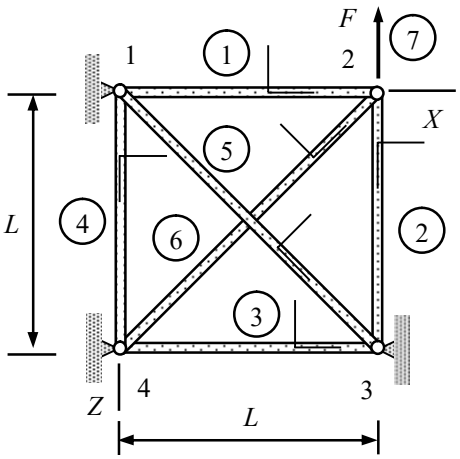
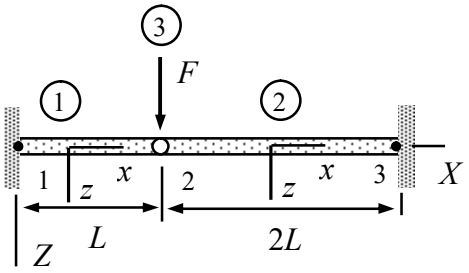
MEC-E8001 Finite Element Analysis, week 2/2019

1. Determine the nodal displacements when force F is acting on the structure as shown. The cross-sectional area of bars 1,2,3 and 4 is A and the cross-sectional area of bars 5 and 6 is $2\sqrt{2}A$. Young's modulus of the material is E. Use the principle of virtual work.



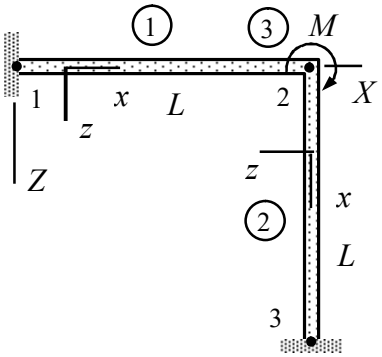
Answer $u_{X2} = -\frac{1}{3}\frac{FL}{EA}$, $u_{Z2} = -\frac{2}{3}\frac{FL}{EA}$

2. Joint between the beams of the figure is frictionless. Force F acting on the joint and displacement of the beam are restricted to the XZ-plane. Determine the rotations and displacement at the joint. Use two beam elements. The second moment of area I and Young's modulus of the material E are constants.



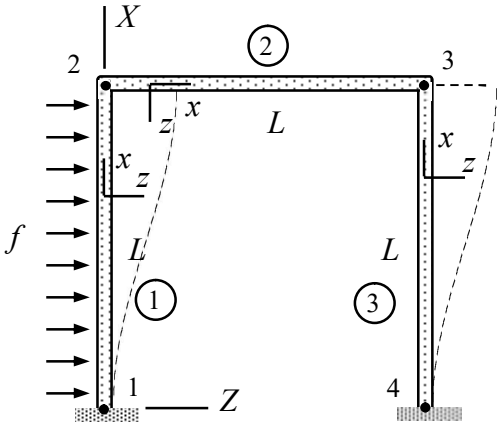
Answer $u_{Z2} = \frac{8}{27}\frac{FL^3}{EI}$, $\theta_{Y2} = -\frac{4}{9}\frac{FL^2}{EI}$ (short), and $\theta_{Y2} = \frac{2}{9}\frac{FL^2}{EI}$ (long).

3. Determine the rotation θ_{Y2} at node 2 of the structure loaded by a point moment (magnitude M) acting on node 2. Use beam elements (1) and (2) of equal length and a point moment element (3). Assume that the beams are inextensible in the axial directions. Young's modulus of material E and the second moment of area I are constants.



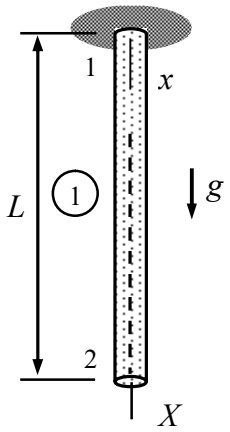
Answer $\theta_{Y2} = -\frac{1}{8}\frac{LM}{EI}$

4. Frame of the figure consists of a rigid body (2) and beam elements (1) and (3). Determine the non-zero displacements and rotations. The beams are identical and can be assumed rigid in the axial directions. Displacements are confined to the XZ-plane. Young's modulus E, second moment of area I, and distributed force f acting on element 1 are constants.



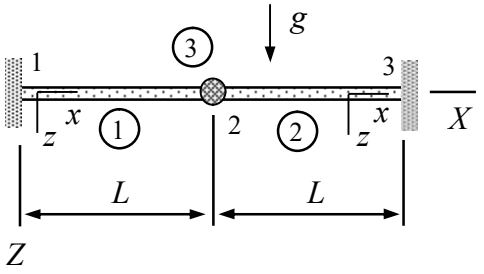
Answer $u_{Z2} = \frac{1}{48}\frac{fL^4}{EI}$

5. Consider a bar of length L loaded by its own weight (figure). Determine the displacement u_{X2} at the free end. Start with the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ and approximation $u = (1 - x/L)u_{x1} + (x/L)u_{x2}$. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.



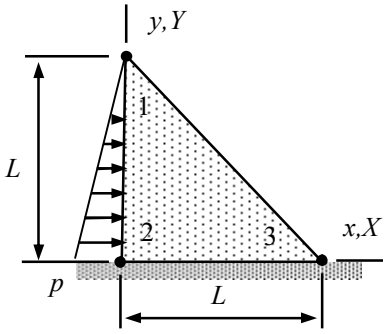
Answer $u_{X2} = \dfrac{\rho g L^2}{2E}$

6. The XZ-plane structure shown consists of two *massless* beams and a homogeneous disk considered as a rigid body. Determine the displacement u_{Z2} and rotation θ_{Y2} at node 2. Young's modulus E of the beam material and the second moment of area I are constants.



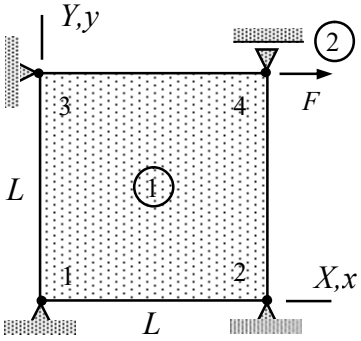
Answer $u_{Z2} = \dfrac{1}{24}\dfrac{mgL^3}{EI}$ and $\theta_{Y2} = 0$

7. A long dam of homogeneous, isotropic, linearly elastic material, is subjected to water pressure on one side. Material properties E and ν are constants. Determine the displacement components u_{X1} and u_{Y1} of node 1. Nodes 2 and 3 are fixed. Use a three-node element and assume plane strain conditions. Consider a slab of thickness t in calculations. The peak value of the linearly varying pressure is p.



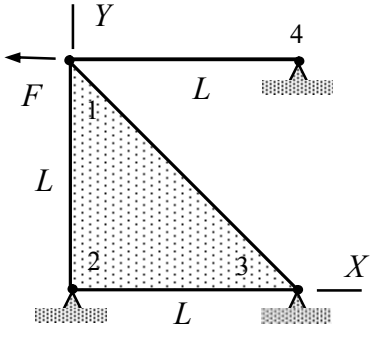
Answer $u_{X1} = \dfrac{2}{3}\dfrac{pL}{E}(1+\nu)$, $u_{Y1} = 0$

8. A thin slab (1) of square shape is loaded by a point force (2) as shown in the figure. Derive the relationship between the force F and the displacement u_{X4} of its point of action. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. The external distributed forces are zeros. Assume plane-stress conditions and use bilinear approximation.



Answer $u_{X4} = \dfrac{6F}{Et}\dfrac{1-\nu^2}{3-\nu}$

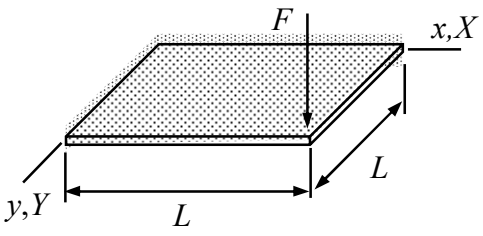
9. A structure, consisting of a thin slab and a bar, is loaded by a horizontal force F acting on node 1. Material properties are E and ν, thickness of the slab is t, and the cross-sectional area of



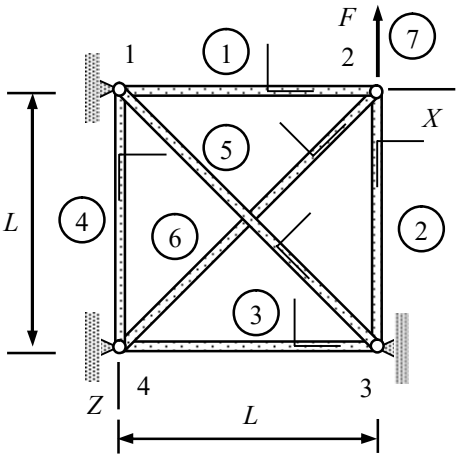
the bar A are constants. Determine displacement components u_{X1} and u_{Y1} of node 1 by using a linear bar element and a linear plane-stress element.

Answer $u_{X1} = -4\dfrac{L(1+\nu)}{Lt+4A(1+\nu)}\dfrac{F}{E}$ and $u_{Y1} = 0$

10. A plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y) = a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the plate bending mode with constant E, ν, ρ and t.



Answer $a_0 = 6\dfrac{FL^2}{Et^3}(1+\nu)$, $w(\dfrac{L}{2},\dfrac{L}{2}) = \dfrac{3}{2}\dfrac{FL^2}{Et^3}(1+\nu)$



Determine the nodal displacements when force F is acting on the structure as shown. The cross-sectional area of bars 1,2,3 and 4 is A and the cross-sectional area of bars 5 and 6 is $2\sqrt{2}A$. Young's modulus of the material is E. Use the principle of virtual work.

Solution

Element and node tables contain the information needed in displacement and stress analysis of the structure. In hand calculations, it is often enough to complete the figure by the material coordinate systems and express the nodal displacements/rotations in terms symbols for the nodal displacements and rotations and/or values known a priori. The components in the material coordinate systems can also be deduced directly form the figure (in simple cases). Virtual work expression of the bar element is given by

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \left(\frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right).$$

Nodal displacements/rotations of the structure are zeros except u_{X2} and u_{Z2}. Element contributions in their virtual work forms are (nodal displacements of the material coordinate system need to be expressed in terms of the structural system components)

Bar 1: $u_{x1} = 0$, $u_{x2} = u_{X2}$: $\quad \delta W^1 = -\delta u_{X2} \dfrac{EA}{L} u_{X2}$,

Bar 2: $u_{x2} = u_{Z2}$, $u_{x3} = 0$: $\quad \delta W^2 = -\delta u_{Z2} \dfrac{EA}{L} u_{Z2}$,

Bar 3: $u_{x4} = 0$ and $u_{x3} = 0$: $\quad \delta W^3 = 0$,

Bar 4: $u_{x1} = 0$ and $u_{x4} = 0$: $\quad \delta W^4 = 0$,

Bar 5: $u_{x1} = 0$ and $u_{x3} = 0$: $\quad \delta W^5 = 0$,

Bar 6: $u_{x4} = 0$, $u_{x2} = \dfrac{1}{\sqrt{2}}(u_{X2} - u_{Z2})$: $\quad \delta W^6 = -(\delta u_{X2} - \delta u_{Z2}) \dfrac{EA}{L} (u_{X2} - u_{Z2})$

Force 7: $\quad \delta W^7 = -\delta u_{Z2} F$.

Virtual work expression of the structure is sum of the element contributions

$$\delta W = \sum\nolimits_e \delta W^e = \delta W^1 + \delta W^2 + \delta W^3 + \delta W^4 + \delta W^5 + \delta W^6 + \delta W^7 \;\Rightarrow$$

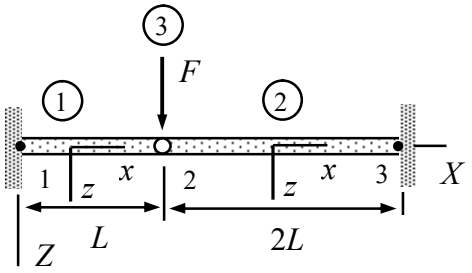
$$\delta W = -\delta u_{X2}\frac{EA}{L}u_{X2} - \delta u_{Z2}\frac{EA}{L}\mathrm{a}_2 + 0 + 0 + 0 - (\delta u_{X2} - \delta u_{Z2})\frac{EA}{L}(u_{X2} - u_{Z2}) - \delta u_{Z2}F \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EA}{L}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix} - \begin{Bmatrix}0\\ -F\end{Bmatrix}\right).$$

Principle of virtual work $\delta W = 0 \;\forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta\mathbf{a}^{\mathrm{T}}\mathbf{R} = 0 \;\; \forall \delta\mathbf{a} \;\Leftrightarrow\; \mathbf{R} = 0$ imply

$$\frac{EA}{L}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix} - \begin{Bmatrix}0\\ -F\end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix} = \frac{FL}{EA}\frac{1}{3}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}0\\ -1\end{Bmatrix} = -\frac{FL}{EA}\begin{Bmatrix}1/3\\ 2/3\end{Bmatrix}.$$

Joint between the beams of the figure is frictionless. Force F acting on the joint and displacement of the beam are restricted to the XZ-plane. Determine the rotations and displacement at the joint. Use two beam elements. The second moment of area I and Young's modulus of the material E are constants.



Solution

Only the displacement in $Z-$direction and rotation in $Y-$direction matter in the planar beam bending problem. From the figure, the non-zero displacement component is u_{Z2}. Rotation may not be continuous at node 2 and, therefore, a superscript is used in θ_{Y2}^1 and θ_{Y2}^2 to denote the rotations at node 2 of element 1 and element 2.

For element 1, the non-zero displacement/rotation components of the material coordinate system are $u_{z2}=u_{Z2}$ and $\theta_{y2}=\theta_{Y2}^1$. The element contribution of a $xz-$plane beam in bending (formulae collection) is

$$\delta W^1=-\begin{Bmatrix}0\\0\\\delta u_{Z2}\\\delta\theta_{Y2}^1\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12&-6L&-12&-6L\\-6L&4L^2&6L&2L^2\\-12&6L&12&6L\\-6L&2L^2&6L&4L^2\end{bmatrix}\begin{Bmatrix}0\\0\\u_{Z2}\\\theta_{Y2}^1\end{Bmatrix}=-\begin{Bmatrix}\delta u_{Z2}\\\delta\theta_{Y2}^1\\\delta\theta_{Y2}^2\end{Bmatrix}^{\mathrm{T}}\frac{EI}{2L^3}\begin{bmatrix}24&12L&0\\12L&8L^2&0\\0&0&0\end{bmatrix}\begin{Bmatrix}u_{Z2}\\\theta_{Y2}^1\\\theta_{Y2}^2\end{Bmatrix}.$$

For element 2, the non-zero displacement/rotation components of the material coordinate system are $u_{z2}=u_{Z2}$ and $\theta_{y2}=\theta_{Y2}^2$. The element contribution is

$$\delta W^2=-\begin{Bmatrix}\delta u_{Z2}\\\delta\theta_{Y2}^2\\0\\0\end{Bmatrix}^{\mathrm{T}}\frac{EI}{2L^3}\begin{bmatrix}3&-3L&-3&-3L\\-3L&4L^2&3&2L^2\\-3&3L&3&3L\\-3L&2L^2&3L&4L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\\theta_{Y2}^2\\0\\0\end{Bmatrix}=-\begin{Bmatrix}\delta u_{Z2}\\\delta\theta_{Y2}^1\\\delta\theta_{Y2}^2\end{Bmatrix}^{\mathrm{T}}\frac{EI}{2L^3}\begin{bmatrix}3&0&-3L\\0&0&0\\-3L&0&4L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\\theta_{Y2}^1\\\theta_{Y2}^2\end{Bmatrix}.$$

Element 3 is a point force whose virtual work expression follows from the definition of work

$$\delta W^3=F\delta u_{Z2}=\begin{Bmatrix}\delta u_{Z2}\\\delta\theta_{Y2}^1\\\delta\theta_{Y2}^2\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}F\\0\\0\end{Bmatrix}.$$

Virtual work expression of a structure is the sum of the element contributions

$$\delta W=\delta W^1+\delta W^2+\delta W^3=-\begin{Bmatrix}\delta u_{Z2}\\\delta\theta_{Y2}^1\\\delta\theta_{Y2}^2\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{2L^3}\begin{bmatrix}27&12L&-3L\\12L&8L^2&0\\-3L&0&4L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\\theta_{Y2}^1\\\theta_{Y2}^2\end{Bmatrix}-\begin{Bmatrix}F\\0\\0\end{Bmatrix}\right).$$

Principle of virtual work $\delta W=0\ \ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus imply the linear equation system

$$\frac{EI}{2L^3}\begin{bmatrix} 27 & 12L & -3L \\ 12L & 8L^2 & 0 \\ -3L & 0 & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta^1_{Y2} \\ \theta^2_{Y2} \end{Bmatrix} - \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix} = 0 .$$

Solving a system of linear equations is one of the basic tasks in FEM (reduction to a triangular system by row operations works in hand calculations). Multiply first row by 4 and third row by 3/L

$$\frac{EI}{2L^3}\begin{bmatrix} 108 & 48L & -12L \\ 12L & 8L^2 & 0 \\ -9 & 0 & 12L \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta^1_{Y2} \\ \theta^2_{Y2} \end{Bmatrix} - \begin{Bmatrix} 4F \\ 0 \\ 0 \end{Bmatrix} = 0 .$$

Add last row to the first row to get

$$\frac{EI}{2L^3}\begin{bmatrix} 99 & 48L & 0 \\ 12L & 8L^2 & 0 \\ -9 & 0 & 12L \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta^1_{Y2} \\ \theta^2_{Y2} \end{Bmatrix} - \begin{Bmatrix} 4F \\ 0 \\ 0 \end{Bmatrix} = 0 .$$

Then multiply second row by -6

$$\frac{EI}{2L^3}\begin{bmatrix} 99 & 48L & 0 \\ -72 & -48L & 0 \\ -9 & 0 & 12L \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta^1_{Y2} \\ \theta^2_{Y2} \end{Bmatrix} - \begin{Bmatrix} 4F \\ 0 \\ 0 \end{Bmatrix} = 0 .$$

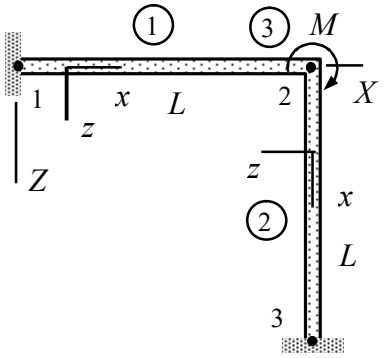
Add the second row to the first row

$$\frac{EI}{2L^3}\begin{bmatrix} 27 & 0 & 0 \\ -72 & -48L & 0 \\ -9 & 0 & 12L \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta^1_{Y2} \\ \theta^2_{Y2} \end{Bmatrix} - \begin{Bmatrix} 4F \\ 0 \\ 0 \end{Bmatrix} = 0 .$$

After these steps, the matrix is lower diagonal one, and solution follows by considering the equations in a proper order one at a time:

$$u_{Z2} = \frac{8FL^3}{27EI}, \quad \theta^1_{Y2} = -\frac{72}{48}\frac{1}{L}u_{Z2} = -\frac{4}{9}\frac{FL^2}{EI} \quad \text{and} \quad \theta^2_{Y2} = \frac{9}{12L}u_{Z2} = \frac{2}{9}\frac{FL^2}{EI} .$$

Determine the rotation θ_{Y2} at node 2 of the structure loaded by a point moment (magnitude M) acting on node 2. Use beam elements (1) and (2) of equal length and a point moment element (3). Assume that the beams are inextensible in the axial directions. Young's modulus of material E and the second moment of area I are constants.



Solution

In a planar problem, torsion and out-plane bending deformation modes can be omitted. As beams are assumed to be inextensible in the axial direction and there are no axial distributed forces, the bar mode virtual work expression vanishes. Virtual work expressions of the beam $xz-$plane bending element and point force/moment elements are given by

$$\delta W = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI_{yy}}{h^3}\begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix}\begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}\right),$$

$$\delta W = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_{X1} \\ F_{Y1} \\ F_{Z1} \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X1} \\ \delta\theta_{Y1} \\ \delta\theta_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_{X1} \\ M_{Y1} \\ M_{Z1} \end{Bmatrix}.$$

Nodal displacements/rotations of the structure are clearly zeros except those for node 2. Displacement at node 2 vanishes also as both beams are inextensible in the axial directions. Therefore, the only non-zero displacement/rotation component of the structure is θ_{Y2}.

Beam 1: $u_{z1}=0$, $\theta_{y1}=0$, $u_{z2}=0$, and $\theta_{y2}=\theta_{Y2}$

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2}.$$

Beam 2: $u_{z1}=0$, $\theta_{y1}=\theta_{Y2}$, $u_{z2}=0$, and $\theta_{y2}=0$

$$\delta W^2 = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2}.$$

Point moment 3:

$$\delta W^3 = \begin{Bmatrix} \delta u_{X2} \\ \delta u_{Y2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_{X2} \\ F_{Y2} \\ F_{Z2} \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X2} \\ \delta\theta_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_{X2} \\ M_{Y2} \\ M_{Z2} \end{Bmatrix} = -\delta\theta_{Y2} M \, .$$

Virtual work expression of the structure is sum of the element contributions

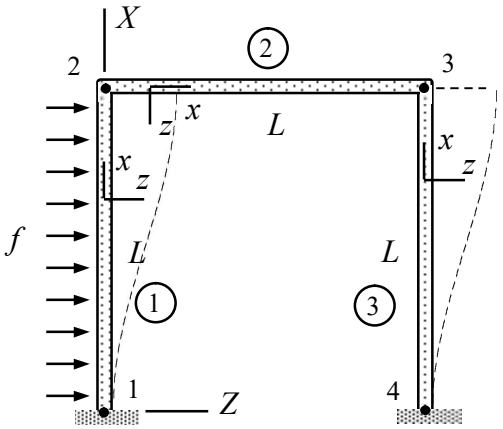
$$\delta W = \sum\nolimits_e \delta W^e = \delta W^1 + \delta W^2 + \delta W^3 \;\Rightarrow$$

$$\delta W = -\delta\theta_{Y2} 4 \frac{EI}{L} \theta_{Y2} - \delta\theta_{Y2} 4 \frac{EI}{L} \theta_{Y2} + 0 - \delta\theta_{Y2} M = -\delta\theta_{Y2} (8 \frac{EI}{L} \theta_{Y2} + M) \, .$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^{\mathrm{T}} \mathbf{R} = 0 \;\; \forall \delta \mathbf{a} \;\Leftrightarrow\; \mathbf{R} = 0$ imply

$$8 \frac{EI}{L} \theta_{Y2} + M = 0 \quad \Leftrightarrow \quad \theta_{Y2} = -\frac{1}{8} \frac{ML}{EI} \, .$$

The frame of the figure consists of a rigid body (2) and beam elements (1) and (3). Determine the non-zero displacements and rotations. The beams are identical and can be assumed rigid in the axial directions. Displacements are confined to the $XZ-$plane. Young's modulus E, second moment of area I, and distributed force f acting on element 1 are constants.



Solution

As element 2 is a rigid body and the other beam are rigid in the axial directions, only the horizontal displacement components $u_{Z3}=u_{Z2}$ are non-zeros. Element contributions to the virtual work expression are

$$\delta W^1=-\begin{Bmatrix}0\\0\\\delta u_{Z2}\\0\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{L^3}\begin{bmatrix}12&-6L&-12&-6L\\-6L&4L^2&6L&2L^2\\-12&6L&12&6L\\-6L&2L^2&6L&4L^2\end{bmatrix}\begin{Bmatrix}0\\0\\u_{Z2}\\0\end{Bmatrix}-\frac{fL}{12}\begin{Bmatrix}6\\-L\\6\\L\end{Bmatrix}\right)=-\delta u_{Z2}\left(12\frac{EI}{L^3}u_{Z2}-\frac{fL}{2}\right),$$

$$\delta W^2=0,$$

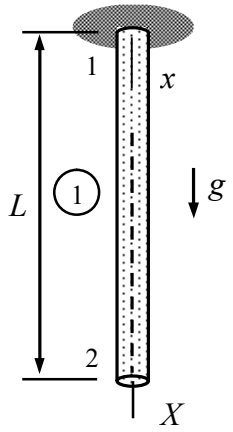
$$\delta W^3=-\begin{Bmatrix}0\\0\\\delta u_{Z2}\\0\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12&-6L&-12&-6L\\-6L&4L^2&6L&2L^2\\-12&6L&12&6L\\-6L&2L^2&6L&4L^2\end{bmatrix}\begin{Bmatrix}0\\0\\u_{Z2}\\0\end{Bmatrix}=-\delta u_{Z2}12\frac{EI}{L^3}u_{Z2}.$$

Virtual work expression of the structure is the sum of element contributions

$$\delta W=\delta W^1+\delta W^2+\delta W^3=-\delta u_{Z2}\left(24\frac{EI}{L^3}u_{Z2}-\frac{fL}{2}\right).$$

Principle of virtual work $\delta W=0\ \forall\delta\mathrm{a}$ and the fundamental lemma of variation calculus imply

$$u_{Z2}=\frac{1}{48}\frac{fL^4}{EI}\ .$$



Consider a bar of length L loaded by its own weight (figure). Determine the displacement u_{X2} at the free end. Start with the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ and approximation $u = (1-x/L)u_{x1} + (x/L)u_{x2}$. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.

Solution

The concise representation of the element contribution consists of a virtual work density expression and approximations to the displacement and rotation components. Approximations are just substituted into the density expression followed by integration over the domain occupied by the element (line segment, triangle etc.). Here the two building blocks are

$$\delta w_\Omega = -\frac{d\delta u}{dx} EA \frac{du}{dx} + \delta u f_x \quad \text{and} \quad u = (1-\frac{x}{L})u_{x1} + \frac{x}{L}u_{x2}.$$

The quantities needed in the virtual work density are the axial displacement, variation of the axial displacement, and variation of the derivative of the axial displacement

$$u = \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \Rightarrow \quad \delta u = \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix},$$

$$\frac{du}{dx} = \frac{1}{L}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \Rightarrow \quad \frac{d\delta u}{dx} = \frac{1}{L}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix} = \frac{1}{L}\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}.$$

When the approximation is substituted there, virtual work density expression of the bar model takes the form

$$\delta w_\Omega = -\frac{d\delta u}{dx} EA \frac{du}{dx} + \delta u f_x = -\frac{1}{L}\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} EA \frac{1}{L}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} + \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix} f \quad \Leftrightarrow$$

$$\delta w_\Omega = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \frac{1}{L} EA \frac{1}{L} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix} f \right) \quad \Leftrightarrow$$

$$\delta w_\Omega = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix} f \right).$$

Finally, integration over the element gives the virtual work expression of the bar element

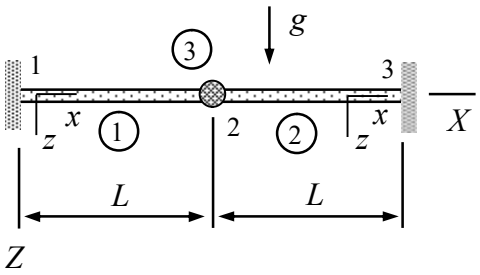
$$\delta W = \int_0^L \delta w_\Omega dx = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{fL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right).$$

Finding the displacement of the free end follows the usual lines. Here, $f_x = \rho g A$, $u_{x1} = u_{X1} = 0$, and $u_{x2} = u_{X2}$

$$\delta W = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \frac{\rho g A L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right) = -\delta u_{X2}\left(\frac{EA}{L}u_{X2} - \frac{\rho g A L}{2}\right) = 0 \quad \forall \delta u_{X2} \quad \Leftrightarrow$$

$$\frac{EA}{L}u_{X2} - \frac{\rho g A L}{2} = 0 \quad \Leftrightarrow \quad u_{X2} = \frac{\rho g L^2}{2E}.$$

The $XZ-$plane structure shown consists of two *massless* beams and a homogeneous disk of mass m considered as a rigid body. Determine the displacement u_{Z2} and rotation θ_{Y2} at node 2. Young's modulus E of the beam material and the second moment of area I are constants.



Solution

Only the displacement in the $Z-$direction and rotation in the $Y-$direction matter in the planar beam bending problem. From the figure, the non-zero displacement and rotation components are u_{Z2} and θ_{Y2}. For element 1, the non-zero displacement/rotation components of the material coordinate system are $u_{z2}=u_{Z2}$ and $\theta_{y2}=\theta_{Y2}$. The element contribution of a plane beam in bending (formulae collection) is

$$\delta W^1=-\begin{Bmatrix}0\\0\\\delta u_{Z2}\\\delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12&-6L&-12&-6L\\-6L&4L^2&6L&2L^2\\-12&6L&12&6L\\-6L&2L^2&6L&4L^2\end{bmatrix}\begin{Bmatrix}0\\0\\u_{Z2}\\\theta_{Y2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{Z2}\\\delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12&6L\\6L&4L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\\theta_{Y2}\end{Bmatrix}.$$

For element 2, the non-zero displacement/rotation components of the material coordinate system are $u_{z2}=u_{Z2}$ and $\theta_{y2}=\theta_{Y2}$. The element contribution of a *xz*-plane beam in bending is

$$\delta W^2=-\begin{Bmatrix}\delta u_{Z2}\\\delta\theta_{Y2}\\0\\0\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12&-6L&-12&-6L\\-6L&4L^2&6L&2L^2\\-12&6L&12&6L\\-6L&2L^2&6L&4L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\\theta_{Y2}\\0\\0\end{Bmatrix}=-\begin{Bmatrix}\delta u_{Z2}\\\delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12&-6L\\-6L&4L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\\theta_{Y2}\end{Bmatrix}.$$

Element 3 is a rigid body. In static displacement analysis, only the weight acting at the mass centroid matters. Virtual work expression of the point force of magnitude mg follows from the definition of work

$$\delta W^3=mg\delta u_{Z2}=\begin{Bmatrix}\delta u_{Z2}\\\delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}mg\\0\end{Bmatrix}.$$

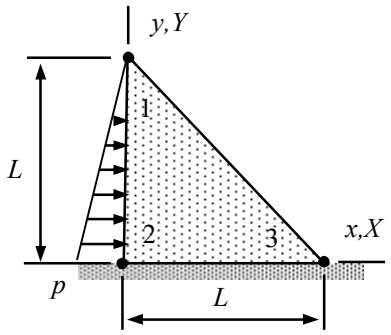
Virtual work expression of a structure is the sum of the element contributions

$$\delta W=\delta W^1+\delta W^2+\delta W^3=-\begin{Bmatrix}\delta u_{Z2}\\\delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{L^3}\begin{bmatrix}24&0\\0&8L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\\theta_{Y2}\end{Bmatrix}-\begin{Bmatrix}mg\\0\end{Bmatrix}\right).$$

Principle of virtual work $\delta W=0\ \ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus imply the linear equation system

$$\frac{EI}{L^3}\begin{bmatrix}24&0\\0&8L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\\theta_{Y2}\end{Bmatrix}-\begin{Bmatrix}mg\\0\end{Bmatrix}=0\ \Leftrightarrow\ u_{Z2}=\frac{1}{24}\frac{mgL^3}{EI}\ \text{ and }\ \theta_{Y2}=0.$$

A long dam of homogeneous, isotropic, linearly elastic material, is subjected to water pressure on one side. Material properties E and ν are constants. Determine the displacement components u_{X1} and u_{Y1} of node 1. Nodes 2 and 3 are fixed. Use a three-node element and assume plane strain conditions. Consider a slab of thickness t in calculations. The peak value of the linearly varying pressure is p.



Solution

Under the plane strain conditions, the virtual work densities of thin slab are

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\varepsilon} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_{\varepsilon} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}.$$

The external forces t_x and t_y (force per unit length in this case) acting on the element edges can be taken into account by a separate force element with the density expression (per unit length)

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}$$

although the expression is actually part of the thin slab model. The approximation on the boundary is just the restriction of the element approximation to the boundary.

Only the shape function for node 1 is needed as the other nodes are fixed (displacement vanishes). In terms of the displacement components u_{X1} and u_{Y1} of node 1, element approximations of the displacement components and their derivatives are

$$u = \frac{y}{L}u_{X1} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = \frac{1}{L}u_{X1},$$

$$v = \frac{y}{L}u_{Y1} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = \frac{1}{L}u_{Y1}.$$

When the approximation is substituted there, the virtual work densities simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ \delta u_{Y1}/L \\ \delta u_{X1}/L \end{Bmatrix}^{\mathrm{T}} \frac{Et}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{Y1}/L \\ u_{X1}/L \end{Bmatrix} \Rightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\dfrac{Et}{2(1+\nu)L^2} & 0\\ 0 & \dfrac{Et(1-\nu)}{(1+\nu)(1-2\nu)L^2}\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix},$$

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix}\delta u\\ \delta v\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}t_x\\ t_y\end{Bmatrix} = \begin{Bmatrix}\delta u_{X1}y/L\\ \delta u_{Y1}y/L\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}pt(1-y/L)\\ 0\end{Bmatrix} = \begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}pt(1-y/L)y/L\\ 0\end{Bmatrix}.$$

Integrations over the element and edge 2-1 give the virtual work expressions (notice that the virtual work density of internal forces is constant)

$$\delta W^{\text{int}} = \int_{\Omega}\delta w_{\Omega}^{\text{int}}\,d\Omega = -\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\dfrac{Et}{4(1+\nu)} & 0\\ 0 & \dfrac{Et(1-\nu)}{2(1+\nu)(1-2\nu)}\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix},$$

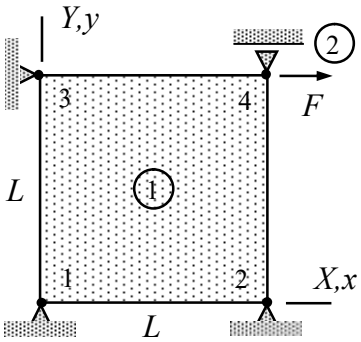
$$\delta W^{\text{ext}} = \int_0^L \delta w_{\partial\Omega}^{\text{ext}}\,dy = \begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}ptL/6\\ 0\end{Bmatrix}.$$

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\end{Bmatrix}^{\mathrm{T}}\left(\begin{bmatrix}\dfrac{Et}{4(1+\nu)} & 0\\ 0 & \dfrac{Et(1-\nu)}{2(1+\nu)(1-2\nu)}\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix} - \begin{Bmatrix}ptL/6\\ 0\end{Bmatrix}\right) = 0 \qquad \Rightarrow$$

$$\begin{bmatrix}\dfrac{Et}{4(1+\nu)} & 0\\ 0 & \dfrac{Et(1-\nu)}{2(1+\nu)(1-2\nu)}\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix} - \begin{Bmatrix}ptL/6\\ 0\end{Bmatrix} = 0 \quad \Leftrightarrow \quad u_{X1} = \frac{2}{3}\frac{pL}{E}(1+\nu) \;\text{ and }\; u_{Y1} = 0.$$

A thin slab (1) of square shape is loaded by a point force (2) as shown in the figure. Derive the relationship between the force F and the displacement u_{X4} of its point of action. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. The external distributed forces are zeros. Assume plane stress conditions and use bilinear approximation.



Solution

Let us start with the shape functions of element 1 and approximations. As nodes 1, 2, and 3 are fixed, it is enough to deduce the shape function of node 4

$$N_4 = \frac{xy}{L^2} \ .$$

Approximations to the displacement components and their derivatives with respect to x and y are

$$u = \frac{xy}{L^2} u_{X4}, \quad \frac{\partial u}{\partial x} = \frac{y}{L^2} u_{X4}, \text{ and } \frac{\partial u}{\partial y} = \frac{x}{L^2} u_{X4}$$

$$v = 0, \quad \frac{\partial v}{\partial x} = 0, \text{ and } \frac{\partial v}{\partial y} = 0 .$$

When the approximations are substituted there, the virtual work density of thin slab model simplifies to (plane stress conditions, only the internal part is needed)

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial \delta u / \partial x \\ \partial \delta v / \partial y \\ \partial \delta u / \partial y + \partial \delta v / \partial x \end{Bmatrix}^{\mathrm{T}} \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_\Omega^{\text{int}} = -\delta u_{X4} \frac{tE}{1-\nu^2} \frac{1}{L^4} (y^2 + \frac{1-\nu}{2} x^2) u_{X4} .$$

Integration over the domain occupied by the element gives the element contribution

$$\delta W^1 = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dx dy = -\delta u_{X4} \frac{Et}{6} \frac{3-\nu}{1-\nu^2} u_{X4} .$$

Virtual work expression of the point force (element 2) follows from the definition of work

$$\delta W^2 = \delta u_{X4} F .$$

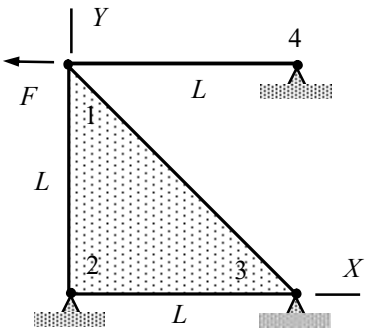
Virtual work expression of a structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^2 = \delta u_{X4} (-\frac{Et}{6} \frac{3-\nu}{1-\nu^2} u_{X4} + F) .$$

Finally, principle of virtual work in the form $\delta W = 0 \quad \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus imply that

$$u_{X4} = \frac{6F}{Et}\frac{1-\nu^2}{3-\nu}.$$

A structure, consisting of a thin slab and a bar, is loaded by a horizontal force F acting on node 1. Material properties are E and ν, thickness of the slab is t, and the cross-sectional area of the bar A are constants. Determine displacement components u_{X1} and u_{Y1} of node 1 by using a linear bar element and a linear plane-stress element.



Solution

Under the plane stress conditions, the virtual work densities (virtual works per unit area) of the thin slab model

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress) and external forces acting on the element domain. Notice that the components f_x and f_y are external forces per unit area. Forces acting on the element edges can be taken into account by separate force elements.

Element contribution for the thin slab needs to be derived from approximation and virtual work densities. Approximations to the displacement components depend only on the shape function associated with node 1 as the other nodes are fixed (displacement vanishes). In terms of the displacement components u_{X1} and u_{Y1}

$$u = u_{X1}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = u_{X1}\frac{1}{L},$$

$$v = u_{Y1}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = u_{Y1}\frac{1}{L}.$$

Virtual work density of the internal forces simplifies to (when the approximations are substituted there)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ \delta u_{Y1} \\ \delta u_{X1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2}\frac{Et}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{Y1} \\ u_{X1} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2}\frac{Et}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}$$

Virtual work expression is the integral of density over the domain occupied by the element (note that the virtual work density is constant in this case). Therefore

$$\delta W^1 = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{2}\frac{Et}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

Virtual work expression of the bar element is given in the formula collection with $u_{x1} = u_{X1}$ and $u_{x2} = 0$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X1} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

Virtual work expression of the point force follows e.g. directly from the definition (force multiplied by the virtual displacement in its direction)

$$\delta W^3 = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F \\ 0 \end{Bmatrix}.$$

Virtual work expression of the structure is the sum of element contributions $\delta W = \delta W^1 + \delta W^2 + \delta W^3$

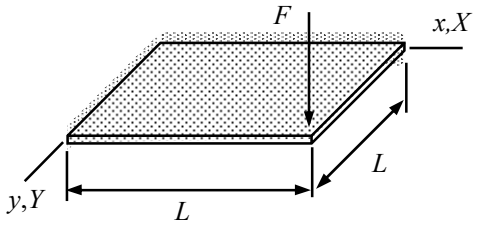
$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\frac{1}{2}\frac{Et}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \end{Bmatrix}\right) \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \frac{1}{4}\frac{Et}{1+\nu} + \frac{EA}{L} & 0 \\ 0 & \frac{1}{2}\frac{Et}{1-\nu^2} \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \end{Bmatrix}\right).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\begin{bmatrix} \frac{1}{4}\frac{Et}{1+\nu} + \frac{EA}{L} & 0 \\ 0 & \frac{1}{2}\frac{Et}{1-\nu^2} \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad u_{X1} = -\frac{4(1+\nu)L}{tL + 4(1+\nu)A}\frac{F}{E} \quad \text{and} \quad u_{Y1} = 0.$$

A plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y) = a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the Kirchhoff plate model with constant E, ν, ρ and t.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, it is enough to consider the virtual work densities of the bending mode only

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix}, \quad \delta w_\Omega^{\text{ext}} = \delta w f_z .$$

in which the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

In the present case, distributed force vanishes i.e. $f_z = 0$ and the point force is taken into account by a point force element.

Approximation to the transverse displacement is chosen to be (a_0 is not associated with any point but it just a parameter of the approximation)

$$w(x,y) = a_0 \frac{x}{L}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 0, \ \frac{\partial^2 w}{\partial y^2} = 0, \text{ and } \frac{\partial^2 w}{\partial x\partial y} = \frac{1}{L^2}a_0 .$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} = -\delta a_0 \frac{Et^3}{6(1+\nu)}\frac{1}{L^4}a_0 .$$

Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element

$$\delta W^1 = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{Et^3}{6(1+\nu)}\frac{1}{L^2}a_0 .$$

Virtual work expression of the point force (element 2 here) follows from the definition of work (notice the use of virtual displacement of the point of action $x = y = L$)

$\delta W^2 = \delta w(L,L)F = \delta a_0 F$.

Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^1 + \delta W^2 = -\delta a_0 (\frac{Et^3}{6(1+\nu)} \frac{1}{L^2} a_0 - F) = 0 \quad \Rightarrow \quad a_0 = 6(1+\nu)\frac{FL^2}{Et^3} .$$

Displacement at the center point

$$w(\frac{L}{2},\frac{L}{2}) = a_0 \frac{1}{4} = \frac{3}{2}(1+\nu)\frac{FL^2}{Et^3} .$$