end{Bmatrix}$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \hline \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix} \frac{f_y h}{12} \begin{Bmatrix} 6 \\ h \\ \hline 6 \\ -h \end{Bmatrix}$$



Above, f_y , I_{zz} and E are assumed to be constants.

EXAMPLE 1.4. The Bernoulli beam of the figure is loaded by its own weight and a point force acting on the right end. Determine the displacement and rotation of the right end starting with the virtual density of the Bernoulli beam model. The x-axis of the material coordinate system is placed at the geometric centroid of the rectangle cross-section. Beam properties A, $I_{zz} = I$, and E are constants.



Answer: $u_{X2} = \dfrac{FL}{EA}$ and $\theta_{Y2} = \dfrac{1}{48}\dfrac{\rho g A L^3}{EI}$

- Bernoulli beam element of the Mathematica code requires the orientation of the $y-$axis unless $y-$axis and $Y-$axis are aligned. Orientation is given by additional parameter defining the components of $\vec{j}$ in the structural coordinate system:

	type	properties	geometry
1	BEAM	{{E, G}, {A, Iyy, Izz, {0, 0, 1}}, {0, f, 0}}	Line[{1, 2}]
2	FORCE	{-F, 0, 0}	Point[{2}]

	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, θY[2], 0}

$$\left\{uX[2] \to -\frac{F\,L}{A\,E},\ \theta Y[2] \to \frac{f\,L^3}{48\,E\,Izz}\right\}$$

PLATE MODEL

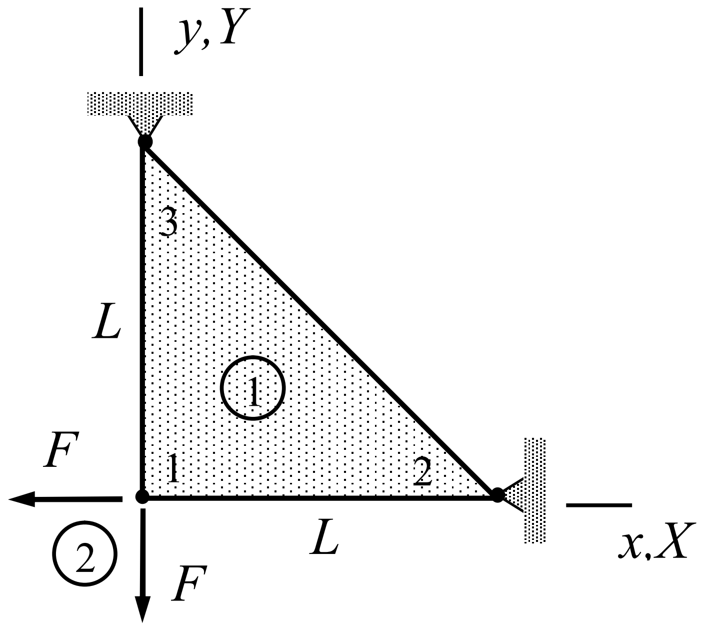
Virtual work densities combine the plane-stress and plate bending modes. Assuming that the material coordinate system is placed at the geometric centroid

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} - \begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_{\sigma} \times$$

$$\times \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \text{ and } \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

Approximation to the displacement components $u(x,y)$, $v(x,y)$, $w(x,y)$ should be continuous and $w(x,y)$ should also have continuous derivatives at the element interfaces.

EXAMPLE 1.5. Consider the thin triangular structure shown. Young's modulus E, Poisson's ratio ν, and thickness t are constants. Distributed external force vanishes. Assume plane-stress conditions, $XY-$plane deformation and determine the displacement of node 1 when the force components acting on the node are as shown in the figure.



Answer: $$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\frac{F}{Et}\frac{(1+\nu)(1-2\nu)}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

- Nodes 2 are 3 are fixed and the non-zero displacement components are u_{X1} and u_{Y1}. Linear shape functions $N_1 = (L - x - y)/L$, $N_2 = x/L$ and $N_3 = y/L$ are easy to deduce from the figure. Therefore

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{L - x - y}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial x \end{Bmatrix} = -\frac{1}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \text{ and } \begin{Bmatrix} \partial u / \partial y \\ \partial v / \partial y \end{Bmatrix} = -\frac{1}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

- Virtual work density of internal forces is given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1} + \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2} \frac{tE}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1} + u_{Y1} \end{Bmatrix}.$$

- Integration over the triangular domain gives (integrand is constant)

$$\delta W^1 = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1} + \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{2} \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1} + u_{Y1} \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta W^1 = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{4} \frac{tE}{1-\nu^2} \begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

- Virtual work expression for the point forces follows from the definition of work

$$\delta W^2 = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -F \\ -F \end{Bmatrix} .$$

- Principle of virtual work in the form $\delta W = \delta W^1 + \delta W^2 = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W=-\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\end{Bmatrix}^{\mathrm{T}}\left(\frac{1}{4}\frac{tE}{1-\nu^2}\begin{bmatrix}3-\nu & 1+\nu\\ 1+\nu & 3-\nu\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix}+F\begin{Bmatrix}1\\ 1\end{Bmatrix}\right)=0\quad\forall\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\end{Bmatrix}\quad\Rightarrow$$

$$\frac{1}{4}\frac{tE}{1-\nu^2}\begin{bmatrix}3-\nu & 1+\nu\\ 1+\nu & 3-\nu\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix}+F\begin{Bmatrix}1\\ 1\end{Bmatrix}=0\quad\Leftrightarrow$$

$$\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix}=-\frac{F}{tE}(1-\nu^2)\begin{Bmatrix}1\\ 1\end{Bmatrix}.$$ ←

The point forces acting on a thin slab should be considered as “equivalent nodal forces” i.e. just representations of tractions acting on some part of the boundary. Under the action of an actual point force, displacement becomes non-bounded. In practice, numerical solution to the displacement at the point of action increases when the mesh is refined.

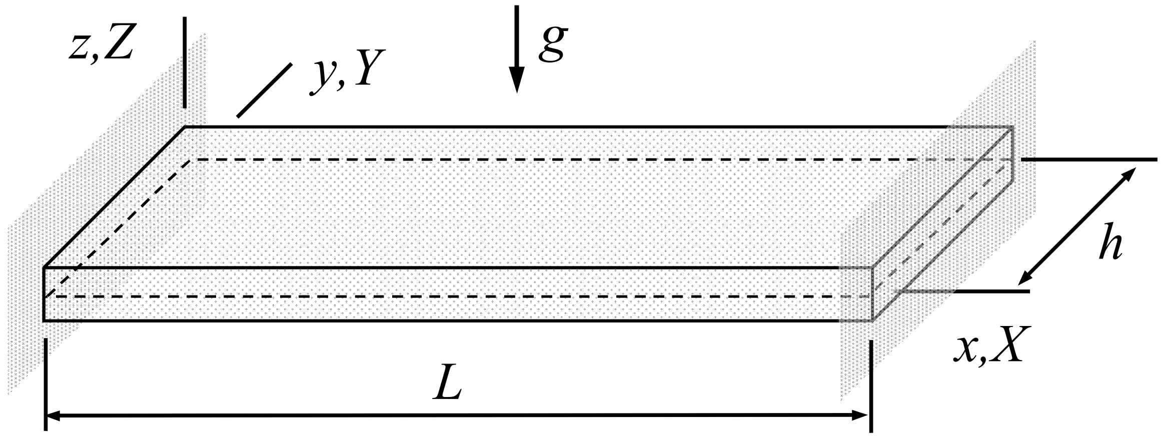
- In Mathematica code of the course, the problem description is given by

	type	properties	geometry
1	PLANE	{{E, ν}, {t}, {0, 0, 0}}	Triangle[{1, 2, 3}]
2	FORCE	{-F, -F, 0}	Point[{1}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{uX[1], uY[1], 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, 0}	{0, 0, 0}
3	{0, L, 0}	{0, 0, 0}	{0, 0, 0}

$$\left\{uX[1] \to \frac{F\,(-1+\nu)\,(1+\nu)}{t\,E},\ uY[1] \to -\frac{F - F\,\nu^2}{t\,E}\right\}$$

EXAMPLE 1.6. Consider a plate strip loaded by its own weight. Determine the deflection w according to the Kirchhoff model. Thickness, length and width of the plate are t, L, and h, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Use the one parameter approximation $w(x) = a_0(1 - x/L)^2(x/L)^2$.



Answer: $$w = -\frac{\rho g L^4}{2Et^2}(1-\nu^2)(1-\frac{x}{L})^2(\frac{x}{L})^2$$

- Approximation satisfies the boundary conditions ‘a priori’ and contains a free parameter a_0 (not associated with any node) to be solved by using the principle of virtual work:

$$w = a_0 (1-\frac{x}{L})^2 (\frac{x}{L})^2 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = a_0 \frac{2}{L^2}[1-6\frac{x}{L}+6(\frac{x}{L})^2] \text{ and } \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x \partial y} = 0 .$$

- When the approximation is substituted there, virtual work densities simplify to

$$\delta w_{\Omega}^{\text{int}} = -a_0 \delta a_0 \frac{Et^3}{3(1-\nu^2)} \frac{1}{L^4}[1-6\frac{x}{L}+6(\frac{x}{L})^2]^2 ,$$

$$\delta w_{\Omega}^{\text{ext}} = -\delta a_0 (1-\frac{x}{L})^2 (\frac{x}{L})^2 \rho g t .$$

- Integrations over the domain $\Omega =]0, L[\times]0, h[$ give the virtual works of internal and external forces

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} d\Omega = -a_0 \delta a_0 \frac{1}{15} \frac{hEt^3}{L^3(1-\nu^2)},$$

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega = -\delta a_0 \frac{1}{30} \rho g t L h.$$

- Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give finally $\forall \delta a_0$

$$\delta W = -\delta a_0 \left(\frac{1}{15} \frac{hEt^3}{L^3(1-\nu^2)} a_0 + \frac{1}{30} \rho g t L h\right) = 0 \quad \Leftrightarrow \quad a_0 = -\frac{1}{2} \frac{\rho g t L^4}{Et^2}(1-\nu^2).$$ 🡸