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ANALYTICAL METHODS IN CONDUCTION HEAT TRANSFER

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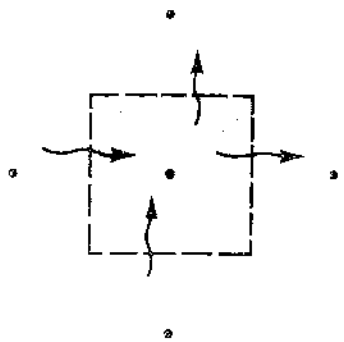


FINITE DIFFERENCES

8·0 INTRODUCTION

The widespread use of the digital computer has made finite-difference methods extremely valuable for solving problems that are not susceptible to the analytical methods discussed in earlier chapters. As we have noticed, analytical methods are usually restricted to very simple geometries and boundary conditions. For the more complex problems, finite differences is a feasible method of attack.

Finite-difference methods are also quite useful in problems involving nonlinearities, such as radiation. Analytical methods rarely work in these cases. Problems with variable properties must often be handled numerically.

It should be mentioned here, however, that, although the majority of practical problems may require finite-difference methods to obtain detailed answers, analytical methods are still important. In setting up a complex problem using finite differences, limiting cases are often considered as a check on the computations. These limiting cases often have analytical solu-

tions that can be used for comparison to the finite-difference results. Asymptotic, analytical solutions can often be used in conjunction with the computer to provide better solutions.

This chapter is intended to provide the reader with the basic ideas behind finite-difference methods for solving differential equations. To give added insight to the engineer, the problem is formulated from both a physical point of view and a mathematical point of view. Several simple problems are discussed to elucidate the techniques and point out some of the pitfalls. Section 8·1 is an introduction to the mathematical concepts needed in finite differences. The solution of one-dimensional, steady-state problems is discussed in Sec. 8·2. These considerations are then extended to two-dimensions in Sec. 8·3. One-dimensional, transient problems are discussed in Sec. 8·4. The extension to more dimensions should be apparent from these sections. Some more advanced problems are discussed in Sec. 8·5.

8·1 FUNDAMENTAL CONCEPTS

You have already been exposed to most of the basic ideas that will be needed for the finite-difference method. The energy balance and rate equation will be used either directly to arrive at a finite-difference formulation (physical formulation) or to arrive at the governing differential equation for the problem as in Chap. 1. The differential equation can then be converted into finite-difference form (mathematical formulation).

The only new idea that is used is the approximation of derivatives in terms of differences. Let us consider the function $t(x)$ shown in Fig. 8·1. The exact definition of the derivative at x_m is given as

$$\left.\frac{dt}{dx}\right|_{x_m} \triangleq \lim_{\Delta x \to 0} \frac{t(x_m + \Delta x) - t(x_m)}{\Delta x}$$

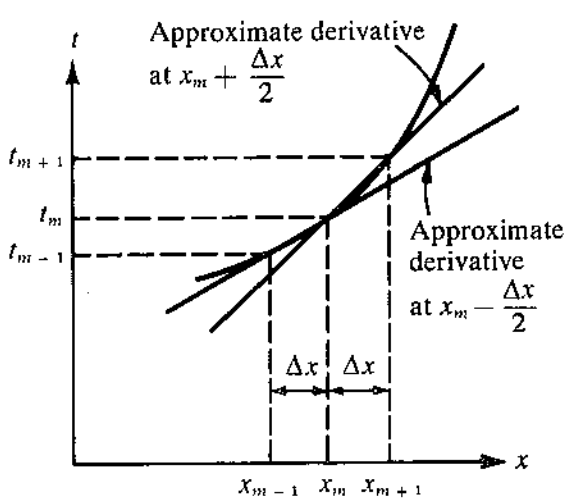


FIG. 8·1 *Approximation of derivatives.*

As an approximation, let us not go through the limiting process. Then we could write

$$\left.\frac{dt}{dx}\right|_{x_m} = \frac{t(x_m + \Delta x) - t(x_m)}{\Delta x} = \frac{t_{m+1} - t_m}{\Delta x}$$

This would be an approximate expression for the derivative at x_m, or, more appropriately, we could consider it to be an approximation of the derivative midway between x_m and x_{m+1}. That is,

$$\left.\frac{dt}{dx}\right|_{m+\frac{1}{2}} = \frac{t_{m+1} - t_m}{\Delta x} \tag{8·1·1}$$

Similarly, the slope at $x_m - \Delta x/2$ may be written as

$$\left.\frac{dt}{dx}\right|_{m-\frac{1}{2}} = \frac{t_m - t_{m-1}}{\Delta x} \tag{8·1·2}$$

The second derivative of a function is simply the rate of change of the first derivative. Thus the second derivative at x_m may be approximated as

$$\left.\frac{d^2t}{dx^2}\right|_m = \frac{\left.\dfrac{dt}{dx}\right|_{x_m + \Delta x/2} - \left.\dfrac{dt}{dx}\right|_{x_m - \Delta x/2}}{\Delta x}$$

Equations (8·1·1) and (8·1·2) may now be substituted into the above to give

$$\left.\frac{d^2t}{dx^2}\right|_m = \frac{\dfrac{t_{m+1} - t_m}{\Delta x} - \dfrac{t_m - t_{m-1}}{\Delta x}}{\Delta x}$$

which reduces to

$$\left.\frac{d^2t}{dx^2}\right|_m = \frac{t_{m-1} - 2t_m + t_{m+1}}{(\Delta x)^2} \tag{8·1·3}$$

The above results may also be deduced by considering a Taylor series expansion of $t(x)$. Expanding $t(x)$ about $x_m + \Delta x/2$, the value of $t(x)$ at x_{m+1} (that is, t_{m+1}) is given by

$$t_{m+1} = t_{m+\frac{1}{2}} + \left.\frac{dt}{dx}\right|_{m+\frac{1}{2}} \frac{\Delta x}{2} + \frac{1}{2} \left.\frac{d^2t}{dx^2}\right|_{m+\frac{1}{2}} \left(\frac{\Delta x}{2}\right)^2 + O[(\Delta x)^3]$$

The symbol $O[(\Delta x)^3]$ means that the order of magnitude of the remaining terms is $(\Delta x)^3$. The value at x_m is similarly found to be

$$t_m = t_{m+\frac{1}{2}} - \left.\frac{dt}{dx}\right|_{m+\frac{1}{2}} \frac{\Delta x}{2} + \frac{1}{2} \left.\frac{d^2t}{dx^2}\right|_{m+\frac{1}{2}} \left(\frac{\Delta x}{2}\right)^2 + O[(\Delta x)^3]$$

These equations may now be subtracted. Observe that the first and third terms will cancel out. Thus

$$t_{m+1} - t_m = \left.\frac{dt}{dx}\right|_{m+\frac{1}{2}} \Delta x + O[(\Delta x)^3]$$

This may now be solved for the derivative as

$$\left.\frac{dt}{dx}\right|_{m+\frac{1}{2}} = \frac{t_{m+1} - t_m}{\Delta x} + O[(\Delta x)^2] \tag{8·1·4}$$

This result is identical to Eq. (8·1·1). As Δx becomes small, the remainder terms, whose order of magnitude is $(\Delta x)^2$, may be neglected. A similar expression may be found corresponding to Eq. (8·1·2).

The second derivative of $t(x)$ may be found by considering the Taylor series expansion about the point x_m. The values of t_{m+1} and t_{m-1} may be written as

$$t_{m+1} = t_m + \left.\frac{dt}{dx}\right|_m \Delta x + \frac{1}{2}\left.\frac{d^2t}{dx^2}\right|_m (\Delta x)^2 + \frac{1}{6}\left.\frac{d^3t}{dx^3}\right|_m (\Delta x)^3 + O[(\Delta x)^4]$$

$$t_{m-1} = t_m - \left.\frac{dt}{dx}\right|_m \Delta x + \frac{1}{2}\left.\frac{d^2t}{dx^2}\right|_m (\Delta x)^2 - \frac{1}{6}\left.\frac{d^3t}{dx^3}\right|_m (\Delta x)^3 + O[(\Delta x)^4]$$

Upon adding these two equations, the odd powers of Δx will cancel to give

$$t_{m+1} + t_{m-1} = 2t_m + \left.\frac{d^2t}{dx^2}\right|_m (\Delta x)^2 + O[(\Delta x)^4]$$

This may now be solved to give the second derivative as

$$\left.\frac{d^2t}{dx^2}\right|_m = \frac{t_{m-1} - 2t_m + t_{m+1}}{(\Delta x)^2} + O[(\Delta x)^2] \tag{8·1·5}$$

This is the same result as given by Eq. (8·1·3) when Δx becomes small.

8·2 ONE-DIMENSIONAL, STEADY-STATE PROBLEMS

Mathematically speaking, one-dimensional, steady-state conduction problems are classed as *boundary-value problems*. A boundary-value problem is one in which the value of the unknown

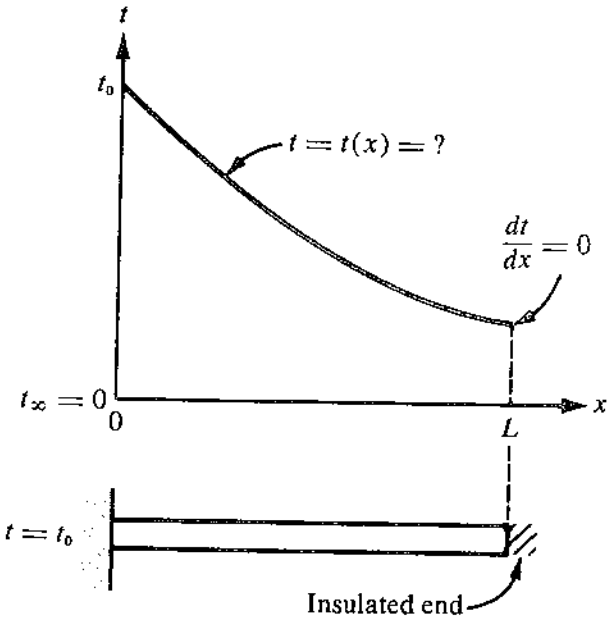


FIG. 8·2 *The thin-rod problem.*

function (or some other information) is specified on the boundary of the region of interest, and the problem is to determine the behavior of the function within the region. The discussion in this section will be representative of any boundary-value problem.

As an example, let us consider the familiar thin-rod problem whose analytical solution was discussed in Chap. 1. Figure 8·2 describes the problem. Due to the linearity of the problem, it is entirely adequate to take the ambient temperature to be zero. The same result would be obtained by defining a new variable $T = t - t_\infty$ as used in Sec. 1·2·4.

The finite-difference formulation of the problem may be deduced in either of two basic ways. The physical formulation begins from scratch and uses energy balances and rate equations in approximate form. The mathematical formulation starts with the differential equation (as derived in Chap. 1) and then approximates the derivatives that appear. Both methods will be discussed because knowledge of each approach is valuable to the engineer.

8·2·1 Physical formulation

We can start directly to set up a finite-difference approximation to the problem by dividing the rod into a number of equally spaced nodes as shown in Fig. 8·3. We can then take a typical interior chunk of material, located about node m, at location x_m and write an energy balance for it. For the energy terms shown in Fig. 8·3 we would write

$$q_{m-1,m} = q_{c_m} + q_{m,m+1}$$

In the above, the notation $q_{m-1,m}$ refers to the energy transferred into the system from node $m - 1$ to node m. A similar convention

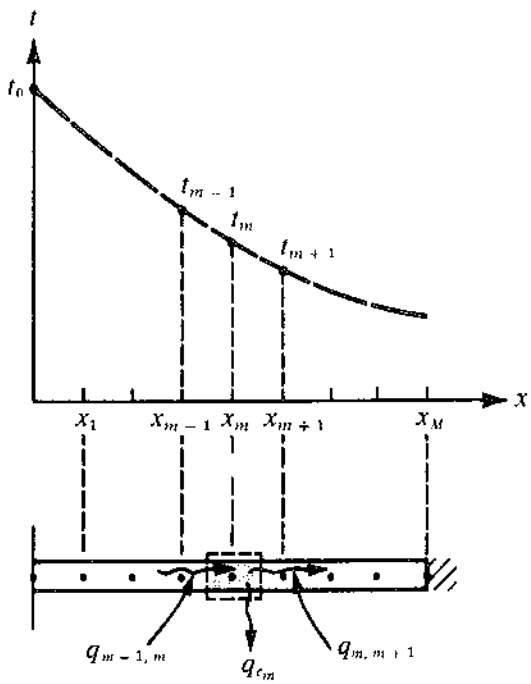


FIG. 8-3 *Typical thin-rod nodal system with important energy terms indicated.*

explains $q_{m,m+1}$ as the conduction heat transfer from node m to node $m + 1$. The order of the subscripts indicates the assumed direction of heat flow.

As in setting up differential equations, the next step is to write the rate equations for each term appearing in the energy balance. The expressions for the two conduction terms may be written as

$$q_{m-1,m} = -kA\frac{dt}{dx} = -kA\frac{t_m - t_{m-1}}{\Delta x} = kA\frac{t_{m-1} - t_m}{\Delta x}$$

and

$$q_{m,m+1} = -kA\frac{dt}{dx} = -kA\frac{t_{m+1} - t_m}{\Delta x} = kA\frac{t_m - t_{m+1}}{\Delta x}$$

The convective loss from the system is most easily approximated by assuming that the average temperature of this chunk of material is t_m. Then we can write

$$q_{c_m} = hp\,\Delta x\,(t_m - t_\infty) = hp\,\Delta x\,t_m$$

where t_∞ has been taken to be zero in the expression for q_{c_m}.

Substitution of the rate equations into the energy balance gives

$$\frac{kA}{\Delta x}(t_{m-1} - t_m) = hp\,\Delta x\,t_m + \frac{kA}{\Delta x}(t_m - t_{m+1})$$

Multiplying by $\Delta x/kA$ and rearranging gives

$$t_{m-1} - 2t_m - \frac{hp}{kA}(\Delta x)^2 t_m + t_{m+1} = 0$$

Finally, the equation may be rewritten as

$$-t_{m-1} + Dt_m - t_{m+1} = 0 \tag{8·2·1}$$

where

$$D = 2 + \frac{hp}{kA}(\Delta x)^2 = 2 + \mathrm{m}^2(\Delta x)^2 \tag{8·2·2}$$

In Eq. (8·2·2) we have used $\mathrm{m}^2 = hp/kA$, as we did in Chap. 1. The m should not be confused with the m used to denote a node as in Eq. (8·2·1). It should be observed that, for $m = 1$, the t_0 that appears in Eq. (8·2·1) is the known temperature specified by the boundary condition at $x = 0$.

To obtain a relation at x_M, we will write an energy balance on the end of the rod as shown in Fig. 8·4. The energy balance, assuming an insulated end, is given by

$$q_{M-1,M} = q_{c_M}$$

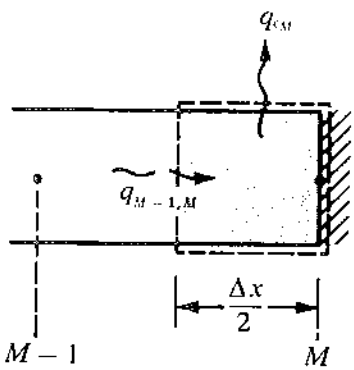


FIG. 8·4 *Nodal system and energy terms at tip of thin rod.*

The rate equations are approximated by

$$q_{M-1,M} = kA\frac{t_{M-1} - t_M}{\Delta x}$$

and

$$q_{c_M} = hp\frac{\Delta x}{2}(t_M - t_\infty) = \frac{hp\,\Delta x}{2}t_M$$

Thus, upon substituting the rate equations into the energy balance,

$$\frac{kA}{\Delta x}(t_{M-1} - t_M) = \frac{hp\,\Delta x}{2}t_M$$

or

$$2(t_{M-1} - t_M) = \frac{hp}{kA}(\Delta x)^2 t_M$$

Finally, upon rearranging and using the definition of D given above,

$$-2t_{M-1} + Dt_M = 0 \tag{8·2·3}$$

This equation is valid at node M.

We have thus arrived at a set of M algebraic equations for M unknowns $(t_1, t_2, \ldots, t_{M-1}, t_M)$ by writing an energy balance on a finite piece of material surrounding each node whose temperature is unknown. These equations are represented by Eqs. (8·2·1) and (8·2·3). The solution to these equations will be discussed later in Sec. 8·2·3.

8·2·2 Mathematical formulation

A completely mathematical approach to the problem begins with the normalized differential equation and its boundary conditions. The normalized differential equation is given by

$$\frac{d^2u}{dx^2} - (mL)^2 u = 0 \tag{8·2·4}$$

where u has been defined to be $(t - t_\infty)/(t_0 - t_\infty)$. The normalized boundary conditions are given by

$$u(0) = 1 \qquad \text{and} \qquad u'(1) = 0$$

We want to find a curve $u = u(x)$, as in Fig. 8·5, which passes through $u = 1$ at $x = 0$ and has zero slope at $x = 1$. This curve must also satisfy Eq. (8·2·4) at every value of x.

To formulate the problem by finite differences, we first divide the x direction into equally spaced nodes as shown in Fig. 8·5. We then replace Eq. (8·2·4) by an approximate difference equation. The second derivative at x_m is given by Eq. (8·1·3). Thus Eq. (8·2·4) is replaced by

$$\frac{u_{m-1} - 2u_m + u_{m+1}}{(\Delta x)^2} - (mL)^2 u_m = 0$$

Rearranging,

$$u_{m-1} - [2 + (mL)^2(\Delta x)^2]u_m + u_{m+1} = 0$$

This may be rewritten as

$$-u_{m-1} + Du_m - u_{m+1} = 0 \tag{8·2·5}$$

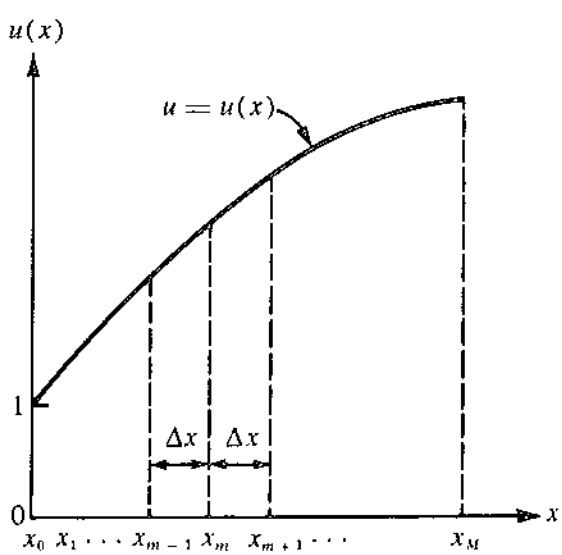


FIG. 8·5 *Normalized thin-rod problem.*

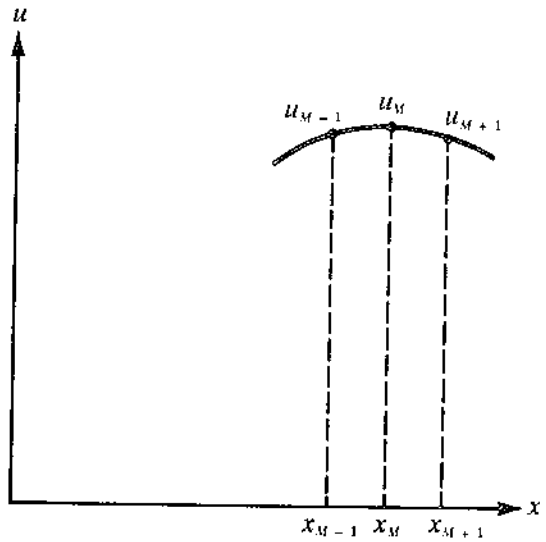


FIG. 8·6 *Fictitious nodal point at x_{M+1} with $u_{M+1} = u_{M-1}$ will provide zero derivative at x_M.*

where

$$D = 2 + (mL)^2(\Delta x)^2 \tag{8·2·6}$$

This equation is recognized as a nondimensional form of Eq. (8·2·1). It represents a set of relations that must be satisfied for each interior node between $x = 0$ and $x = 1$. At the first interior node $m = 1$, the boundary condition $u_0 = 1$ may be used to simplify this expression. The boundary condition at $x = 1$ can be approximately satisfied by taking $u_{M+1} = u_{M-1}$. As shown in Fig. 8·6, this makes the slope of $u(x)$ equal to zero at $x = 1$ by providing for symmetry about the endpoint x_M. Another way of looking at this is to think of having a mirror-image extension of the rod beyond $x = 1$ as shown in Fig. 8·7. Symmetry demands that $u_{M-1} = u_{M+1}$. At x_M, Eq. (8·2·5) then becomes

$$-u_{M-1} + Du_M - u_{M-1} = 0$$

or

$$-2u_{M-1} + Du_M = 0 \tag{8·2·7}$$

Equations (8·2·5) and (8·2·7) represent M algebraic relations among the M unknown u_m that must be satisfied to obtain an

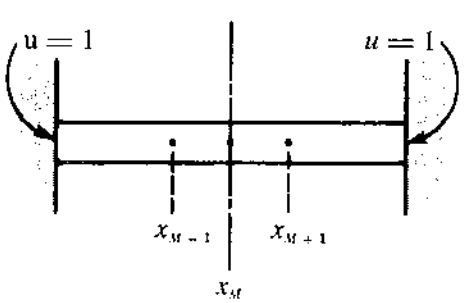


FIG. 8·7 *Mirror image of thin rod.*

approximate solution to the problem. They are the nondimensional counterparts of Eqs. (8·2·1) and (8·2·3).

For the case in which $M = 4$, these equations may be written as

$$\begin{aligned} Du_1 - u_2 &= 1 \\ -u_1 + Du_2 - u_3 &= 0 \\ - u_2 + Du_3 - u_4 &= 0 \\ - 2u_3 + Du_4 &= 0 \end{aligned} \tag{8·2·8}$$

Observe that the problem has now been reduced to the solution of a set of four simultaneous algebraic equations rather than a single ordinary differential equation. These equations can be easily solved on the digital computer as discussed in the next section.

8·2·3 Solution by gaussian elimination

Systems of equations such as those that appear in this problem are most efficiently solved by systematically eliminating the unknowns. It is particularly simple in this case since no more than three unknowns appear in one equation. In addition the unknowns and equations are arranged in a *tridiagonal* pattern.

As an example, consider the case of a fin with $mL = 2$. For this case $D = 2 + (2)^2(0.25)^2 = 2.25$, and the system of equations becomes

$$\begin{aligned} 2.25u_1 - u_2 &= 1 \\ -u_1 + 2.25u_2 - u_3 &= 0 \\ - u_2 + 2.25u_3 - u_4 &= 0 \\ - 2u_3 + 2.25u_4 &= 0 \end{aligned} \tag{8·2·9}$$

To solve these equations let us first eliminate all the terms below the main diagonal. The first equation can be multiplied by 1/2.25 and added to the second equation to give

$$\begin{aligned} 2.25u_1 - u_2 &= 1 \\ 1.81u_2 - u_3 &= 0.444 \\ - u_2 + 2.25u_3 - u_4 &= 0 \\ - 2u_3 + 2.25u_4 &= 0 \end{aligned}$$

Observe that u_1 has now been eliminated from the second equation. This process may be continued to eliminate u_2 from the third equation and u_3 from the last equation. The result is the

bidiagonal form shown below:

$$\begin{aligned} 2.25u_1 - u_2 &= 1 \\ 1.81u_2 - u_3 &= 0.444 \\ 1.70u_3 - u_4 &= 0.246 \\ 1.07u_4 &= 0.290 \end{aligned} \tag{8·2·10}$$

Next, the last equation may be solved for $u_4 = 0.290/1.07 = 0.271$. This value may be substituted into the third equation to find u_3, which in turn may be substituted into the second equation to find u_2, etc.

This elimination process is easily programmed for the digital computer. The execution time for this problem on the IBM 360/67 was less than a tenth of a second. This is obviously a fast way to solve this system of equations.

8·2·4 Numerical results

The results of this solution are compared to the exact solution of the differential equation in Table 8·1. Observe that reasonably good results have been obtained by using only four nodal points. Better accuracy can be obtained by increasing the number of nodes. Figure 8·8 shows how the finite-difference solution progresses toward the exact solution for $M = 1$, 2, and 4.

TABLE 8·1 *Finite-difference solution for thin rod*

Node	*Exact*	*Finite difference*	*Error, exact-approx.*	*Error, % of u(0)*
1	0.6252	0.6289	−0.0037	−0.37
2	0.4102	0.4150	−0.0048	−0.48
3	0.2998	0.3049	−0.0051	−0.51
4	0.2658	0.2710	−0.0052	−0.52

As mentioned in Sec. 1·2·4, one of the important uses of the thin-rod temperature distribution is to estimate the heat loss from the rod. This may be done by summing up the convective energy losses from each chunk of fin. Recall that for the purposes of evaluating the convection in setting up the finite-difference equations it was assumed that the entire chunk of material could be considered as being at its central temperature. Thus the total

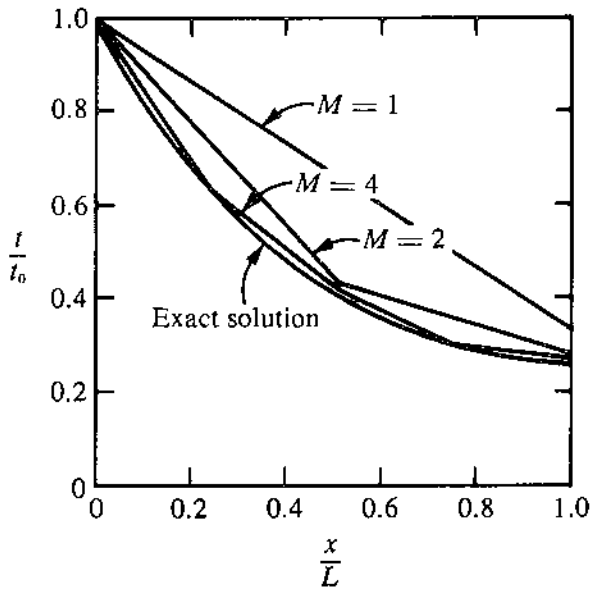


FIG. 8-8 *Effect of nodal-point spacing on finite-difference solution for a thin rod with* $mL = 2$.

heat transfer by convection in this example would be given by

$$q_0 = \sum_{m=0}^{4} q_{c_m}$$

$$= hp\frac{\Delta x}{2}t_0 + hp\,\Delta x\,t_1 + hp\,\Delta x\,t_2 + hp\,\Delta x\,t_3 + hp\frac{\Delta x}{2}t_4$$

In arriving at this equation we have taken $t_\infty = 0$ for simplicity. Upon normalizing to obtain the fin efficiency, and setting $\Delta x = L/4$,

$$\eta = \frac{q_0}{hpLt_0} = \tfrac{1}{4}(\tfrac{1}{2} + u_1 + u_2 + u_3 + \tfrac{1}{2}u_4)$$

Here we have defined $u = t/t_0$ and recognized that $u_0 = 1$. The result as obtained by finite differences is 0.4961 as compared to the exact result of 0.4820. The same result could have been obtained by writing an energy balance about the node at $x = 0$ as shown in Fig. 8-9. The energy balance is given by

$$q_0 = kA\frac{t_0 - t_1}{\Delta x} + hp\frac{\Delta x}{2}t_0$$

In normalized form, for $\Delta x = L/4$, this becomes

$$\eta = \frac{q_0}{hpLt_0} = \frac{4}{(mL)^2}(1 - u_1) + \frac{1}{8}$$

This gives the same numerical result (0.4961) for the finite-difference solution. These numbers show that reasonably accurate heat-transfer rates can be calculated with relatively few nodal

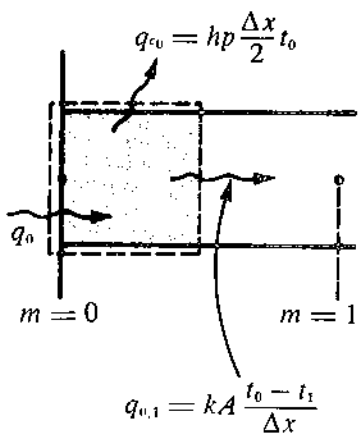


FIG. 8-9 *Nodal system and energy terms at root of thin rod.*

points. The adequacy of a given number of nodes, however, depends upon the values of the parameters involved in the problem (mL in this case). Figure 8·10 shows how the normalized heat-transfer rate predicted with $M = 1$, 2, and 4 compares with the exact value for various values of mL. Observe that the prediction gets worse as mL increases. This is because at larger mL the temperature profile is more curved, and four straight-line segments cannot do a very good job of approximating it. Also, notice that the prediction improves as the number of nodes increases for a given value of mL.

As a practical matter, the engineer is always faced with the problem of determining how many nodal points he should take to get a reasonable answer. The answer to this question is most often found by trial. You first work the problem using what you feel is a reasonable number of nodes based on the accuracy you desire and the amount of computing you want to do. Then you either double or halve the number of nodes, rework the problem, and compare your results. If the two answers seem to agree, you take the one with the greater number of nodes as being the best value and stop. If the answers are not acceptably close, increase the number of nodes until you get acceptable answers.

8·2·5 Matrix representation

Systems of linear algebraic equations are a common occurrence in finite-difference methods. They will arise in the next two sections where two-dimensional and transient problems are considered. It is very helpful to have a shorthand way to write systems of linear algebraic equations in order to simplify life. An introduction to matrices will be quite helpful to us later on.

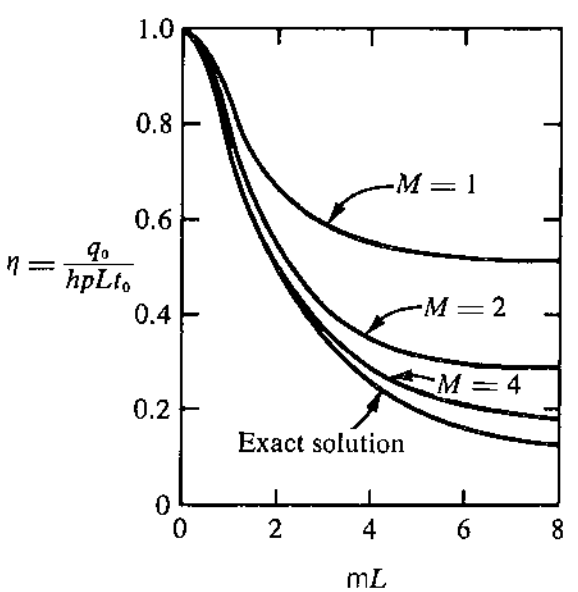


FIG. 8·10 *Finite-difference solutions for thin-rod fin efficiency.*

For simplicity, let us consider the following system of two algebraic equations for the two unknowns u_1 and u_2:

$$\begin{aligned} a_{11}u_1 + a_{12}u_2 &= c_1 \\ a_{21}u_1 + a_{22}u_2 &= c_2 \end{aligned} \tag{8·2·11}$$

The coefficients a_{11}, a_{12}, a_{21}, and a_{22} and the right-hand sides c_1 and c_2 are constants. The matrix representation for these equations is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{8·2·12}$$

The coefficients a_{ij} in the algebraic equations are collected together as the *components* in the *coefficient matrix*. The unknowns u_i are written in the *column matrix* alongside the coefficient matrix. The right-hand sides c_i are put in another column matrix on the other side of the equal sign. This matrix equation has the same meaning as the system of equations (8·2·11) and avoids the writing of the unknowns as many times. An even shorter notation is to let single symbols stand for each of the three matrices. Thus we might write Eq. (8·2·12) as

$$\mathbf{Au} = \mathbf{c} \tag{8·2·13}$$

where **A** stands for the square coefficient matrix, and **u** and **c** stand for the column matrices of u_i and c_i.

We will find it useful to be familiar with a few of the mathematical manipulations that can be done with matrices. This will enable us to handle systems of equations in a more efficient manner. Some of these operations are discussed in the remainder of this section.

The left-hand sides of Eqs. (8·2·12) and (8·2·13) can be thought of as matrix multiplication. The elements of the product matrix, the c_i in this case, are given by the following rule:

$$c_i = \sum_{j=1}^{2} a_{ij}u_j \tag{8·2·14}$$

As a numerical illustration of matrix multiplication, consider the following example:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\mathbf{Au} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1(2) + 2(3) \\ 3(2) + 4(3) \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix}$$

Square matrices (equal number of rows and columns) may also be multiplied together ($\mathbf{P} = \mathbf{AB}$) by the following rule:

$$p_{ik} = \sum_j a_{ij} b_{jk} \tag{8·2·15}$$

As a numerical illustration, consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix}$$

Then

$$\mathbf{P} = \mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-5) + 2(4) & 1(3) + 2(2) \\ 3(-5) + 4(4) & 3(3) + 4(2) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 7 \\ 1 & 17 \end{bmatrix}$$

A matrix with m rows and n columns is called an $m \times n$ *matrix*. Two matrices can be multiplied together only if the number of columns in the first matrix is equal to the number of rows in the second matrix. The resulting product is a matrix with the same number of rows as the first matrix and the same number of columns as the second matrix. That is,

$$\mathbf{A}_{m \times n} \mathbf{B}_{n \times p} = \mathbf{C}_{m \times p} \tag{8·2·16}$$

This rule can be extended to the product of three or more matrices as follows:

$$\mathbf{A}_{m \times n} \mathbf{B}_{n \times p} \mathbf{C}_{p \times q} = \mathbf{D}_{m \times q} \tag{8·2·17}$$

This triple product may be computed as either $(\mathbf{AB})\mathbf{C}$ or $\mathbf{A}(\mathbf{BC})$.

It is also important to know that, in general, $\mathbf{AB} \neq \mathbf{BA}$. This means that we must be careful to keep matrix multiplications in the correct order.

Matrices of equal size may be added together ($\mathbf{S} = \mathbf{A} + \mathbf{B}$) simply by adding the components according to the following rule:

$$s_{ij} = a_{ij} + b_{ij} \tag{8·2·18}$$

Thus

$$\mathbf{S} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 3 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-5 & 2+3 \\ 3+4 & 4+2 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ 7 & 6 \end{bmatrix}$$

It can also be shown that $\mathbf{AB} + \mathbf{AC} = \mathbf{A}(\mathbf{B} + \mathbf{C})$.

Matrices may be multiplied by a constant (often called a *scalar*) by simply multiplying each of the components by the constant. That is,

$$3\mathbf{A} = 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

A *diagonal matrix* is a matrix that has nonzero components only along the main diagonal. A special diagonal matrix $\mathbf{I}$ is the *identity matrix*. Its diagonal terms are all equal to 1. That is, for a 2×2 case,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The *transpose* of a matrix is obtained by interchanging rows and columns. That is, the rows in a matrix are the columns in the transpose of the matrix. Thus if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

its transpose $\mathbf{A}^{\mathrm{T}}$ is given as

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

A matrix is said to be *symmetric* if it is equal to its own transpose. This means that component a_{ij} must equal component a_{ji}. An example of a symmetric matrix would be

$$\begin{bmatrix} 1 & 3 & 0 \\ 3 & 4 & 5 \\ 0 & 5 & 2 \end{bmatrix}$$

The *inverse* of a matrix is another matrix such that, when a matrix and its inverse are multiplied together, their product is the identity matrix. That is, if the inverse of the matrix $\mathbf{A}$ is called $\mathbf{A}^{-1}$, then

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \quad \text{or} \quad \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{8·2·19}$$

For example, the inverse of the matrix $\mathbf{A}$ that we have been

using is

$$\mathbf{A}^{-1} = \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix}$$

This may be verified by multiplying $\mathbf{A}$ and $\mathbf{A}^{-1}$,

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The *determinant* of a matrix $\mathbf{A}$ is the determinant of the components of the matrix. For the matrix $\mathbf{A}$ we have been using,

$$\det \mathbf{A} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3) = -2$$

There is one last operation we should mention in regard to a matrix. The *eigenvalues* of a matrix $\mathbf{A}$ are the values of λ which satisfy the following equation:

$$\det (\mathbf{A} - \lambda \mathbf{I}) = 0 \qquad (8{\cdot}2{\cdot}20)$$

We have

$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix}$$

Then

$$\det (\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{vmatrix} = (1 - \lambda)(4 - \lambda) - 2(3) = 0$$

This is a quadratic equation for λ. Upon rearranging the terms,

$$\lambda^2 - 5\lambda - 2 = 0$$

The solutions to this equation are

$$\lambda_1 = \frac{5 + \sqrt{33}}{2} \quad \text{and} \quad \lambda_2 = \frac{5 - \sqrt{33}}{2}$$

The above concepts of matrix multiplication, addition, the

identity matrix, the inverse matrix, and matrix eigenvalues have all been illustrated with 2 × 2 examples. These ideas are all valid for $n \times n$ matrices. The computations are more involved, however. The remainder of this chapter will begin to use matrix representation to simplify the presentation of systems of equations. You will not need to know any more about matrices, however, than we have illustrated in this section.

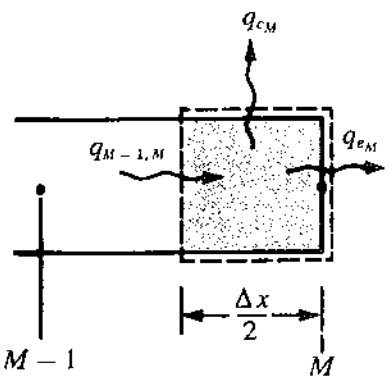


FIG. 8·11 *Nodal system and energy terms at tip of thin rod with end convection included.*

8·2·6 Convection boundary condition

The adiabatic boundary condition at the tip of the rod may be replaced by a convective boundary condition with no additional complications in the finite-difference formulation. The only change is that the equation for the node at $x = L$ must be modified. The pertinent energy terms are shown in Fig. 8·11. The energy balance may be written as

$$q_{M-1,M} = q_{c_M} + q_{e_M}$$

The rate equations may be approximated by

$$q_{M-1,M} = kA\frac{t_{M-1} - t_M}{\Delta x}$$

$$q_{c_M} = hp\frac{\Delta x}{2}t_M$$

$$q_{e_M} = h_e A t_M$$

where we have taken $t_\infty = 0$ in the two convective terms. We have also allowed the end heat-transfer coefficient h_e to be different from the one on the periphery. These equations may be combined with the energy balance in the usual way to give (after some rearrangement)

$$-2t_{M-1} + \left(D + \frac{2h_e\,\Delta x}{k}\right)t_M = 0 \tag{8·2·21}$$

This equation replaces Eq. (8·2·3), which was found for the adiabatic-end case.

The normalized version of the set of equations to be solved in place of Eqs. (8·2·8) may be written in matrix form as

$$\begin{bmatrix} D & -1 & & \\ -1 & D & -1 & \\ & -1 & D & -1 \\ & & -2 & D + \dfrac{H}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{8·2·22}$$

In the above we have defined $H = h_e L/k$ and set $\Delta x = L/4$ in the term involving h_e.

Mathematically we have the same type of problem to solve. The only difference is that the numerical value of one of the components of the coefficient matrix has been changed. The same computational procedures will work.

8·3 TWO-DIMENSIONAL, STEADY-STATE PROBLEMS

The solution to two-dimensional, steady-state conduction problems by finite-difference methods is simply an extension of the techniques discussed in Sec. 8·2. In fact, this section should almost seem like a review of Sec. 8·2. You should occasionally refer back to Sec. 8·2 to note the striking similarities.

As an example, let us consider the steady-state conduction of heat in a square region with uniform energy generation as described in Fig. 8·12. It is assumed that there is no temperature variation in the z direction. The temperature distribution $t(x,y)$ is the desired result.

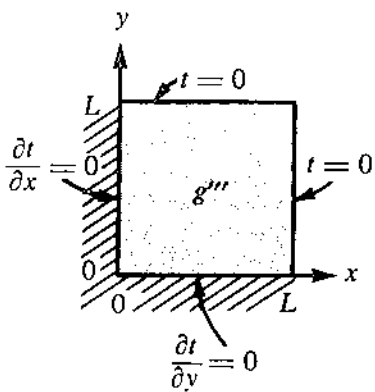


FIG. 8·12 *Uniform energy generation in a square region.*

8·3·1 Physical formulation

The first step in the physical formulation of the problem is to subdivide the region into a finite number of small sections by setting up a system of nodes as shown in Fig. 8·13.

An expanded picture around a typical interior node located at the point (x_m, y_n) is shown in Fig. 8·14, with the "call letters" of the nodes shown in parentheses.[1]

[1] The index m denotes the x position, and the index n denotes the y position.

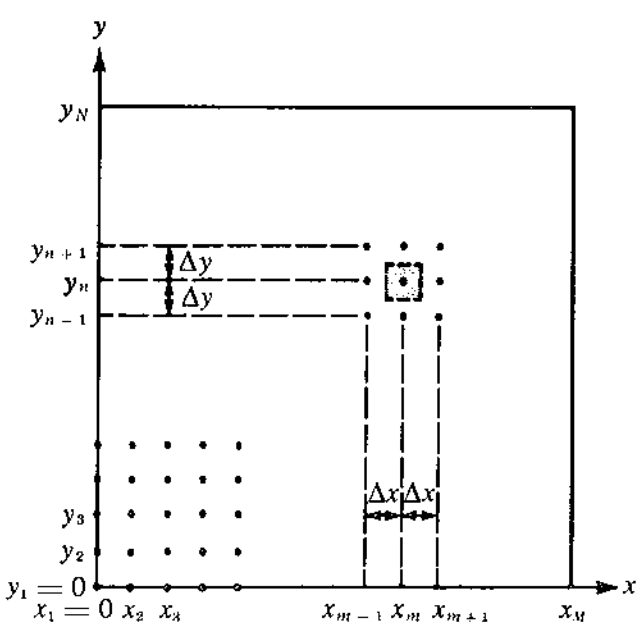


FIG. 8·13 *Nodal-point arrangement for conduction in a square region.*

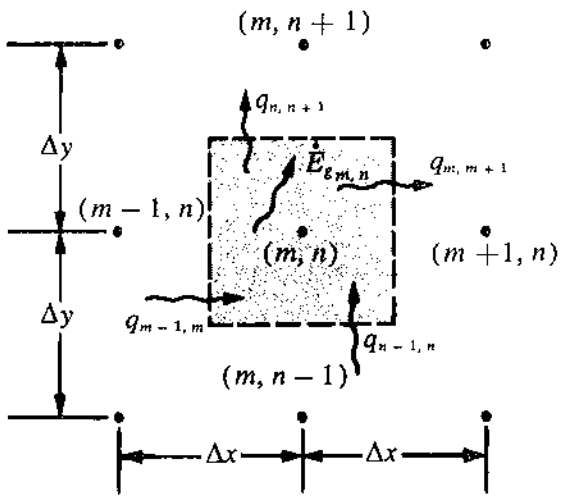


FIG. 8·14 *Typical nodal system with important energy terms indicated.*

Next, as in previous analytical solutions, we will define a system and show the pertinent energy terms as pictured in Fig. 8·14. The subscript notation for the heat-flow terms should be familiar from the discussion in Sec. 8·2·1. Observe that conduction through the four sides of the system and generation are the only energy transfers being considered.

The energy balance may be simply written as

$$q_{m-1,m} + q_{n-1,n} + \dot{E}_{g_{m,n}} = q_{m,m+1} + q_{n,n+1}$$

The rate equations can be approximated by

$$q_{m-1,m} = k\,\Delta y \frac{t_{m-1,n} - t_{m,n}}{\Delta x}$$

$$q_{n-1,n} = k\,\Delta x \frac{t_{m,n-1} - t_{m,n}}{\Delta y}$$

$$q_{m,m+1} = k\,\Delta y \frac{t_{m,n} - t_{m+1,n}}{\Delta x}$$

$$q_{n,n+1} = k\,\Delta x \frac{t_{m,n} - t_{m,n+1}}{\Delta y}$$

The generation term may be written as

$$\dot{E}_{g_{m,n}} = g'''\,\Delta x\,\Delta y$$

Substituting these rate equations into the energy balance gives

$$\frac{k\,\Delta y}{\Delta x}(t_{m-1,n} - t_{m,n}) + \frac{k\,\Delta x}{\Delta y}(t_{m,n-1} - t_{m,n}) + g'''\,\Delta x\,\Delta y$$

$$= \frac{k\,\Delta y}{\Delta x}(t_{m,n} - t_{m+1,n}) + \frac{k\,\Delta x}{\Delta y}(t_{m,n} - t_{m,n+1})$$

If Δx is taken to be equal to Δy, the above can be rearranged to give

$$-t_{m-1,n} - t_{m,n-1} + 4t_{m,n} - t_{m,n+1} - t_{m+1,n} = \frac{g'''(\Delta x)^2}{k} \tag{8·3·1}$$

Equation (8·3·1) contains five unknown temperatures. Such an equation can be written for each node whose temperature is unknown to give a set of equations equal to the number of unknowns.

The nodes along the adiabatic boundaries must be given special attention. Consider a typical node along the $x = 0$ boundary as shown in Fig. 8·15. The energy balance may be written as

$$q_{n-1,n} + \dot{E}_{g_{1,n}} = q_{1,2} + q_{n,n+1}$$

The rate equations may be approximated in the following way:

$$q_{n-1,n} = k\frac{\Delta x}{2}\frac{t_{1,n-1} - t_{1,n}}{\Delta y}$$

$$q_{1,2} = k\,\Delta y\frac{t_{1,n} - t_{2,n}}{\Delta x}$$

$$q_{n,n+1} = k\frac{\Delta x}{2}\frac{t_{1,n} - t_{1,n+1}}{\Delta y}$$

$$\dot{E}_{g_{1,n}} = g'''\frac{\Delta x\,\Delta y}{2}$$

Upon combining the energy balance and rate equations, setting $\Delta x = \Delta y$, and rearranging, the following equation is arrived at:

$$-t_{1,n-1} + 4t_{1,n} - t_{1,n+1} - 2t_{2,n} = \frac{g'''(\Delta x)^2}{k} \tag{8·3·2}$$

This equation could also have been found from Eq. (8·3·1) by using symmetry and setting $t_{m-1,n} = t_{m+1,n}$ and then setting $m = 1$.

In a similar fashion the equations along $y = 0$ are of the form

$$-t_{m-1,1} + 4t_{m,1} - 2t_{m,2} - t_{m+1,1} = \frac{g'''(\Delta x)^2}{k} \tag{8·3·3}$$

At the point $(x,y) = (0,0)$ the equation is given by

$$4t_{1,1} - 2t_{1,2} - 2t_{2,1} = \frac{g'''(\Delta x)^2}{k} \tag{8·3·4}$$

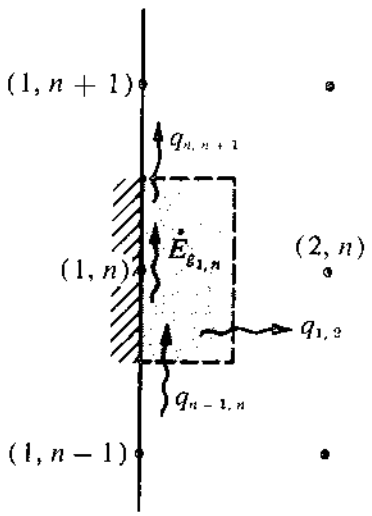


FIG. 8·15 *Nodal system along $x = 0$ with important energy terms indicated.*

In this particular example the temperature at each of the nodes on the other boundaries is known so it is not necessary to write special equations at the boundary.[1] Equations (8·3·1) to (8·3·4) represent the system of algebraic equations that must be solved for the unknown nodal temperatures.

8·3·2 Mathematical formulation

The mathematical formulation of the problem by finite-difference methods begins with the governing partial differential equation and boundary conditions. The equation is the following:

$$u_{xx} + u_{yy} + 1 = 0 \tag{8·3·5}$$

The boundary conditions are that

$$u_x(0,y) = 0 \qquad u_y(x,0) = 0$$

$$u(1,y) = 0 \qquad u(x,1) = 0$$

In this normalized problem, $u = t/(g'''L^2/k)$ and x and y are really x/L and y/L, respectively.

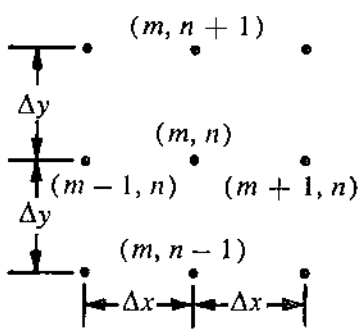


FIG. 8·16 *Typical two-dimensional nodal points.*

The first step is to set up a system of nodal points in the region of interest just as was done in the physical formulation of this problem. A typical interior set of nodal points is shown in Fig. 8·16.

Based upon the discussion in Sec. 8·1, the two second derivatives may be approximated in the following way:

$$u_{xx} = \frac{u_{m-1,n} - 2u_{m,n} + u_{m+1,n}}{(\Delta x)^2}$$

$$u_{yy} = \frac{u_{m,n-1} - 2u_{m,n} + u_{m,n+1}}{(\Delta y)^2}$$

These approximations may now be substituted into Eq. (8·3·5) to give

$$\frac{u_{m-1,n} - 2u_{m,n} + u_{m+1,n}}{(\Delta x)^2} + \frac{u_{m,n-1} - 2u_{m,n} + u_{m,n+1}}{(\Delta y)^2} + 1 = 0$$

If Δx is taken to be equal to Δy, this equation may be rearranged to give

$$-u_{m-1,n} - u_{m,n-1} + 4u_{m,n} - u_{m,n+1} - u_{m+1,n} = (\Delta x)^2 \tag{8·3·6}$$

This result is seen to be the normalized equivalent to Eq. (8·3·1).

An equation corresponding to Eq. (8·3·2) may be found by setting $u_{m-1,n} = u_{m+1,n}$ in Eq. (8·3·6) and then setting $m = 1$.

[1] Two other types of boundary conditions are discussed later.

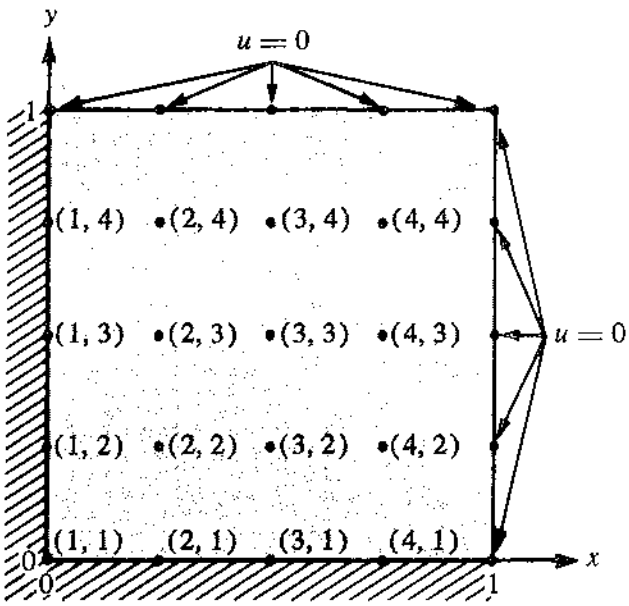


FIG. 8·17 *Nodal-point arrangement for square generation.*

Equations corresponding to Eqs. (8·3·3) and (8·3·4) may also be easily obtained.

Just as in Sec. 8·2, the finite-difference scheme has resulted in a set of simultaneous, linear, algebraic equations to solve. As an illustration of what is involved, let us consider an example in which $\Delta x = \frac{1}{4}$, just as we did in the one-dimensional case. This nodal-point arrangement is shown in Fig. 8·17. Observe that the nodal points (1,5), . . . , (5,1) which are not labeled are all at zero temperature as given by the boundary conditions of the problem. Thus there are 16 unknown temperatures to find.

Since there are 16 unknown temperatures, there will be 16 equations to solve. Using the matrix representation for these equations, the following is obtained from the normalized form of Eqs. (8·3·1) to (8·3·4):

$$\begin{bmatrix}
4 & -2 & 0 & 0 & -2 & & & & & & & & & & & \\
-1 & 4 & -1 & 0 & 0 & -2 & & & & & & & & & & \\
0 & -1 & 4 & -1 & 0 & 0 & -2 & & & & & & & & & \\
0 & 0 & -1 & 4 & 0 & 0 & 0 & -2 & & & & & & & & \\
-1 & 0 & 0 & 0 & 4 & -2 & 0 & 0 & -1 & & & & & & & \\
 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & & & & & & \\
 & & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & & & & & \\
 & & & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & & & & \\
 & & & & -1 & 0 & 0 & 0 & 4 & -2 & 0 & 0 & -1 & & & \\
 & & & & & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & & \\
 & & & & & & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & \\
 & & & & & & & -1 & 0 & 0 & -1 & 4 & -2 & 0 & 0 & \\
 & & & & & & & & -1 & 0 & 0 & -1 & 4 & -1 & 0 & \\
 & & & & & & & & & -1 & 0 & 0 & -1 & 4 & -1 & \\
 & & & & & & & & & & -1 & 0 & 0 & -1 & 4 &
\end{bmatrix}
\begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{1,4} \\ u_{2,1} \\ u_{2,2} \\ u_{2,3} \\ u_{2,4} \\ u_{3,1} \\ u_{3,2} \\ u_{3,3} \\ u_{3,4} \\ u_{4,1} \\ u_{4,2} \\ u_{4,3} \\ u_{4,4} \end{bmatrix}
=
\begin{bmatrix} \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \end{bmatrix} \tag{8·3·7}$$

There are several things that should be pointed out about these equations before discussing methods of solution. First, notice that by dividing the x direction into only 4 nodes (also, the y direction has 4 nodes) there are $4 \times 4 = 16$ simultaneous equations to solve. In other words, relatively coarse subdivisions lead to many equations. In a practical situation the number of equations can be in the hundreds or more. Second, observe that the matrix is *banded*. That is, the nonzero components only appear in a band on either side of the main diagonal. The practical significance of this is that special subroutines can be written to solve this problem in less computer time than if the matrix was filled with nonzero components. Another feature is that the zero matrix components outside the band do not need to be stored in the computer. This is of considerable importance in large problems where computer memory size becomes a limiting factor.

Two comments should be made regarding the nodal numbering system. First, the double subscripting of the nodal temperatures is not necessary. Single subscripts, 1 through 16 in this example, would be much simpler. Second, the nodal numbering system can affect the bandwidth of the system of equations. For example, if nodes numbered 1 and 16 were interchanged, the upper bandwidth would increase from 5 to 15. It is therefore advantageous to number the nodal points so as to make the bandwidth as small as possible.

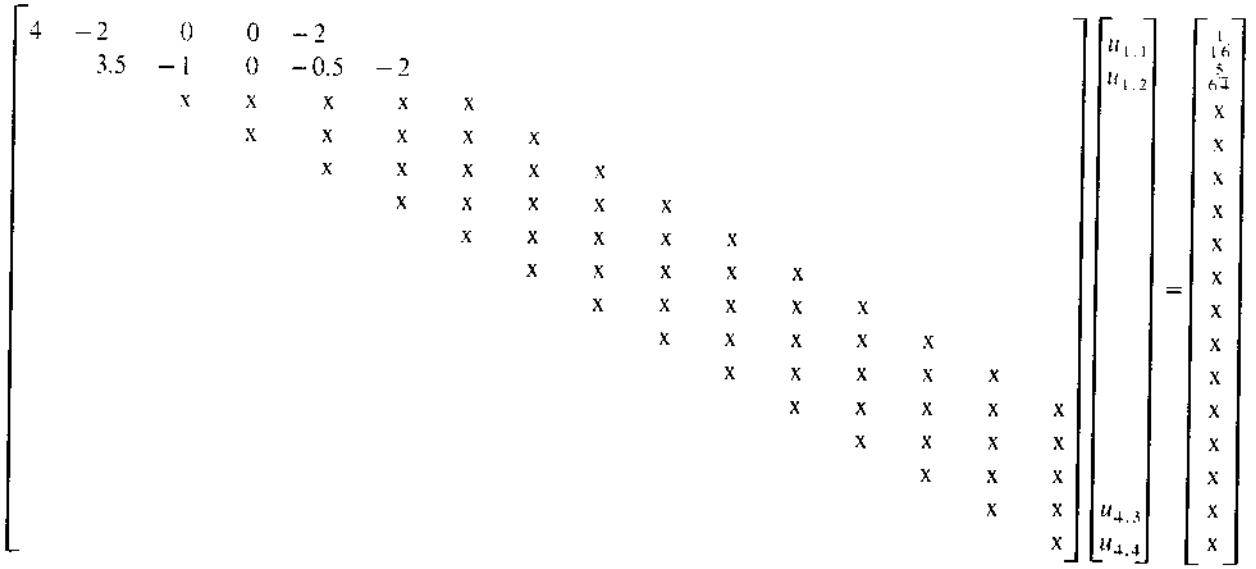
Two ways that can be used to solve these equations on the computer are gaussian elimination and iteration.

8·3·3 Solution by gaussian elimination

The elimination procedure is basically the same as that discussed for tridiagonal matrices in Sec. 8·2·3. The first equation can be used to eliminate all the nonzero elements in the first column. In this example the first equation can be multiplied by $\frac{1}{4}$ and added to the second and fifth equations (rows) to give

$$\begin{bmatrix} 4 & -2 & 0 & 0 & -2 & & & & & \\ 0 & 3.5 & -1 & 0 & -0.5 & -2 & & & & \\ 0 & -1 & 4 & -1 & 0 & 0 & -2 & & & \\ 0 & 0 & -1 & 4 & 0 & 0 & 0 & -2 & & \\ 0 & -0.5 & 0 & 0 & 3.5 & -2 & 0 & 0 & -1 & \\ & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{1,4} \\ u_{1,5} \\ u_{1,6} \\ \cdots \end{bmatrix} = \begin{bmatrix} \frac{1}{16} \\ \frac{5}{64} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{5}{64} \\ \frac{1}{16} \\ \cdots \end{bmatrix}$$

Observe that all the components in the first column are now zero except for the one in the first row. The second equation (row) can now be used to eliminate all the nonzero components below the second row in the second column. This process is continued until an *upper triangular, band matrix* has been formed as indicated below.



Now we are ready to solve the last equation for $u_{4,4}$, substitute it into the next-to-last equation to find $u_{4,3}$, and continue to work our way back up to the first equation where $u_{1,1}$ is found. This completes the solution of the problem.

It is interesting to observe that the original banded matrix in this example required computer storage space for 124 components within the band rather than the 256 spaces that would have been required to store the entire matrix. This can be of considerable importance in large problems where computer storage is critical. Of course in this physically symmetrical problem we could save work by recognizing that $u_{i,j} = u_{j,i}$. This would reduce the number of equations from 16 to 10.

8·3·4 Iterative methods of solution

There is another feature of the original band matrix, given in Eq. (8·3·7), that should be observed. Notice that many of the components within the band itself are zero. In this example, 60 of the 124 band components are zero. These components must still be stored, however, if gaussian elimination is to be used to solve the problem, because during the elimination process they will, in general, change to nonzero values. If computer storage is critical, one might like to use a method which does not require

storage of these zero components in the band. The Gauss-Seidel iteration procedure is one method of doing this.

In this procedure each of the rows in Eq. (8·3·7) would first be solved for the main diagonal unknown. Thus in this example we would have

$$\begin{aligned}
u_{1,1} &= \tfrac{1}{4}(\tfrac{1}{16} + 2u_{1,2} + 2u_{2,1}) \\
u_{1,2} &= \tfrac{1}{4}(\tfrac{1}{16} + u_{1,1} + u_{1,3} + 2u_{2,2}) \\
u_{1,3} &= \tfrac{1}{4}(\tfrac{1}{16} + u_{1,2} + u_{1,4} + 2u_{2,3}) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u_{4,3} &= \tfrac{1}{4}(\tfrac{1}{16} + u_{3,3} + u_{4,2} + u_{4,4}) \\
u_{4,4} &= \tfrac{1}{4}(\tfrac{1}{16} + u_{3,4} + u_{4,3})
\end{aligned} \tag{8·3·8}$$

The computational procedure begins with initial guesses $u_{m,n}^{(0)}$ for all the unknowns. An improved value for each of the unknowns $u_{m,n}^{(1)}$ is then computed from Eqs. (8·3·8) in the following way:

$$\begin{aligned}
u_{1,1}^{(1)} &= \tfrac{1}{4}[\tfrac{1}{16} + 2u_{1,2}^{(0)} + 2u_{2,1}^{(0)}] \\
u_{1,2}^{(1)} &= \tfrac{1}{4}[\tfrac{1}{16} + u_{1,1}^{(1)} + u_{1,3}^{(0)} + 2u_{2,2}^{(0)}] \\
u_{1,3}^{(1)} &= \tfrac{1}{4}[\tfrac{1}{16} + u_{1,2}^{(1)} + u_{1,4}^{(0)} + 2u_{2,3}^{(0)}] \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u_{4,3}^{(1)} &= \tfrac{1}{4}[\tfrac{1}{16} + u_{3,3}^{(1)} + u_{4,2}^{(1)} + u_{4,4}^{(0)}] \\
u_{4,4}^{(1)} &= \tfrac{1}{4}[\tfrac{1}{16} + u_{3,4}^{(1)} + u_{4,3}^{(1)}]
\end{aligned} \tag{8·3·9}$$

Notice that in the computation of the new values $u_{m,n}^{(1)}$ the latest values of the terms on the right-hand side are used.

One initial guess that could be used would be to take every $u_{m,n}^{(0)} = 0$ since this is the value of the temperature along two of the boundaries. The solutions for $u_{m,n}^{(1)}$ would then be obtained as indicated below.

$$\begin{aligned}
u_{1,1}^{(1)} &= \tfrac{1}{4}[\tfrac{1}{16} + 2(0) + 2(0)] = 0.0156 \\
u_{1,2}^{(1)} &= \tfrac{1}{4}[\tfrac{1}{16} + 0.0156 + 0 + 2(0)] = 0.0195 \\
u_{1,3}^{(1)} &= \tfrac{1}{4}[\tfrac{1}{16} + 0.0195 + 0 + 2(0)] = 0.0205 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u_{4,3}^{(1)} &= \tfrac{1}{4}[\tfrac{1}{16} + 0.0292 + 0.0276 + 0] = 0.0298 \\
u_{4,4}^{(1)} &= \tfrac{1}{4}[\tfrac{1}{16} + 0.0298 + 0.0298] = 0.0305
\end{aligned}$$

This process can be carried out again and again (*iteration*) until it converges. That is, you would continue until $u_{m,n}^{(i+1)}$ is as close as you want to $u_{m,n}^{(i)}$.

One of the problems with the Gauss-Seidel method of solving the equations is that it is relatively slow to converge to the solution. In this example, $u_{1,1}$ requires 42 iterations before it converges to three significant figures.

A method that can be used to accelerate convergence of the iteration process is *successive overrelaxation*. In this technique, Eqs. (8·3·9) would be replaced by the following set of equations:

$$u_{1,1}^{(1)} = \tfrac{1}{4}\left[\tfrac{1}{16} + 2u_{1,2}^{(0)} + 2u_{2,1}^{(0)}\right]\omega + (1 - \omega)u_{1,1}^{(0)}$$

$$u_{1,2}^{(1)} = \tfrac{1}{4}\left[\tfrac{1}{16} + u_{1,1}^{(1)} + u_{1,3}^{(0)} + 2u_{2,2}^{(0)}\right]\omega + (1 - \omega)u_{1,2}^{(0)}$$

$$u_{1,3}^{(1)} = \tfrac{1}{4}\left[\tfrac{1}{16} + u_{1,2}^{(1)} + u_{1,4}^{(0)} + 2u_{2,3}^{(0)}\right]\omega + (1 - \omega)u_{1,3}^{(0)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (8{\cdot}3{\cdot}10)$$

$$u_{4,3}^{(1)} = \tfrac{1}{4}\left[\tfrac{1}{16} + u_{3,3}^{(1)} + u_{4,2}^{(1)} + u_{4,4}^{(0)}\right]\omega + (1 - \omega)u_{4,3}^{(0)}$$

$$u_{4,4}^{(1)} = \tfrac{1}{4}\left[\tfrac{1}{16} + u_{3,4}^{(1)} + u_{4,3}^{(1)}\right]\omega + (1 - \omega)u_{4,4}^{(0)}$$

The *relaxation factor* ω may be thought of as a weighting factor. For $\omega = 0$ the new value of u would be identical to the old value, and hence no progress would be made. For $\omega = 1$ the new value of u would be the same as calculated in the Gauss-Seidel procedure. Values of ω between 0 and 1 would represent an *underrelaxation* or an interpolation between the old value of u and the Gauss-Seidel value. Values of ω greater than unity would represent an *overrelaxation* or an extrapolation beyond the Gauss-Seidel value for u. It is this extrapolation process that can accelerate the convergence of the iterative process. The best value of ω will be the one that gives the maximum rate of convergence. This optimum value will depend upon the particular problem being considered and is determined largely by experience. Since the iterative process will not converge for values of ω greater than 2, the value of ω will be between 1 and 2. In this particular example, a value of $\omega = 1.4$ will give convergence of $u_{1,1}$ to three significant figures after 16 iterations instead of the 42 iterations required for the Gauss-Seidel method. This value of ω is close to the optimum for this example.

8·3·5 Numerical results

The solution to this example by gaussian elimination is compared to the exact analytical solution (see Exercise 3·38) in

Table 8·2. These results show that the finite-difference solution falls slightly below the exact analytical solution.

TABLE 8·2 *Finite-difference solution for square generation*

Node	*Exact*	*Finite difference*	*Error, exact-approx.*	*Error, % of u(0,0)*
1,1	0.2947	0.2911	0.0036	1.22
1,2	0.2789	0.2755	0.0034	1.15
1,3	0.2293	0.2266	0.0027	0.92
1,4	0.1397	0.1381	0.0016	0.54
2,1	0.2789	0.2755	0.0034	1.15
2,2	0.2642	0.2609	0.0033	1.12
2,3	0.2178	0.2151	0.0027	0.92
2,4	0.1333	0.1317	0.0016	0.54
3,1	0.2293	0.2266	0.0027	0.92
3,2	0.2178	0.2151	0.0027	0.92
3,3	0.1811	0.1787	0.0024	0.81
3,4	0.1127	0.1110	0.0017	0.58
4,1	0.1397	0.1381	0.0016	0.54
4,2	0.1333	0.1317	0.0016	0.54
4,3	0.1127	0.1110	0.0017	0.58
4,4	0.0728	0.0711	0.0017	0.58

Figure 8·18 shows how the finite-difference solution progressively approaches the exact solution as the number of nodal points is increased. The number of nodal points could be in-

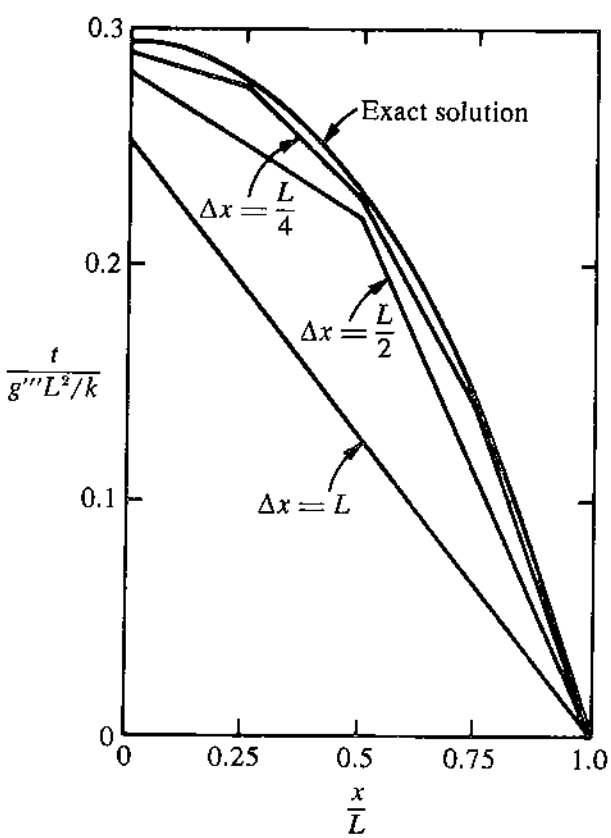


FIG. 8·18 *Effect of nodal-point spacing on finite-difference solution for square generation, $y/L = 0.0$.*

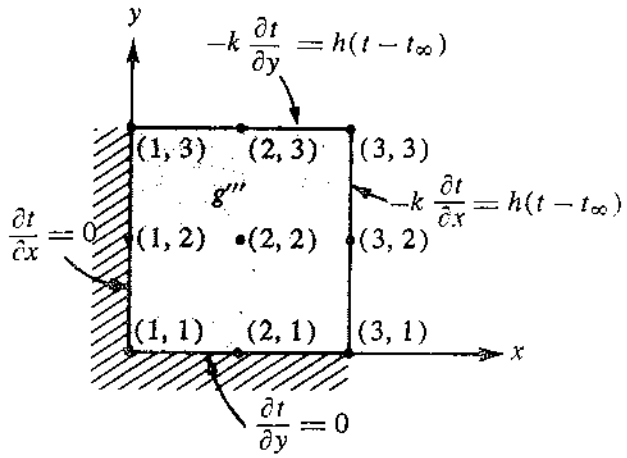


FIG. 8·19 *Nodal-point arrangement with convection boundary condition.*

creased even more for greater precision. Indeed, the most common method of determining the accuracy of a finite-difference solution is to experiment with the number of nodal points.

Problems in which the region of interest is irregular require more work, but the underlying features are the same as those we have discussed in this section. The modifications which must be made to handle the more complicated regions are discussed later in Sec. 8·5.

8·3·6 Convection boundary condition

Convection boundary conditions are easily handled by the finite-difference method with no additional difficulty. As an example, let us consider our generation problem again but with convective boundary conditions replacing the specified temperature boundary conditions. This new problem is shown in Fig. 8·19. For simplicity let us take only four interior nodes. These can be handled just as before.

Since the surface temperatures are not now specified, we will need to find an additional equation for each of them. This is easily accomplished by writing an energy balance on each node. Figure 8·20 shows the energy terms associated with node (3,2). The energy balance for the system shown is

$$q_a + q_b + \dot{E}_g = q_c + q_{\text{conv}}$$

The rate equations may be written as

$$q_a = k\frac{\Delta x}{2}\frac{t_{3,1} - t_{3,2}}{\Delta y}$$

$$q_b = k\,\Delta y\,\frac{t_{2,2} - t_{3,2}}{\Delta x}$$

$$\dot{E}_g = g'''\frac{\Delta x}{2}\Delta y$$

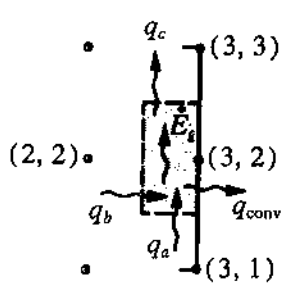


FIG. 8·20 *Nodal system along x = L with surface convection*

$$q_c = k\frac{\Delta x}{2}\frac{t_{3,2} - t_{3,3}}{\Delta y}$$

and

$$q_{\text{conv}} = h\,\Delta y(t_{3,2} - t_\infty)$$

Following the usual procedure, the rate equations may be substituted into the energy balance to arrive at an algebraic equation involving the four nodal temperatures. For $\Delta x = \Delta y$, the final result is given by

$$-2t_{2,2} - t_{3,1} + \left(4 + \frac{2h\,\Delta x}{k}\right)t_{3,2} - t_{3,3} = \frac{g'''(\Delta x)^2}{k} + \frac{2h\,\Delta x}{k}t_\infty$$

To further simplify this equation we can define $T = t - t_\infty$. This reduces the equation to

$$-2T_{2,2} - T_{3,1} + \left(4 + \frac{2h\,\Delta x}{k}\right)T_{3,2} - T_{3,3} = \frac{g'''(\Delta x)^2}{k}$$

Next, we can normalize the equation by defining $u = T/(g'''L^2/k)$. This gives

$$-2u_{2,2} - u_{3,1} + (4 + 2H\,\Delta x)u_{3,2} - u_{3,3} = (\Delta x)^2 \qquad (8\cdot3\cdot11)$$

where Δx really means $\Delta(x/L)$ and $H = hL/k$.

Similar equations may be found for each of the surface nodes. These five surface equations along with the four interior equations, represented by Eq. (8·3·6), result in the system of nine equations shown below in Eq. (8·3·12).

$$\begin{bmatrix}
4 & -2 & 0 & -2 & & & & & \\
-1 & 4 & -1 & 0 & -2 & & & & \\
0 & -2 & 4+H & 0 & 0 & -2 & & & \\
-1 & 0 & 0 & 4 & -2 & 0 & -1 & & \\
 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & \\
 & & -1 & 0 & -2 & 4+H & 0 & 0 & -1 \\
 & & & -2 & 0 & 0 & 4+H & -2 & 0 \\
 & & & & -2 & 0 & -1 & 4+H & -1 \\
 & & & & & -2 & 0 & -2 & 4+2H
\end{bmatrix}
\begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{2,1} \\ u_{2,2} \\ u_{2,3} \\ u_{3,1} \\ u_{3,2} \\ u_{3,3} \end{bmatrix}
=
\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \qquad (8\cdot3\cdot12)$$

In this equation $\Delta x = \frac{1}{2}$ has been used.

This set of equations may be solved by exactly the same techniques that were discussed earlier in this section. Observe that we again have a banded matrix just as was obtained in Eq. (8·3·7).

The only additional difficulty is that we now have nine

equations, whereas for the same nodal spacing there would be only four equations if the surface temperatures had been specified. This approximately doubles the amount of computations that must be done in this particular problem. In more practical problems, however, the ratio of surface nodes to interior nodes will be much smaller. For example, if we had used $\Delta x = \frac{1}{10}$ in this problem, there would have been 81 interior nodes and only 19 additional surface nodes that must be considered because of the convection boundary condition.

8·4 ONE-DIMENSIONAL, TRANSIENT PROBLEMS

The solution of the heat equation as discussed in this section is done in two steps. The first step is to obtain a system of ordinary differential equations to approximate the behavior of the heat equation. Again, this may be done by either a physical formulation or a mathematical formulation. The second step is to obtain a numerical solution to this system of ordinary differential equations.

There is some parallelism to the steady-state problem discussed in Sec. 8·3 that should be mentioned here. In the steady-state problem, *finite differencing* the spatial behavior (physical) or the second derivatives (mathematical) gave rise to a system of *algebraic* equations. In the transient problem we are about to consider, *finite differencing* the spatial behavior or the second derivatives in the heat equation will give rise to a system of *ordinary differential* equations. In the steady-state case we had to learn how to solve systems of algebraic equations. Now we will have to learn how to solve systems of ordinary differential equations. In the steady-state case we were able to obtain the exact solution to the algebraic equations. In the transient case it is also possible to obtain the exact solution to the system of differential equations. This is only practical for systems with relatively few equations, however. For the more typical case in which there are many equations, we will have to resort to approximate numerical techniques.

As a specific example, the following sections will discuss the solution for a plane wall, insulated on one face, that is subjected to a step change in surface temperature on the other face. Figure 8·21 describes the problem.

8·4·1 Physical formulation

The first step in the finite-difference formulation is to subdivide the x direction into equally spaced nodes, Δx apart, as shown

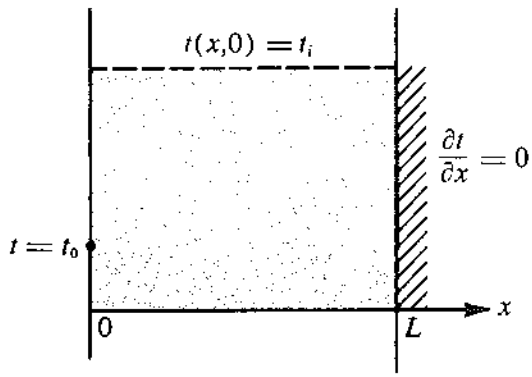


FIG. 8·21 *Plane-wall transient.*

in Fig. 8·22. As an approximation we will let the temperature of each node represent the temperature of a thin plane wall Δx thick surrounding the node. A typical set of three consecutive nodes is labeled in Fig. 8·22.

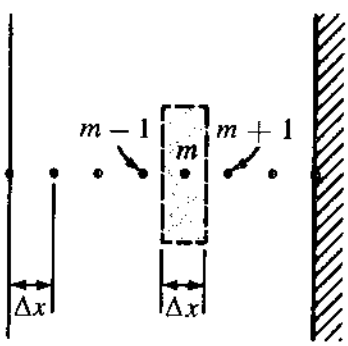


FIG. 8·22 *Nodal-point arrangement for plane-wall transient.*

An energy balance can now be written for the thin wall about node m. The system shown in Fig. 8·22 is elaborated in Fig. 8·23 to show the pertinent energy terms. The energy balance says that

$$q_{m-1,m} = q_{m,m+1} + \dot{E}_{s_m} \tag{8·4·1}$$

Next we must use the rate equations. These may be approximately stated as

$$q_{m-1,m} = kA\,\frac{t_{m-1} - t_m}{\Delta x} \tag{8·4·2}$$

$$q_{m,m+1} = kA\,\frac{t_m - t_{m+1}}{\Delta x} \tag{8·4·3}$$

and

$$\dot{E}_{s_m} = \rho cA\,\Delta x\,\frac{dt_m}{d\theta} \tag{8·4·4}$$

Substituting the rate equations into the energy balance gives

$$\frac{kA}{\Delta x}(t_{m-1} - t_m) = \frac{kA}{\Delta x}(t_m - t_{m+1}) + \rho cA\,\Delta x\,\frac{dt_m}{d\theta} \tag{8·4·5}$$

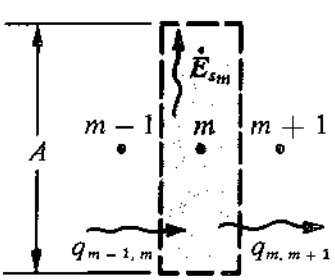


FIG. 8·23 *Typical nodal system with important energy terms indicated.*

This may be rearranged to give

$$\rho cA\,\Delta x\,\frac{dt_m}{d\theta} = \frac{kA}{\Delta x}(t_{m-1} - 2t_m + t_{m+1}) \tag{8·4·6}$$

A similar equation can be written for each of the interior nodes.

The node M at the insulated surface must be treated separately. Since there is no energy conducted out the insulated face, the energy balance is simply

$$q_{M-1,M} = \dot{E}_{s_M}$$

The rate equations are

$$q_{M-1,M} = kA \frac{t_{M-1} - t_M}{\Delta x}$$

$$\dot{E}_{s_M} = \rho c A \frac{\Delta x}{2} \frac{dt_M}{d\theta}$$

The resulting ordinary differential equation is found to be

$$\frac{\rho c A \, \Delta x}{2} \frac{dt_M}{d\theta} = \frac{kA}{\Delta x}(t_{M-1} - t_M) \tag{8·4·7}$$

The set of equations represented by Eqs. (8·4·6) and (8·4·7) must be satisfied for all values of time beginning with the initial condition. The problem has now been reduced from the solution of a single partial differential equation to the simultaneous solution of a set of ordinary differential equations. The solution of these equations will be discussed after we first see how to formulate the problem from a mathematical point of view in the next section.

8·4·2 Mathematical formulation

The mathematical formulation of the finite-difference method begins with the partial differential equation plus its boundary and initial conditions. For this problem they may be written in normalized form as

$$u_{xx} = u_\theta$$

$$u(0,\theta) = 0$$

$$u_x(1,\theta) = 0$$

$$u(x,0) = 1$$

In these equations, $u = (t - t_0)/(t_i - t_0)$ and x and θ are really x/L and $\alpha\theta/L^2$, respectively.

A nodal mesh may be set up just as was done in the physical formulation of this problem. Following the discussion in Sec. 8·1, the second derivative in the differential equation may be replaced by its finite-difference approximation to give

$$\frac{u_{m-1} - 2u_m + u_{m+1}}{(\Delta x)^2} = \frac{du_m}{d\theta} \tag{8·4·8}$$

This equation is seen to be the nondimensional equivalent of Eq. (8·4·6). It is applicable to all the interior nodal points.

The equation for the node at $x = 1$ may be obtained from Eq. (8·4·8) by recognizing the symmetry involved. Replacing the

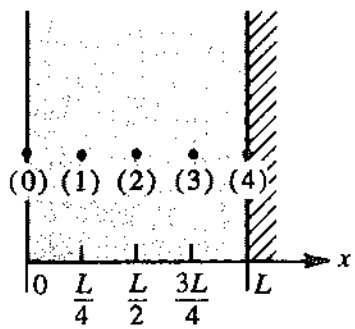


FIG. 8·24 *Nodal-point arrangement for* $\Delta x = L/4$.

adiabatic condition with another wall that is the mirror image of our problem does not change the problem. Thus u_{M+1} may be set equal to u_{M-1} in Eq. (8·4·8) to give the equation for node M as

$$\frac{2u_{M-1} - 2u_M}{(\Delta x)^2} = \frac{du_M}{d\theta} \tag{8·4·9}$$

Again, we recognize this as the nondimensional version of Eq. (8·4·7).

To see more clearly what the problem has now become, let us consider four divisions in the x direction just as we have done in Secs. 8·2 and 8·3 (see Fig. 8·24). There will then be four simultaneous ordinary differential equations to solve. These are shown below in normalized form.

$$\begin{aligned}
\frac{du_1}{d\theta} &= \frac{1}{(\Delta x)^2}(-2u_1 + u_2) \\
\frac{du_2}{d\theta} &= \frac{1}{(\Delta x)^2}(u_1 - 2u_2 + u_3) \\
\frac{du_3}{d\theta} &= \frac{1}{(\Delta x)^2}(u_2 - 2u_3 + u_4) \\
\frac{du_4}{d\theta} &= \frac{1}{(\Delta x)^2}(2u_3 - 2u_4)
\end{aligned} \tag{8·4·10}$$

In these equations Δx should really be set equal to $\frac{1}{4}$. This has not been done, however, to retain some generality for later discussions. In arriving at the first of these equations, it has been recognized that $u_0 = 0$ because of the boundary condition at $x = 0$.

This system of ordinary differential equations may be classed as an *initial-value problem* as opposed to boundary-value problems, such as were discussed in Secs. 8·2 and 8·3. This is because these equations are to be solved for the unknowns as a function of time beginning with an initial value for each of the unknowns. In this case the initial values are obtained from the initial temperature distribution in the plane wall. Thus, in normalized form, the initial conditions that go with Eqs. (8·4·10) are

$$u_1(0) = 1 \qquad u_3(0) = 1$$

$$u_2(0) = 1 \qquad u_4(0) = 1$$

Methods for solving these equations are discussed in the next four sections.

8·4·3 Analog and exact methods of solution

These two methods can be used to solve the system of equations without further numerical approximations. The analog method is based on the thermal-circuit notion. The exact solution uses the theory of ordinary differential equations to solve the system.

Analog method The use of the thermal-circuit notion is most easily seen by rewriting the dimensional form of the general nodal equation (8·4·5) as

$$\frac{t_{m-1} - t_m}{\Delta x/kA} = \frac{t_m - t_{m+1}}{\Delta x/kA} + \rho cA\,\Delta x\,\frac{dt_m}{d\theta}$$

The term $\Delta x/kA$ is recognized as the steady-state thermal resistance between two nodal points. The term $\rho cA\,\Delta x$ is recognized as the thermal capacitance of the material surrounding node m. If we denote these quantities by R and C, respectively, the system of four equations, in dimensional form, becomes

$$\begin{aligned}
\frac{t_0 - t_1}{R} &= \frac{t_1 - t_2}{R} + C\frac{dt_1}{d\theta} \\
\frac{t_1 - t_2}{R} &= \frac{t_2 - t_3}{R} + C\frac{dt_2}{d\theta} \\
\frac{t_2 - t_3}{R} &= \frac{t_3 - t_4}{R} + C\frac{dt_3}{d\theta} \\
\frac{t_3 - t_4}{R} &= \frac{C}{2}\frac{dt_4}{d\theta}
\end{aligned} \qquad (8\cdot4\cdot11)$$

The thermal-circuit representation of these equations is shown in Fig. 8·25. Initially, each of the temperatures is at the value t_i. At time zero the switch is suddenly closed and the system begins to respond. This may be set up and solved on an analog computer. It should be pointed out, however, that large systems of equations with hundreds of nodes would require a great many resistors and capacitors and also a lot of work to set up the circuit. Consequently in many practical problems the analog thermal-circuit approach will not be feasible.

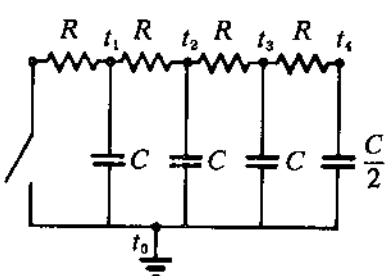


FIG. 8·25 *Thermal circuit for plane-wall transient.*

Exact mathematical solution If the number of simultaneous equations is small enough, it is practical to obtain their solutions by analytical methods. For example, consider the case in which only one node is used (at $x = 1$). In this case Eqs. (8·4·10) reduce

to only one ordinary differential equation to solve:

$$\frac{du_1}{d\theta} = -2u_1 \tag{8·4·12}$$

Here u_1 is now the normalized temperature at $x = 1$. The initial condition is that $u_1(0) = 1$. The exact solution to this equation is

$$u_1 = e^{-2\theta} \tag{8·4·13}$$

This differential equation and its exponential solution should remind you of the lumped-parameter analysis in Chap. 1.

For a two-node system the following pair of ordinary differential equations must be solved simultaneously:

$$\begin{aligned} \frac{du_1}{d\theta} &= -8u_1 + 4u_2 \\ \frac{du_2}{d\theta} &= 8u_1 - 8u_2 \end{aligned} \tag{8·4·14}$$

Now, u_1 is the temperature at $x = 0.5$ and u_2 is the temperature at the insulated surface $x = 1$.

Since these are linear equations with constant coefficients, let us follow Appendix A and seek solutions of the following form:

$$u_1 = ae^{\lambda\theta} \qquad u_2 = be^{\lambda\theta} \tag{8·4·15}$$

These assumed solutions may be substituted into Eqs. (8·4·14) to give

$$a\lambda e^{\lambda\theta} = -8ae^{\lambda\theta} + 4be^{\lambda\theta}$$

$$b\lambda e^{\lambda\theta} = 8ae^{\lambda\theta} - 8be^{\lambda\theta}$$

The time dependence may be divided out and the equations rearranged to give the following result:

$$(\lambda + 8)a - 4b = 0$$

$$-8a + (\lambda + 8)b = 0$$

These equations have the trivial solution that $a = b = 0$, which is of no value to us. They have a nontrivial solution only when the determinant of the coefficients is zero. Thus we will demand that λ have a value such that

$$\begin{vmatrix} \lambda + 8 & -4 \\ -8 & \lambda + 8 \end{vmatrix} = 0$$

Expanding the determinant,

$$(\lambda + 8)(\lambda + 8) - (-4)(-8) = 0$$

Rearranging,

$$(\lambda + 8)^2 = 32$$

Taking the square root of both sides,

$$\lambda + 8 = \pm \sqrt{32} = \pm 4\sqrt{2}$$

or

$$\lambda = -8 \pm 4\sqrt{2}$$

We now see that there are two values of λ that make the determinant equal to zero. These values are given by

$$\lambda_1 = -8 + 4\sqrt{2} \qquad \lambda_2 = -8 - 4\sqrt{2} \tag{8·4·16}$$

This means we have two solutions of the form given by Eq. (8·4·15), one using λ_1, the other using λ_2. Because of the linearity of the system of equations we can superimpose these two solutions to get the general solution. Thus the general solution may be written as

$$\begin{aligned} u_1 &= a_1 e^{\lambda_1 \theta} + a_2 e^{\lambda_2 \theta} \\ u_2 &= b_1 e^{\lambda_1 \theta} + b_2 e^{\lambda_2 \theta} \end{aligned} \tag{8·4·17}$$

The four constants a_1, a_2, b_1, and b_2 must now be determined from the initial conditions of the problem. The initial values of u are given as

$$u_1(0) = 1 \qquad \text{and} \qquad u_2(0) = 1$$

These values may be substituted into Eqs. (8·4·14) to find the initial values of the time derivatives. Thus

$$\left.\frac{du_1}{d\theta}\right|_{\theta=0} = -8(1) + 4(1) = -4$$

$$\left.\frac{du_2}{d\theta}\right|_{\theta=0} = 8(1) - 8(1) = 0$$

These four conditions are enough to determine the values of the four constants. The result is that

$$a_1 = \frac{2 + \sqrt{2}}{4} \qquad a_2 = \frac{2 - \sqrt{2}}{4}$$

$$b_1 = \frac{1 + \sqrt{2}}{2} \qquad b_2 = \frac{1 - \sqrt{2}}{2}$$

The final solution is then given by the following pair of equations:

$$u_1 = \tfrac{1}{4}\{(2 + \sqrt{2}) \exp [-(8 - 4\sqrt{2})\theta] + (2 - \sqrt{2}) \exp [-(8 + 4\sqrt{2})\theta]\} \tag{8·4·18}$$

$$u_2 = \tfrac{1}{2}\{(1 + \sqrt{2}) \exp [-(8 - 4\sqrt{2})\theta] + (1 - \sqrt{2}) \exp [-(8 + 4\sqrt{2})\theta]\}$$

These are the exact solutions to the system of two ordinary differential equations which resulted from a two-node finite-difference approximation to the original problem.

The one-node and two-node solutions (8·4·13) and (8·4·18) are compared to the exact solution of the original problem in Fig. 8·26. This shows how the finite-difference approximation approaches the exact solution as the number of nodes is increased. It is interesting to note that the finite-difference solutions fall below the exact solution at the start of the problem but then cross over to the high side as time goes on.

You should observe the increase in complexity in obtaining the two-node analytical solution as compared to the one-node case. In a practical case, where there may be hundreds of equations, an analytical solution of the set of ordinary differential equations is not feasible. Consequently, we are again forced into using finite-difference ideas to solve the system of ordinary differential equations. The next three sections discuss three ways of going about this.

Before continuing on, however, a comparison with the steady-

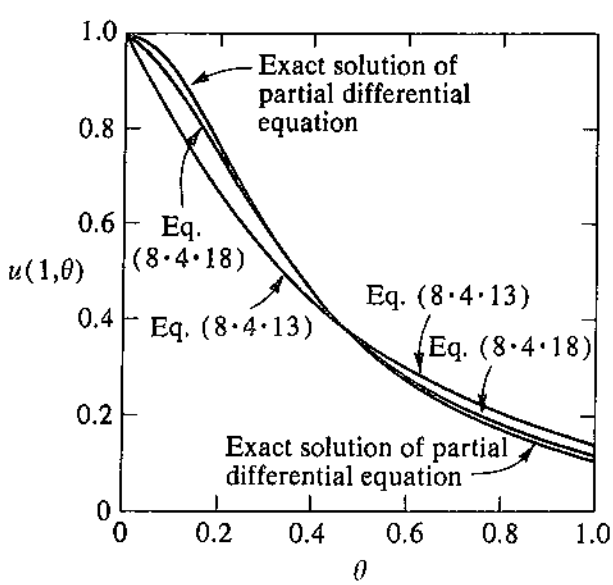


FIG. 8·26 *Exact solutions to finite-difference system of ordinary differential equations for M = 1 and 2 show effect of nodal-point spacing.*

state problems discussed in Secs. 8·2 and 8·3 is worth making. Replacing spatial derivatives with their finite-difference approximations in a steady-state problem reduced a differential equation to a system of algebraic equations. Large systems of such algebraic equations may be solved directly on the digital computer as we discussed earlier. In a transient problem, however, finite differencing the spatial derivatives reduces the governing partial differential equation to a system of ordinary differential equations. Exact analytical solutions to these equations cannot be carried out directly on the digital computer. Some further approximations will be needed. Thus a transient problem is considerably more complex than a steady-state problem.

8·4·4 The Euler method of solution

Since the exact analytical methods for solving systems of ordinary differential equations become rather cumbersome if the number of equations is large, we need approximate numerical methods for finding the solution. A further application of the finite-difference concept can be used.

Since this is an initial-value problem, we will know the solution $u^{(v)}$ at some point in time $\theta^{(v)}$, and we will be seeking to find the solution $u^{(v+1)}$ at some later point in time $\theta^{(v+1)} = \theta^{(v)} + \Delta\theta$, as shown in Fig. 8·27. The simplest way to estimate the solution at $\theta^{(v+1)}$ is to compute the derivative at $\theta^{(v)}$ and then move ahead in time in the following way:

$$u^{(v+1)} = u^{(v)} + \left.\frac{du}{d\theta}\right|^{(v)} \Delta\theta \tag{8·4·19}$$

This can be generalized for the system of equations we are

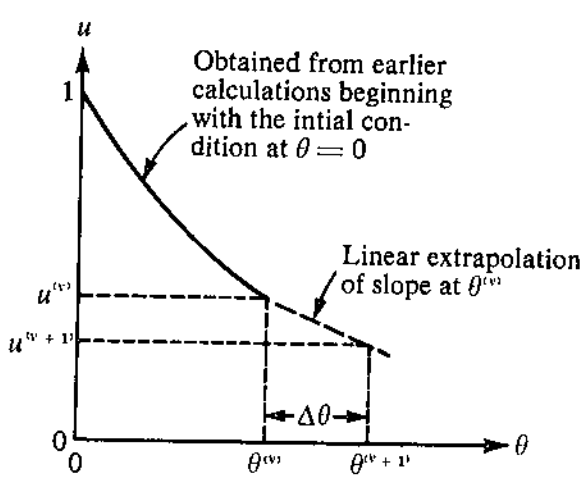


FIG. 8·27 *Linear extrapolation of previously computed solution to move ahead in time.*

considering to give the following:

$$\begin{aligned}
u_1^{(\nu+1)} &= u_1^{(\nu)} + \left.\frac{du_1}{d\theta}\right|^{(\nu)} \Delta\theta \\
u_2^{(\nu+1)} &= u_2^{(\nu)} + \left.\frac{du_2}{d\theta}\right|^{(\nu)} \Delta\theta \\
u_3^{(\nu+1)} &= u_3^{(\nu)} + \left.\frac{du_3}{d\theta}\right|^{(\nu)} \Delta\theta \\
u_4^{(\nu+1)} &= u_4^{(\nu)} + \left.\frac{du_4}{d\theta}\right|^{(\nu)} \Delta\theta
\end{aligned} \tag{8·4·20}$$

The values of the four derivatives may be obtained from the system of differential equations (8·4·10) and substituted into Eq. (8·4·20) to give the following system of algebraic equations:

$$\begin{aligned}
u_1^{(\nu+1)} &= u_1^{(\nu)} + \frac{1}{(\Delta x)^2}\left[-2u_1^{(\nu)} + u_2^{(\nu)}\right]\Delta\theta \\
u_2^{(\nu+1)} &= u_2^{(\nu)} + \frac{1}{(\Delta x)^2}\left[u_1^{(\nu)} - 2u_2^{(\nu)} + u_3^{(\nu)}\right]\Delta\theta \\
u_3^{(\nu+1)} &= u_3^{(\nu)} + \frac{1}{(\Delta x)^2}\left[u_2^{(\nu)} - 2u_3^{(\nu)} + u_4^{(\nu)}\right]\Delta\theta \\
u_4^{(\nu+1)} &= u_4^{(\nu)} + \frac{1}{(\Delta x)^2}\left[2u_3^{(\nu)} - 2u_4^{(\nu)}\right]\Delta\theta
\end{aligned}$$

In this example, of course, Δx has already been chosen as $\frac{1}{4}$ in arriving at four equations. This substitution has not yet been made in order to maintain some generality.

These equations may be rearranged to give

$$\begin{aligned}
u_1^{(\nu+1)} &= (1-2p)u_1^{(\nu)} + pu_2^{(\nu)} \\
u_2^{(\nu+1)} &= pu_1^{(\nu)} + (1-2p)u_2^{(\nu)} + pu_3^{(\nu)} \\
u_3^{(\nu+1)} &= pu_2^{(\nu)} + (1-2p)u_3^{(\nu)} + pu_4^{(\nu)} \\
u_4^{(\nu+1)} &= 2pu_3^{(\nu)} + (1-2p)u_4^{(\nu)}
\end{aligned}$$

where $p = \Delta\theta/(\Delta x)^2$ has been defined for simplicity.

As in Sec. 8·2 and 8·3, where we were dealing with systems of algebraic equations, it is convenient to use a matrix representation for these equations. They can be expressed as follows:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(v+1)} = \begin{bmatrix} 1-2p & p & & \\ p & 1-2p & p & \\ & p & 1-2p & p \\ & & 2p & 1-2p \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(v)} \tag{8·4·21}$$

The tridiagonal matrix on the right-hand side of the equation is known (and constant) once the size of the time step is picked. The known values of u at $\theta^{(v)}$ (that is, $u^{(v)}$) are multiplied by this tridiagonal matrix to obtain the new values of u at $\theta^{(v+1)}$. This matrix multiplication is quite easy to carry out on the computer since only the nonzero terms will contribute to the calculation. This is an *explicit* computation to obtain the new values of u from the old values of u.* The process is repeated over and over again to move ahead in time.

To be more specific, let us consider a numerical example in which $\Delta\theta$ has been selected so that $p = \frac{1}{4}$. That is,

$$\Delta\theta = p(\Delta x)^2 = \tfrac{1}{4}(\tfrac{1}{4})^2 = \tfrac{1}{64}$$

The problem then reduces to

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(v+1)} = \begin{bmatrix} 0.50 & 0.25 & & \\ 0.25 & 0.50 & 0.25 & \\ & 0.25 & 0.50 & 0.25 \\ & & 0.50 & 0.50 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_4 \\ u_4 \end{bmatrix}^{(v)} \tag{8·4·22}$$

The solution begins by substituting the given initial conditions $u^{(0)}$ into the right-hand side and then carrying out the matrix multiplication as shown below.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(1)} = \begin{bmatrix} 0.50 & 0.25 & & \\ 0.25 & 0.50 & 0.25 & \\ & 0.25 & 0.50 & 0.25 \\ & & 0.50 & 0.50 \end{bmatrix} \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.50(1.0) + 0.25(1.0) \\ 0.25(1.0) + 0.50(1.0) + 0.25(1.0) \\ 0.25(1.0) + 0.50(1.0) + 0.25(1.0) \\ 0.50(1.0) + 0.50(1.0) \end{bmatrix}$$

*Because of this fact, the Euler method is usually called the *explicit method* when

which gives

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(1)} = \begin{bmatrix} 0.75 \\ 1.00 \\ 1.00 \\ 1.00 \end{bmatrix}$$

These are the new values at $\theta^{(1)}$. They are now substituted into Eq. (8·4·22) as shown below in order to move on to the next step in time. Thus

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(2)} = \begin{bmatrix} 0.50 & 0.25 & & \\ 0.25 & 0.50 & 0.25 & \\ & 0.25 & 0.50 & 0.25 \\ & & 0.50 & 0.50 \end{bmatrix} \begin{bmatrix} 0.75 \\ 1.00 \\ 1.00 \\ 1.00 \end{bmatrix}$$

The matrix multiplication can again be carried out to give the next values of u. The result is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(2)} = \begin{bmatrix} 0.6250 \\ 0.9375 \\ 1.0000 \\ 1.0000 \end{bmatrix}$$

The digital computer is ideally suited for carrying out this process over and over again to obtain the solution. The numerical results of these computations are compared to the exact solution in Fig. 8·28. Figure 8·28*a* shows the temperature at $x = 1$ as a function of time. Figure 8·28*b* shows the internal temperature distribution at three values of time. In this case the response is hardly distinguishable from the exact solution of the original partial differential equation.

8·4·5 The Crank-Nicolson method of solution

In the Euler method the value of the derivative at the beginning of the time interval was used to move ahead in time. A more accurate method would be to use the arithmetic mean value of the derivatives at the beginning and the end of the time interval. That is, instead of using Eq. (8·4·19) to move ahead in time, one would use

$$u^{(v+1)} = u^{(v)} + \frac{1}{2}\left[\left.\frac{du}{d\theta}\right|^{(v)} + \left.\frac{du}{d\theta}\right|^{(v+1)}\right]\Delta\theta \qquad (8\cdot4\cdot23)$$

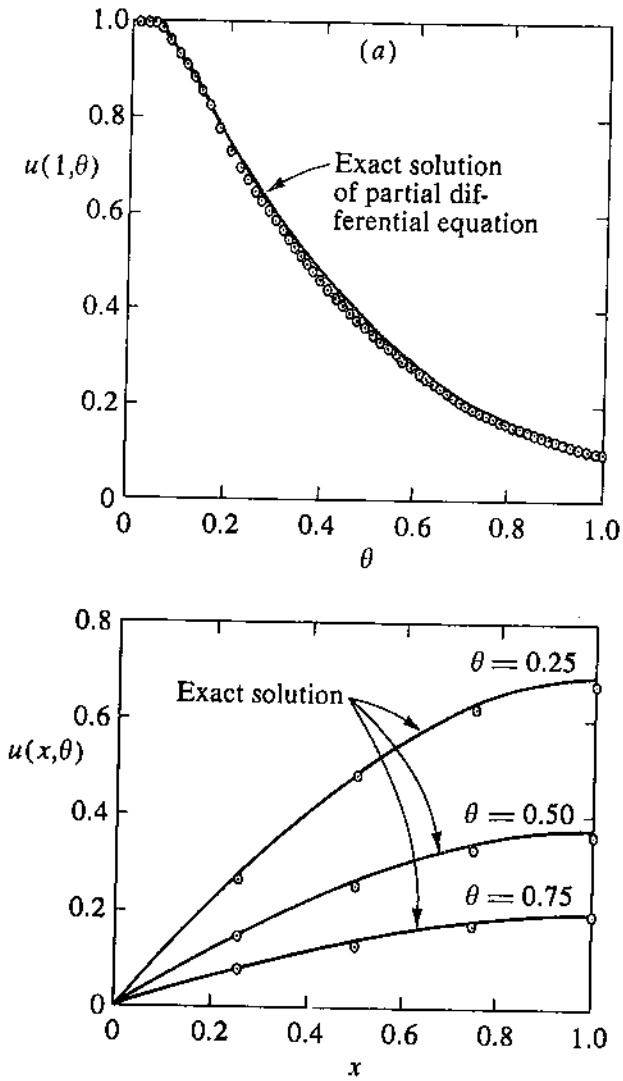


FIG. 8·28 *Euler finite-difference solution for* $M = 4$ *and* $p = 0.25$.

This can be generalized for the system of equations we are considering to give

$$
\begin{aligned}
u_1^{(v+1)} &= u_1^{(v)} + \frac{1}{2}\left[\left.\frac{du_1}{d\theta}\right|^{(v)} + \left.\frac{du_1}{d\theta}\right|^{(v+1)}\right]\Delta\theta \\
u_2^{(v+1)} &= u_2^{(v)} + \frac{1}{2}\left[\left.\frac{du_2}{d\theta}\right|^{(v)} + \left.\frac{du_2}{d\theta}\right|^{(v+1)}\right]\Delta\theta \\
u_3^{(v+1)} &= u_3^{(v)} + \frac{1}{2}\left[\left.\frac{du_3}{d\theta}\right|^{(v)} + \left.\frac{du_3}{d\theta}\right|^{(v+1)}\right]\Delta\theta \\
u_4^{(v+1)} &= u_4^{(v)} + \frac{1}{2}\left[\left.\frac{du_4}{d\theta}\right|^{(v)} + \left.\frac{du_4}{d\theta}\right|^{(v+1)}\right]\Delta\theta
\end{aligned}
\qquad (8\cdot4\cdot24)
$$

The values of the derivatives can be found from the system of

differential equations and substituted into Eqs. (8·4·24) to yield

$$u_1^{(v+1)} = u_1^{(v)} + \frac{\Delta\theta}{2(\Delta x)^2}\left[(-2u_1^{(v)} + u_2^{(v)}) + (-2u_1^{(v+1)} + u_2^{(v+1)})\right]$$

$$u_2^{(v+1)} = u_2^{(v)} + \frac{\Delta\theta}{2(\Delta x)^2}\left[(u_1^{(v)} - 2u_2^{(v)} + u_3^{(v)}) + (u_1^{(v+1)} - 2u_2^{(v+1)} + u_3^{(v+1)})\right]$$

$$u_3^{(v+1)} = u_3^{(v)} + \frac{\Delta\theta}{2(\Delta x)^2}\left[(u_2^{(v)} - 2u_3^{(v)} + u_4^{(v)}) + (u_2^{(v+1)} - 2u_3^{(v+1)} + u_4^{(v+1)})\right]$$

$$u_4^{(v+1)} = u_4^{(v)} + \frac{\Delta\theta}{2(\Delta x)^2}\left[(2u_3^{(v)} - 2u_4^{(v)}) + (2u_3^{(v+1)} - 2u_4^{(v+1)})\right]$$

Observe that the unknown values of $u^{(v+1)}$ now appear on both sides of the equation. This is of no particular difficulty in this case since they can be transferred to the left-hand side and combined with the ones already there. If this is done, the system of equations may be written in matrix form as

$$\begin{bmatrix} 1+p & -\frac{p}{2} & & \\ -\frac{p}{2} & 1+p & -\frac{p}{2} & \\ & -\frac{p}{2} & 1+p & -\frac{p}{2} \\ & & -p & 1+p \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(v+1)}$$

$$= \begin{bmatrix} 1-p & \frac{p}{2} & & \\ \frac{p}{2} & 1-p & \frac{p}{2} & \\ & \frac{p}{2} & 1-p & \frac{p}{2} \\ & & p & 1-p \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(v)} \tag{8·4·25}$$

where $p = \Delta\theta/(\Delta x)^2$ as before.

Just as was the case in the Euler method, the right-hand side can be computed directly because all the components are known. This results in a column matrix as before. The difference comes in the fact that this does not give an explicit result for the unknowns on

the left-hand side. Rather, an *implicit* tridiagonal system of algebraic equations results. This system of algebraic equations must then be solved. The process is repeated at each step in time.

It is immediately apparent that there will be more computational work in the implicit Crank-Nicolson method than there was in the explicit Euler method. There are three reasons, however, that one might be willing to resort to this extra work at each time step. First, by using the average value of the derivative to move ahead in time, the solution should be more accurate than the explicit method. Second, if increasing the accuracy is not too important, larger time steps can be taken, thereby reducing the total number of steps needed to reach a certain point in time. Finally, as we shall discuss in Sec. 8·4·7, the Euler method is much more apt to produce unwanted numerical oscillations than is the Crank-Nicolson method if too large a time step is taken.

As a specific numerical example, let us consider the case in which the time step has been chosen to give $p = 1$. That is,

$$\Delta\theta = p(\Delta x)^2 = 1(\tfrac{1}{4})^2 = \tfrac{1}{16}$$

This means that it will take only 16 steps to reach $\theta = 1$ instead of the 64 used with the explicit example. Equation (8·4·25) then reduces to

$$\begin{bmatrix} 2.0 & -0.5 & & \\ -0.5 & 2.0 & -0.5 & \\ & -0.5 & 2.0 & -0.5 \\ & & -1.0 & 2.0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(v+1)}$$

$$= \begin{bmatrix} 0.0 & 0.5 & & \\ 0.5 & 0.0 & 0.5 & \\ & 0.5 & 0.0 & 0.5 \\ & & 1.0 & 0.0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(v)} \qquad (8\cdot4\cdot26)$$

The solution begins by substituting the known initial conditions $u^{(0)}$ into the right-hand side of the equation. Thus

$$\begin{bmatrix} 2.0 & -0.5 & & \\ -0.5 & 2.0 & -0.5 & \\ & -0.5 & 2.0 & -0.5 \\ & & -1.0 & 2.0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(1)}$$

$$= \begin{bmatrix} 0.0 & 0.5 & & \\ 0.5 & 0.0 & 0.5 & \\ & 0.5 & 0.0 & 0.5 \\ & & 1.0 & 0.0 \end{bmatrix} \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \end{bmatrix}$$

The matrix multiplication on the right-hand side is again easy to carry out because of the tridiagonal nature of the square matrix. The multiplication may be carried out to give

$$\begin{bmatrix} 2.0 & -0.5 & & \\ -0.5 & 2.0 & -0.5 & \\ & -0.5 & 2.0 & -0.5 \\ & & -1.0 & 2.0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(1)} = \begin{bmatrix} 0.5 \\ 1.0 \\ 1.0 \\ 1.0 \end{bmatrix}$$

This tridiagonal system of equations must now be solved to find the values of u at the new time. The solution is most easily obtained using gaussian elimination as was discussed in Sec. 8·2. The following result is obtained:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(1)} = \begin{bmatrix} 0.464 \\ 0.856 \\ 0.959 \\ 0.979 \end{bmatrix}$$

These are the values of temperature after one time step, or at $\theta = \frac{1}{16}$. These values can now be substituted into the right-hand side of Eq. (8·4·26) to begin the process all over again to move ahead to the next point in time.

The results of these calculations are compared to the exact solution in Fig. 8·29. The accuracy is quite good even though the time step was four times as large as in the Euler example. In fact, the accuracy appears to be slightly better than the Euler method shown in Fig. 8·28.

8·4·6 A pure implicit method of solution

One of the difficulties with the explicit method is that numerical oscillations can be introduced which will be unstable if too large a time step is taken. Numerical oscillations also occur in the Crank-Nicolson implicit method, but they never become unstable. They can become large enough, however, to make the solution inaccurate.

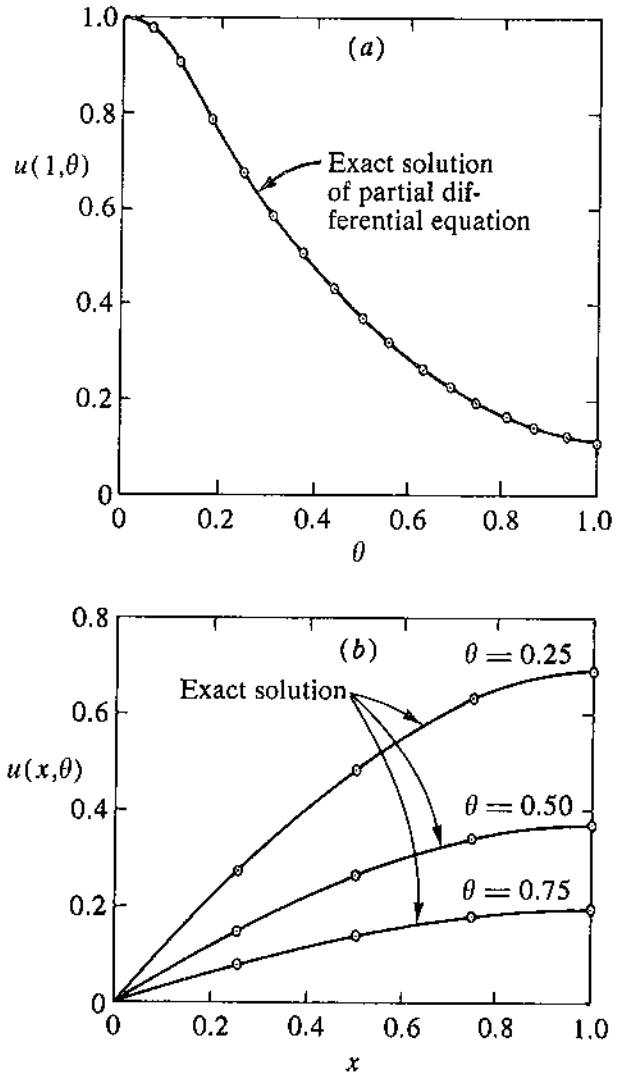


FIG. 8·29 *Crank-Nicolson finite-difference solution for* $M = 4$ *and* $p = 1.0$.

The *pure implicit* numerical method discussed in this section is not as accurate as the Crank-Nicolson method, but numerical oscillations will never appear (no matter how large a time step is taken). This means that the solution will be more accurate than either the explicit method or the Crank-Nicolson method if very large time steps are being used.

In order to see how this method works and at the same time to become more familiar with matrix methods of handling systems of equations, let us rewrite Eqs. (8·4·10) in matrix form. They can be represented in the following simple way:

$$\dot{\mathbf{u}} = -\mathbf{K}\mathbf{u} \tag{8·4·27}$$

where $\dot{\mathbf{u}}$ stands for the column matrix of time derivatives of temperature, and $\mathbf{u}$ stands for the column matrix of temperatures.

The matrix $\mathbf{K}$ is given by

$$\mathbf{K} = \frac{1}{(\Delta x)^2}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -2 & 2 \end{bmatrix} \tag{8·4·28}$$

The easiest way to verify Eq. (8·4·27) is to carry out the matrix multiplication and show that Eqs. (8·4·10) are obtained.

In the Euler method we move ahead in time by using the derivatives evaluated at the old time, Eq. (8·4·19). In the Crank-Nicolson method we moved ahead by using the average of the time derivatives at the old and the new times, Eq. (8·4·23). Now we will use only the time derivative at the new time. Thus, we will be moving ahead according to the matrix equation

$$\mathbf{u}^{(v+1)} = \mathbf{u}^{(v)} + \Delta\theta\dot{\mathbf{u}}^{(v+1)} \tag{8·4·29}$$

Equation (8·4·27) may be evaluated at $\theta^{(v+1)}$ and substituted into Eq. (8·4·29) to eliminate $\dot{\mathbf{u}}^{(v+1)}$. Thus

$$\mathbf{u}^{(v+1)} = \mathbf{u}^{(v)} - \Delta\theta\,\mathbf{K}\mathbf{u}^{(v+1)} \tag{8·4·30}$$

The temperatures at the new time may be transferred to the left-hand side of the equation to give

$$\mathbf{u}^{(v+1)} + \Delta\theta\,\mathbf{K}\mathbf{u}^{(v+1)} = \mathbf{u}^{(v)} \tag{8·4·31}$$

Since $\mathbf{u}^{(v+1)}$ may also be written as $\mathbf{I}\mathbf{u}^{(v+1)}$, Eq. (8·4·31) may be simplified to

$$(\mathbf{I} + \Delta\theta\,\mathbf{K})\mathbf{u}^{(v+1)} = \mathbf{u}^{(v)} \tag{8·4·32}$$

The expanded form of this equation, comparable to Eqs. (8·4·21) or (8·2·25), is

$$\begin{bmatrix} 1+2p & -p & & \\ -p & 1+2p & -p & \\ & -p & 1+2p & -p \\ & & -2p & 1+2p \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(v+1)} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(v)} \tag{8·4·33}$$

This is an implicit set of equations to solve for the new temperatures at each step in time. Computationally it will be a little easier than Eq. (8·4·25) because there is no matrix multiplication on the right-hand side to carry out at each time step.

The numerical solution to Eq. (8·4·33) is compared to the exact solution for $p = 1.0$ in Fig. 8·30. Observe that the accuracy is not as good as the Crank-Nicolson method for the same time step. This is expected since the average derivative used in the Crank-Nicolson method should be more accurate.

8·4·7 Numerically induced oscillations

In Sec. 8·4·3 we discussed the ability of the exact analytical solution to a set of ordinary differential equations to approximate the solution to the original partial differential equation. Now we need to discuss the numerical schemes used to obtain approximate solutions to the set of ordinary differential equations. We have used three numerical methods to solve these equations: the explicit Euler method, the Crank-Nicolson implicit method, and the pure implicit method. In all these cases we expect better results if we take smaller time steps. In practice, however, one is

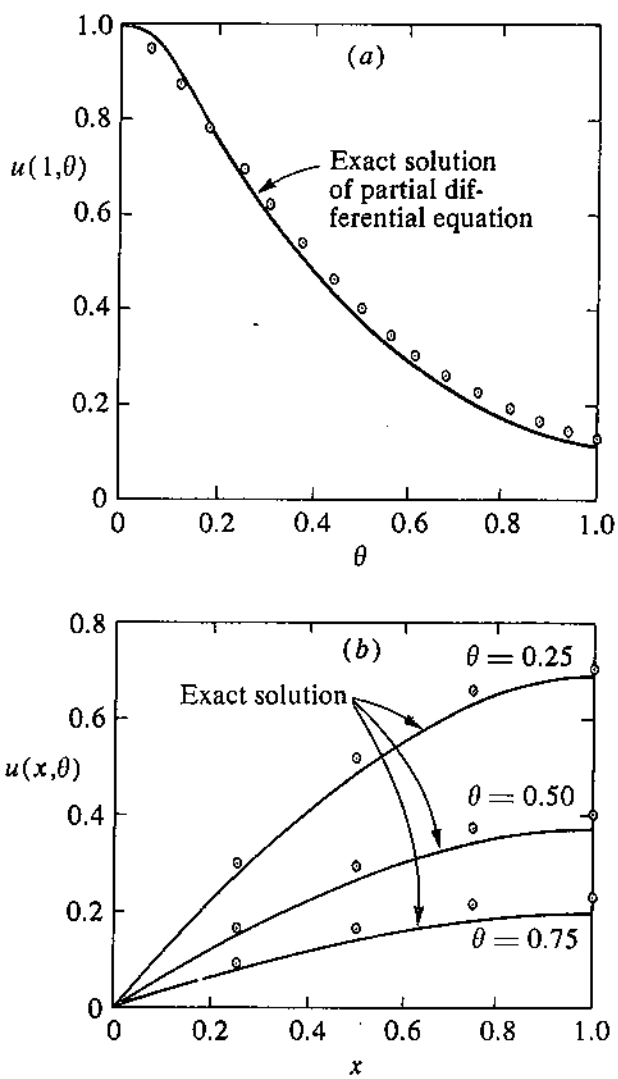


FIG. 8·30 *Pure implicit finite-difference solution for* $M = 4$ *and* $p = 1.0$.

usually trying to take as large a time step as one can to cut down on the amount of computations. In addition to decreasing the accuracy of the solution, larger time steps can introduce some unwanted, numerically induced oscillations into the solution.

A deeper understanding of the types of behavior that the numerical solution of these ordinary differential equations can exhibit begins by considering the special case in which there is only one such equation, Eq. (8·4·12). The exact solution is given by Eq. (8·4·13). The following three equations are obtained corresponding to the three numerical schemes:

Euler: $$u^{(v+1)} = (1 - 2p)u^{(v)}$$

Crank-Nicolson: $$(1 + p)u^{(v+1)} = (1 - p)u^{(v)}$$

Pure implicit: $$(1 + 2p)u^{(v+1)} = u^{(v)}$$

Each of these may be put in the following general form:

$$u^{(v+1)} = \lambda u^{(v)} \tag{8·4·34}$$

where λ has the following definitions:

Euler: $$\lambda = 1 - 2p \tag{8·4·35}$$

Crank-Nicolson: $$\lambda = \frac{1 - p}{1 + p} \tag{8·4·36}$$

Pure implicit: $$\lambda = \frac{1}{1 + 2p} \tag{8·4·37}$$

The value of λ determines the character of the solution. By looking at Eq. (8·4·34) it can be seen that there are four possible types of behavior:

1. $\lambda > 1$ — Steady, unbounded growth (the new value of u has the same sign as the old value and is larger in magnitude)
2. $1 > \lambda > 0$ — Steady decay (the new value of u has the same sign as the old value and is smaller in magnitude)
3. $0 > \lambda > -1$ — Stable oscillations (the new value of u has the opposite sign as the old value and is smaller in magnitude)
4. $\lambda < -1$ — Unstable oscillations (the new value of u has the opposite sign as the old value and is larger in magnitude)

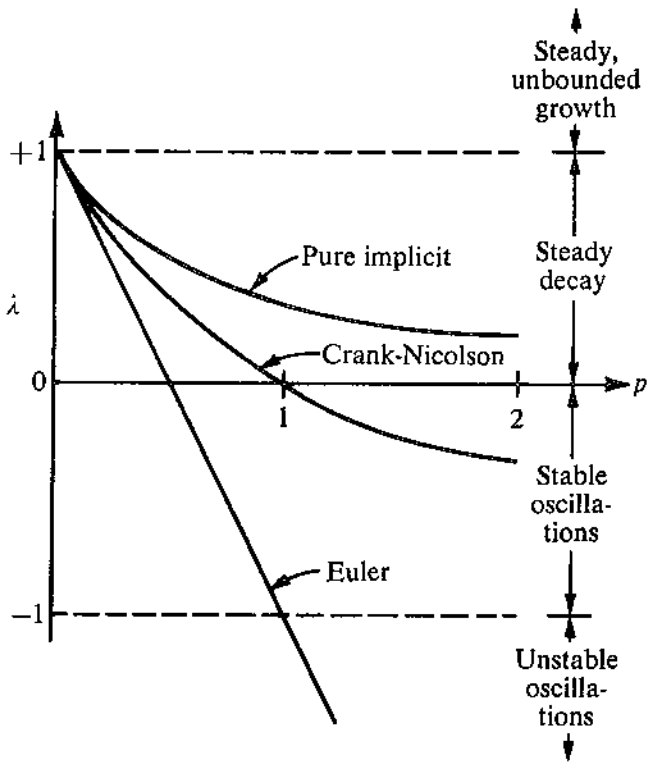


FIG. 8·31 *Finite-difference stability curves for* $M = 1$.

Figure 8·31 shows λ as a function of p ($p = \Delta\theta$ for this special case) for each of the three numerical schemes we have considered, Eqs. (8·4·35) to (8·4·37). It should be observed that all three schemes become identical if the time step is small enough (i.e., as $p \to 0$). As the time step is increased, the solutions begin to differ. The Euler method can have either steady decay, stable oscillations, or unstable oscillations. The Crank-Nicolson method can have either steady decay or stable oscillations. The pure implicit method has only a steadily decaying type of solution.

Figure 8·32 compares the three numerical solutions to the exact solution they are trying to approximate, Eq. (8·4·13). For a stable time step of $p = 0.25$ the Crank-Nicolson method gives very good results. The Euler method is low because it uses too large a time derivative to move ahead at each step. The pure implicit method uses too small a derivative and consequently falls above the exact solution. As the time step is increased to $p = 0.50$ (an oscillatory limit of the Euler method), all three solutions lose accuracy as expected. The Crank-Nicolson is still the best. The explicit solution falls even lower than before, and the pure implicit solution moves farther above than it was previously.

If we increase the time step beyond $p = 0.50$, numerically induced oscillations will begin to appear as shown in Fig. 8·33. For $p = 0.80$ the Euler solution gives a decaying oscillatory

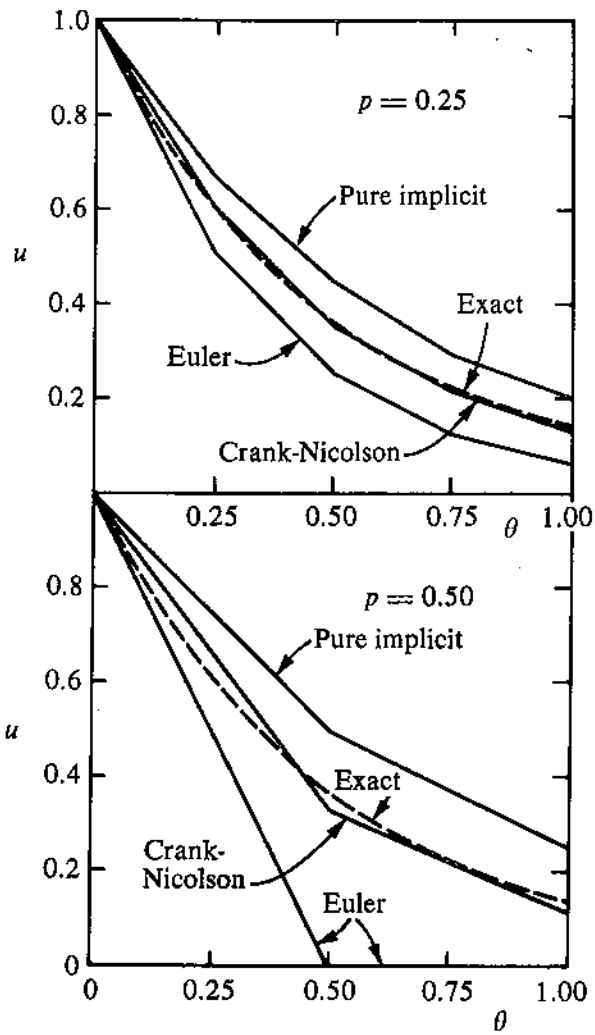


FIG. 8-32 *Finite-difference solutions for* $M = 1$ *with* $p = 0.25$ *and* 0.50.

solution. This is obviously wrong and cannot be used. The Crank-Nicolson solution still looks rather respectable but, of course, less accurate. The pure implicit solution is reasonable but less accurate. Increasing the time step to $p = 1.2$ makes matters worse. The oscillations in the Euler solution now grow without bound, and oscillations are introduced into the Crank-Nicolson solution. The pure implicit solution continues to lose accuracy, but no oscillations appear. Further increases in the time step will just make matters worse. The oscillations in the Crank-Nicolson solution will become more severe, but they will never go unstable. Oscillations will never appear in the pure implicit solution, but it will continue to lose accuracy.

The above discussion may be extended to the more general case in which there is more than one equation. The matrix representation for any of the three numerical schemes can be written as

$$\mathbf{A}\mathbf{u}^{(\nu+1)} = \mathbf{B}\mathbf{u}^{(\nu)} \tag{8-4-38}$$

where the matrices **A** and **B** will depend upon the particular method. This equation can be premultiplied by the inverse of matrix **A** to give

$$\mathbf{u}^{(v+1)} = \mathbf{A}^{-1}\mathbf{B}\mathbf{u}^{(v)} \tag{8·4·39}$$

The eigenvalues λ of the matrix $\mathbf{A}^{-1}\mathbf{B}$ play a similar role to the λ in Eq. (8·4·34). If there are M simultaneous equations being handled, there will be M eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$. These values will determine the character of the solution (see Refs. [3, 5]).

The eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ are determined from the relation

$$\det(\mathbf{A}^{-1}\mathbf{B} - \lambda\mathbf{I}) = 0 \tag{8·4·40}$$

We may now multiply this equation by det (**A**) to get

$$\det(\mathbf{A})\det(\mathbf{A}^{-1}\mathbf{B} - \lambda\mathbf{I}) = 0$$

There is a theorem in matrix theory that says the product of the determinants of two matrices is equal to the determinant of the product of the matrices. Thus the above is equivalent to

$$\det(\mathbf{B} - \lambda\mathbf{A}) = 0 \tag{8·4·41}$$

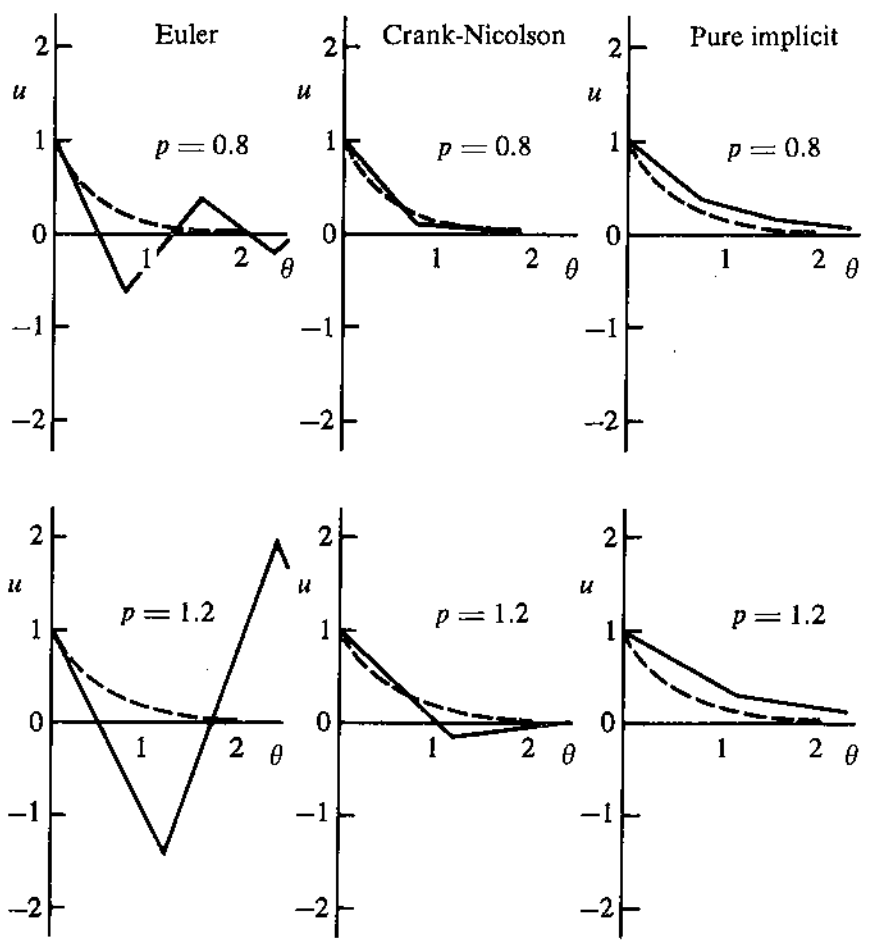


FIG. 8·33 *Finite-difference solutions for* $M = 1$ *with* $p =$ 0.8 *and* 1.2.

This requirement will determine the values of λ for the problem. Equation (8·4·41) is a little easier to use than Eq. (8·4·40) since the inverse of **A** does not have to be found.

There will be three general classes of solutions which will arise in this problem. If all the eigenvalues are between 0 and 1, there will be no oscillations. The solution will gradually approach a steady-state value. If one of the eigenvalues falls between 0 and -1, numerically induced oscillations will appear. If one of the eigenvalues is less than -1 the oscillations will be unstable.

As an example, let us consider the two-node case for the Euler method. The matrices **A** and **B** are then given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 - 2p & p \\ 2p & 1 - 2p \end{bmatrix}$$

Then

$$\mathbf{B} - \lambda\mathbf{A} = \begin{bmatrix} 1 - 2p - \lambda & p \\ 2p & 1 - 2p - \lambda \end{bmatrix}$$

The determinant of this matrix is then given by

$$\det(\mathbf{B} - \lambda\mathbf{A}) = (1 - 2p - \lambda)^2 - 2p^2$$

This is then set equal to zero in accordance with Eq. (8·4·41). The resulting equation is a quadratic in λ. Its solutions turn out to be

$$\lambda_1 = 1 - p(2 + \sqrt{2}) \qquad \lambda_2 = 1 - p(2 - \sqrt{2})$$

The value of λ_1 will determine the character of the solution since it is the one that is most likely to be negative. It is plotted in Fig. 8·34 as a function of p along with the corresponding eigenvalues for the Crank-Nicolson method and the pure implicit method. These curves are comparable to Fig. 8·31 which was obtained for the one-node case. The same general character is exhibited by the solutions, but the curves have shifted to the left so that the critical values of p are smaller than before. The upper limit for stable oscillations of the Euler method is now $2/(2 + \sqrt{2}) = 0.586$ as compared to 1.000 in the one-node case.

In a practical case the matrices **A** and **B** are large. This prohibits an algebraic analytical investigation of the numerical oscillations as we have done here. Qualitatively the same behavior can be expected, however. In the limit, as the number of nodes becomes infinite, it can be shown that the stability limit for the Euler method approaches 0.500.

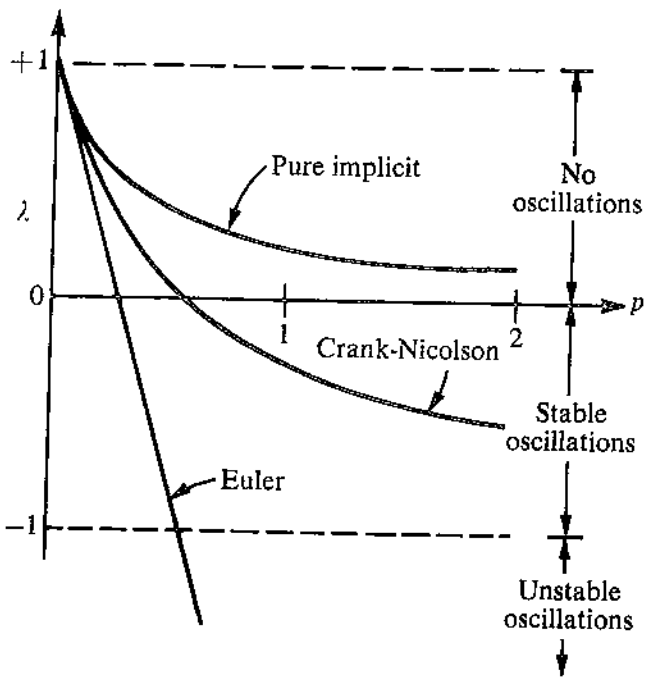


FIG. 8·34 *Finite-difference stability curves for* $M = 2$.

The limiting value of the critical stability limit for the Euler method (0.500) may be derived by considering the physical picture of the Euler method. Let us begin with the dimensional form of the set of differential equations (8·4·11). If, for example, we pick the second one of these to study, it may be rewritten as

$$RC\frac{dt_2}{d\theta} + 2t_2 = t_1 + t_3$$

We are looking for the solution of this equation as a function of time. In the exact solution all three temperatures are varying with time, and this equation is therefore coupled to the other equations in the set. A *semiexplicit* approximation to this equation holds the values of t_1 and t_3 constant at their initial values at the beginning of the time interval, $t_1^{(v)}$ and $t_3^{(v)}$. Thus the equation becomes

$$RC\frac{dt_2}{d\theta} + 2t_2 = t_1^{(v)} + t_3^{(v)} \qquad (8\cdot4\cdot42)$$

This semiexplicit equation now has only one dependent variable t_2, which starts at its initial value $t_2^{(v)}$ and exponentially approaches a steady-state solution. Specifying t_1 and t_3 as constants has uncoupled this equation from the rest of the equations in the set.

Equation (8·4·42) has the rather obvious steady-state solution that the final temperature $\tilde{t}_2$ is the average of the two temperatures

on either side. The exact solution to this semiexplicit equation is

$$\frac{t_2 - \tilde{t}_2}{t_2^{(v)} - \tilde{t}_2} = e^{-\theta/(RC/2)} \tag{8·4·43}$$

This solution is pictured in Fig. 8·35.

In the Euler numerical solution of Eq. (8·4·42), the derivative of t_2 at $\theta^{(v)}$ is used to move t_2 ahead in time according to the relation

$$t_2 = t_2^{(v)} + \left.\frac{dt_2}{d\theta}\right|^{(v)} \Delta\theta$$

We know from the exact solution of Eq. (8·4·42) that the solution for t_2 can never be beyond $\tilde{t}_2$. The first and second laws of thermodynamics could not both be simultaneously satisfied if it did. Thus $\Delta\theta_c$, the largest value of $\Delta\theta$ that can be used without violating one of these laws, must be such that

$$\tilde{t}_2 = t_2^{(v)} + \left.\frac{dt_2}{d\theta}\right|^{(v)} \Delta\theta_c \tag{8·4·44}$$

The initial value of the derivative may be obtained from Eq. (8·4·42) as

$$\left.\frac{dt_2}{d\theta}\right|^{(v)} = \frac{1}{RC}(t_1^{(v)} - 2t_2^{(v)} + t_3^{(v)})$$

Thus Eq. (8·4·44) becomes

$$\frac{t_1^{(v)} + t_3^{(v)}}{2} = t_2^{(v)} + (t_1^{(v)} - 2t_2^{(v)} + t_3^{(v)})\frac{\Delta\theta_c}{RC}$$

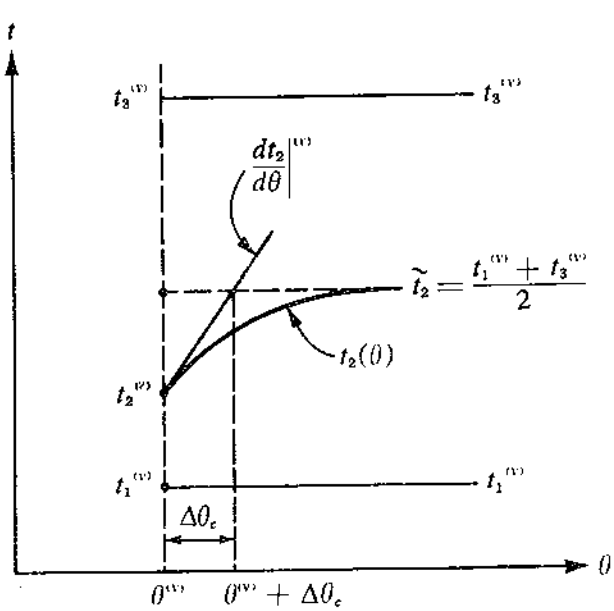


FIG. 8·35 *Semiexplicit Euler solution.*

This may be solved to give

$$\frac{\Delta\theta_c}{RC} = \frac{1}{2} \tag{8·4·45}$$

If each of the other three equations is checked in the same way, the same result is obtained. Thus, by comparison with Eq. (8·4·43), we see that the critical time step $\Delta\theta_c$ is equal to the nodal time constant $RC/2$. It is pleasing, of course, to see that physical significance can be attached to this critical time. It is not obvious that violation of Eq. (8·4·45) will produce numerical oscillations. However, it is not surprising to find mysterious behavior of mathematical solutions when physical principles are not satisfied.

We have seen that the stability limit for the one-node approximation was 1.000. For the two-node approximation we found that the stability limit was reduced to 0.586. In the limit, as the number of nodes becomes infinite, it can be shown that the stability limit approaches 0.500. Thus the critical time step derived from physical considerations is a conservative estimate of the stability limit for a finite number of nodes and an exact value as the number of nodes becomes infinite.

In practice the engineer does not do a great deal of analysis to determine his stability limit. He generally checks a few of the nodal time constants that he expects to be critical and then forges ahead with the solution without exact knowledge of the oscillation limits. The result must then be examined to see if any oscillations are showing up. He may be willing to put up with small oscillations or he may want to decrease his time step to reduce or eliminate them.

It is important to point out that the selection of the spatial mesh size Δx and the size of the time step $\Delta\theta$ are related through the parameter $p = \Delta\theta/(\Delta x)^2$. Cutting the spatial mesh size by a factor of 2 not only doubles the number of nodal points (giving better accuracy) but it also forces you to divide the time step by a factor of 4 if you want to maintain similar stability conditions (for example, $p = 0.5$). Thus what started out as a doubling of the amount of work (i.e., twice as many equations) really amounts to eight times the amount of work (i.e., twice as many equations *and* four times as many calculations to get to a particular point in time).

8·4·8 Convection boundary condition

The finite-difference method may easily be extended to handle a convection boundary condition. As an example, let us consider the same example of a plane-wall transient but with a convective

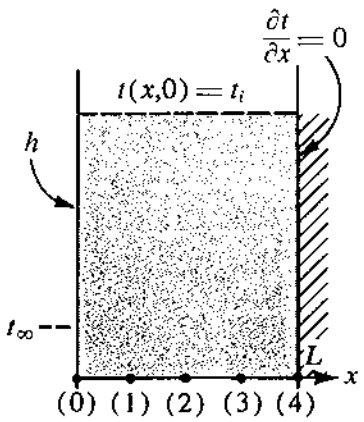


FIG. 8·36 *Nodal-point arrangement with convection boundary condition.*

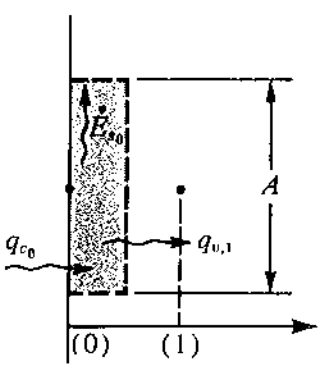


FIG. 8·37 *Nodal system at $x = 0$ with surface convection.*

boundary condition at $x = 0$ rather than a step change in surface temperature. Figure 8·36 shows the nodal system that we will use. We have again taken $\Delta x = \frac{1}{4}$, but now we have fives nodes to consider instead of only four because the surface temperature is now an unknown.

The set of ordinary differential equations for the interior nodal points are obtained in the same way as was done to find Eqs. (8·4·6) and (8·4·7). The surface node must be handled separately because of the convective boundary condition at the surface. Figure 8·37 shows an expanded picture of the surface nodal system and the important energy terms. The energy balance may be written as

$$q_{c_0} = q_{0,1} + \dot{E}_{s_0} \tag{8·4·46}$$

The rate equations may be written as

$$q_{c_0} = hA(t_\infty - t_0) \tag{8·4·47}$$

$$q_{0,1} = kA\frac{t_0 - t_1}{\Delta x} \tag{8·4·48}$$

$$\dot{E}_{s_0} = \rho cA\frac{\Delta x}{2}\frac{dt_0}{d\theta} \tag{8·4·49}$$

Upon combining the energy balance and rate equations in the usual way, the following differential equation is obtained:

$$hA(t_\infty - t_0) = \frac{kA}{\Delta x}(t_0 - t_1) + \rho cA\frac{\Delta x}{2}\frac{dt_0}{d\theta} \tag{8·4·50}$$

This equation and Eqs. (8·4·6) and (8·4·7) provide the set of ordinary differential equations. After normalizing and rearranging, the resulting system may be written as

$$\begin{aligned}
\frac{du_0}{d\theta} &= \frac{1}{(\Delta x)^2}[-2(1 + H\,\Delta x)u_0 + 2u_1] \\
\frac{du_1}{d\theta} &= \frac{1}{(\Delta x)^2}[u_0 - 2u_1 + u_2] \\
\frac{du_2}{d\theta} &= \frac{1}{(\Delta x)^2}[u_1 - 2u_2 + u_3] \\
\frac{du_3}{d\theta} &= \frac{1}{(\Delta x)^2}[u_2 - 2u_3 + u_4] \\
\frac{du_4}{d\theta} &= \frac{1}{(\Delta x)^2}[2u_3 - 2u_4]
\end{aligned} \tag{8·4·51}$$

Observe that the Biot number $H = hL/k$ has appeared in the normalization. This is not surprising since it is an important number that appears in analytical solutions. It should also be observed that the ambient temperature has been absorbed into the nondimensional temperature defined as $u = (t - t_\infty)/(t_i - t_\infty)$. These equations replace Eqs. (8·4·10) as the set of differential equations to be solved.

For the Euler numerical method of solving these equations, the following matrix equation is obtained:

$$\mathbf{u}^{(v+1)} = \mathbf{B}\mathbf{u}^{(v)} \tag{8·4·52}$$

where

$$\mathbf{B} = \begin{bmatrix} 1 - 2p(1 + H\,\Delta x) & 2p & & & \\ p & 1 - 2p & p & & \\ & p & 1 - 2p & p & \\ & & p & 1 - 2p & p \\ & & & 2p & 1 - 2p \end{bmatrix}$$

All the components of the matrix **B** are known and constant for this problem. The values of the temperature at the new time are obtained by simply multiplying the old values of temperature by the tridiagonal matrix **B** as given by Eq. (8·4·52). The only difference between this and the nonconvective boundary condition case (8·4·21) is that we now have one more equation and it involves the heat-transfer coefficient. The computations are carried out in exactly the same manner.

The Crank-Nicolson method may also be used to move ahead in time. In this case the following matrix equation is obtained:

$$\mathbf{A}\mathbf{u}^{(v+1)} = \mathbf{B}\mathbf{u}^{(v)} \tag{8·4·53}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 + p(1 + H\,\Delta x) & -p & & & \\ -p/2 & 1 + p & -p/2 & & \\ & -p/2 & 1 + p & -p/2 & \\ & & -p/2 & 1 + p & -p/2 \\ & & & -p & 1 + p \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} 1 - p(1 + H\,\Delta x) & p & & & \\ p/2 & 1 - p & p/2 & & \\ & p/2 & 1 - p & p/2 & \\ & & p/2 & 1 - p & p/2 \\ & & & p & 1 - p \end{bmatrix}$$

Equation (8·4·53) takes the place of Eq. (8·4·25) for the non-convective boundary case. Again, it should be observed that the solution has not been significantly complicated by the convection boundary condition. Multiplication of the old temperatures by the tridiagonal matrix **B** still gives a set of known values on the right-hand side. You then have a set of simultaneous equations to solve for the new temperatures at the next time.

The most serious limitation that the convection boundary condition imposes is that the critical stability limit for the Euler numerical method of solution is more restrictive. Using the techniques discussed in Sec. 8·4·7, it can be shown that the critical stability limit (beyond which the solution will be unstable) is conservatively given by

$$p_{\text{crit}} = 0.5\,\frac{1}{1 + H\,\Delta x} \tag{8·4·54}$$

This compares to $p_{\text{crit}} = 0.5$ that we derived for the case in which we assumed a step change in surface temperature. For a given nodal spacing, the critical stability limit becomes smaller as $H = hL/k$ increases. To maintain the same stability condition (that is, p = constant), the value of Δx will have to be decreased as the value of H is increased. Since $p = \Delta\theta/(\Delta x)^2$, this means that smaller time steps will have to be used.

One way of easing this stability restriction is to *implicitize* the surface node that is giving the trouble. That is, we will still treat all the interior nodes in an explicit manner, but we will move ahead in time in the implicit way for the surface node at the convective boundary. The resulting matrix equation is

$$\mathbf{A}\mathbf{u}^{(v+1)} = \mathbf{B}\mathbf{u}^{(v)} \tag{8·4·55}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 + p(1 + H\Delta x) & -p & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} 1 - p(1 + H\Delta x) & p & & & \\ p & 1 - 2p & p & & \\ & p & 1 - 2p & p & \\ & & p & 1 - 2p & p \\ & & & 2p & 1 - 2p \end{bmatrix}$$

This is still an explicit solution since no simultaneous algebraic equations must be solved. The values of $u_1^{(\nu+1)}$, $u_2^{(\nu+1)}$, $u_3^{(\nu+1)}$, and $u_4^{(\nu+1)}$ can be solved for immediately. The newly computed value of $u_1^{(\nu+1)}$ can then be substituted into the first equation to find $u_0^{(\nu+1)}$. The solution can then be continued to the next time step. This can easily be extended if both surfaces of the plane wall had convection boundary conditions.

8·5 INTRODUCTION TO ADVANCED PROBLEMS

Thus far, we have considered only problems in which we could have found an exact analytical solution. This was instructive because the approximate finite-difference solutions can then be compared to the exact solution to provide us with an idea of the accuracy of the finite-difference method. It should be apparent that, if enough care is taken (e.g., take small-enough nodal spacings and watch out for stability problems), the finite-difference method will give very good engineering results.

The whole purpose of discussing finite-difference methods is to be able to solve problems either for which there is no analytical solution or for which the analytical solution is too hard to obtain. Even in some cases where you may be able to find an analytical solution, it may be hard to evaluate because of a slowly convergent series that must be summed or some other difficulty. This section discusses a few examples which, in general, must be

handled approximately because of difficulties with exact analytical methods.

Conceptually, there is very little difference in the formulation of these more complicated problems. You have already been exposed to the basic ideas you will need. The only additional difficulties are computational. Although the computations become more involved, the use of the digital computer makes them quite feasible in most cases.

By the time you have studied this section your confidence should have developed to the point where you could tackle almost any conduction problem.

8·5·1 Nonuniform heat-transfer coefficient

The discussion of the thin-rod problem in Sec. 8·2 assumed that the heat-transfer coefficient was a constant along the entire length of the rod. In a more practical case it might be a variable. For example, the flow of air over the rod might be very small near $x = 0$ due to the effect of the wall. This would mean that the heat-transfer coefficient would be low near $x = 0$ and increase further out along the rod. In general such a problem could not be handled analytically.

The finite-difference formulation follows directly from the discussion leading up to Eq. (8·2·1). The only difference is that in the convection rate equation the constant value of h would now have to be replaced by a suitable average value h_m over the system upon which the energy balance is being made. Equation (8·2·1) would then be replaced by

$$-t_{m-1} + D_m t_m - t_{m+1} = 0 \tag{8·5·1}$$

Here, D_m is given by

$$D_m = 2 + \frac{h_m p}{kA}(\Delta x)^2 \tag{8·5·2}$$

Thus the numerical value of D_m would be different for each nodal equation.

After normalization, the resulting system of algebraic equations may be written in matrix form as

$$\mathbf{Au} = \mathbf{c} \tag{8·5·3}$$

where, for four nodes,

$$\mathbf{A} = \begin{bmatrix} D_1 & -1 & & \\ -1 & D_2 & -1 & \\ & -1 & D_3 & -1 \\ & & -2 & D_4 \end{bmatrix}$$

and

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This equation replaces Eq. (8·2·8), which was derived for a uniform heat-transfer coefficient. The solution of Eq. (8·5·3) is obtained in exactly the same manner as it was for Eq. (8·2·8). The only difference is that the numerical values of the diagonal terms have been changed.

8·5·2 Variable conduction area

A thin rod with variable conduction area can sometimes be handled analytically as shown in Chap. 2 (Exercises 2·25 and 2·26). However, if the area does not vary in certain special ways, exact analytical techniques fall down and approximate methods must be employed.

Figure 8·38 shows a typical section of a rod with variable conduction area (and also perimeter). If the cross-sectional area does not vary too rapidly, we can still use a one-dimensional treatment of the problem. As in the constant-area case, the energy balance is given by

$$q_{m-1,m} = q_{m,m+1} + q_{c_m}$$

Now we must modify the rate equations to account for the variable area and perimeter. For the conduction terms we may write

$$q_{m-1,m} = k \frac{A_{m-1} + A_m}{2} \frac{t_{m-1} - t_m}{\Delta x}$$

$$q_{m,m+1} = k \frac{A_m + A_{m+1}}{2} \frac{t_m - t_{m+1}}{\Delta x}$$

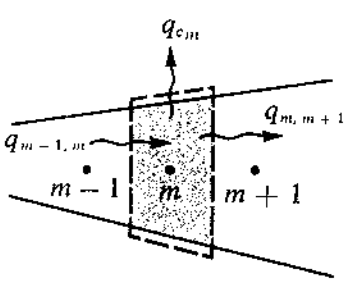


FIG. 8·38 *Typical nodal system for a thin rod with variable conduction area.*

The convection term may be written as

$$q_{c_m} = hp_m \Delta x\, t_m$$

Observe that the perimeter, as well as the cross-sectional area, varies with position. The rate equations can be substituted into the energy balance to arrive at the following equation:

$$-\frac{A_{m-1} + A_m}{2A_m} t_{m-1} + \left[\frac{A_{m-1} + 2A_m + A_{m+1}}{2A_m} + \frac{hp_m}{kA_m}(\Delta x)^2\right] t_m - \frac{A_m + A_{m+1}}{2A_m} t_{m+1} = 0 \quad (8\cdot5\cdot4)$$

Observe that this equation reduces to Eq. (8·2·1) if all the areas are equal. If the areas are different, we will still have a tridiagonal system of equations to solve. For four nodal points, these equations may be written in normalized matrix form as

$$\mathbf{Au} = \mathbf{c} \quad (8\cdot5\cdot5)$$

The normalized temperature is given by $u = (t - t_\infty)/(t_0 - t_\infty)$. For $t_\infty = 0$ this becomes $u = t/t_0$. The matrixes **A** and **c** are given by

$$\mathbf{A} = \begin{bmatrix} D_1 & -b_1 & & \\ -a_2 & D_2 & -b_2 & \\ & -a_3 & D_3 & -b_3 \\ & & -2a_4 & D_4 \end{bmatrix}$$

and

$$\mathbf{c} = \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where the following symbols have been defined:

$$a_m = \frac{A_{m-1} + A_m}{2A_m} \qquad b_m = \frac{A_m + A_{m+1}}{2A_m}$$

and (8·5·6)

$$D_m = \frac{A_{m-1} + 2A_m + A_{m+1}}{2A_m} + \frac{hp_m}{kA_m}(\Delta x)^2$$

The numerical solution of these equations is carried out just as it was for Eqs. (8·2·8). The only difference is that the matrix components have been altered. Each one must be computed separately based upon the cross-sectional-area variation. You should also observe that these equations can incorporate a variable heat-transfer coefficient with relatively little change in complexity. The h in the expression for D_m would have to be replaced by h_m.

8·5·3 Variable thermal properties

In some engineering problems it is necessary to take variations of thermal properties into consideration. When large temperature differences exist, average values of conductivity and specific heat may no longer be acceptable. The temperature dependence of these quantities must be considered. Another possibility is that the material may not be homogeneous in composition. Again, exact analytical methods are not too fruitful an approach in such cases.

As an illustration of how such a problem might be attacked using finite differences, let us consider a one-dimensional, transient problem such as we discussed in Sec. 8·4. Now we will assume that the thermal properties k and c are functions of temperature and position within the wall. The density is a constant in time, but it may vary with position.

Figure 8·23 will still apply as will the energy balance given by Eq. (8·4·1). The alterations will come in writing the rate equations for these energy terms. The conduction terms may be approximated as

$$q_{m-1,m} = \frac{k_{m-1} + k_m}{2} A \frac{t_{m-1} - t_m}{\Delta x}$$

$$q_{m,m+1} = \frac{k_m + k_{m+1}}{2} A \frac{t_m - t_{m+1}}{\Delta x}$$

The thermal energy storage term may be written as[1]

$$\dot{E}_{s_m} = (\rho c)_m A \, \Delta x \frac{dt_m}{d\theta}$$

[1] It is left as an exercise for the student to show that this expression is correct and not

$$\dot{E}_{s_m} = A \, \Delta x \frac{d}{d\theta}(\rho c t)_m$$

Upon substituting the rate equations into the energy balance and rearranging, the following equation is obtained:

$$(\rho c)_m A\,\Delta x\,\frac{dt_m}{d\theta} = \frac{A}{\Delta x}\left(\frac{k_{m-1}+k_m}{2}\,t_{m-1} - \frac{k_{m-1}+2k_m+k_{m+1}}{2}\,t_m + \frac{k_m+k_{m+1}}{2}\,t_{m+1}\right) \tag{8·5·7}$$

Observe that this equation reduces to Eq. (8·4·6) for constant properties if $k_{m-1} = k_m = k_{m+1} = k$ and $(\rho c)_m = \rho c$. Upon dividing by $(\rho c)_m A\,\Delta x$, the above may be written as

$$\frac{dt_m}{d\theta} = \frac{1}{(\Delta x)^2}(a_m t_{m-1} - 2b_m t_m + c_m t_{m+1}) \tag{8·5·8}$$

where the following definitions have been made to simplify matters:

$$a_m = \frac{k_{m-1}+k_m}{2(\rho c)_m}$$

$$2b_m = \frac{k_{m-1}+2k_m+k_{m+1}}{2(\rho c)_m} \tag{8·5·9}$$

$$c_m = \frac{k_m+k_{m+1}}{2(\rho c)_m}$$

The normalization of Eq. (8·5·8) is not as clear cut as it would be for the constant-property case. You will recall that in the constant-property case a nondimensional time was defined as $\bar{\theta} = \alpha\theta/L^2$. In the present case, α is no longer a constant. It is a function of position and time. Some average value (averaged over space and time in some manner) might be used if one insisted on normalizing the equation. Instead of doing this, let us just continue to work with the equation in dimensional form.

The Euler method of solving these equations results in the following matrix relation (for four nodes):

$$\mathbf{t}^{(v+1)} = \mathbf{B}^{(v)}\mathbf{t}^{(v)} + \mathbf{c}^{(v)} \tag{8·5·10}$$

where

$$\mathbf{B}^{(v)} = \begin{bmatrix} 1 - 2pb_1 & pc_1 & & \\ pa_2 & 1 - 2pb_2 & pc_2 & \\ & pa_3 & 1 - 2pb_3 & pc_3 \\ & & 2pa_4 & 1 - 2pb_4 \end{bmatrix}^{(v)}$$

and

$$\mathbf{c}^{(v)} = \begin{bmatrix} pa_1 t_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{(v)}$$

As in previous examples, we have defined $p = \Delta\theta/(\Delta x)^2$, but here p, $\Delta\theta$, and Δx are dimensional since we have not normalized the problem. The matrix $\mathbf{c}^{(v)}$ has appeared since we did not take $t - t_0$ as the temperature variable. If we had done this (or, equivalently, set $t_0 = 0$), the matrix $\mathbf{c}^{(v)}$ would not appear. Even if it is not set equal to zero, we can easily add $\mathbf{c}^{(v)}$ to $\mathbf{B}^{(v)}\mathbf{t}^{(v)}$ at each step in time without much extra work.

The reason for putting the superscript (v) on the matrices $\mathbf{B}^{(v)}$ and $\mathbf{c}^{(v)}$ is that their components may now depend upon time because a_m, b_m, and c_m may now depend upon temperature, which in turn depends upon time. This means that the matrix components of $\mathbf{B}^{(v)}$ and $\mathbf{c}^{(v)}$ will have to be reevaluated at each step in time. That is, given the temperatures $\mathbf{t}^{(v)}$, the components of $\mathbf{B}^{(v)}$ and $\mathbf{c}^{(v)}$ can be calculated. The computations indicated by Eq. (8·5·10) can then be carried out to find $\mathbf{t}^{(v+1)}$. The process is then repeated.

In the constant-property case, Eqs. (8·4·21) may be written as $\mathbf{u}^{(v+1)} = \mathbf{B}\mathbf{u}^{(v)}$. The matrix $\mathbf{B}$ is calculated only once and used over and over again. If the properties vary with position only, the superscript (v) may be dropped from $\mathbf{B}$ and $\mathbf{c}$ in Eq. (8·5·10) because these no longer depend upon temperature. This considerably simplifies the computational problem since, in this case, $\mathbf{B}$ and $\mathbf{c}$ are constants and do not need to be reevaluated at each time step. This is then no more difficult than the constant-property case once the components of $\mathbf{B}$ and $\mathbf{c}$ are computed at the start of the problem.

The Crank-Nicolson method of moving ahead in time leads to more complications than the Euler method mentioned above. The matrix equation for the implicit method turns out to be

$$\mathbf{A}^{(v+1)}\mathbf{t}^{(v+1)} = \mathbf{B}^{(v)}\mathbf{t}^{(v)} + \mathbf{c}^{(v)} + \mathbf{c}^{(v+1)} \tag{8·5·11}$$

where, for four nodes,

$$\mathbf{A}^{(v+1)} = \begin{bmatrix} 1 + pb_1 & -\frac{pc_1}{2} & & \\ -\frac{pa_2}{2} & 1 + pb_2 & -\frac{pc_2}{2} & \\ & -\frac{pa_3}{2} & 1 + pb_3 & -\frac{pc_3}{2} \\ & & -pa_4 & 1 + pb_4 \end{bmatrix}^{(v+1)}$$

$$\mathbf{B}^{(v)} = \begin{bmatrix} 1 - pb_1 & \frac{pc_1}{2} & & \\ \frac{pa_2}{2} & 1 - pb_2 & \frac{pc_2}{2} & \\ & \frac{pa_3}{2} & 1 - pb_3 & \frac{pc_3}{2} \\ & & pa_4 & 1 - pb_4 \end{bmatrix}^{(v)}$$

$$\mathbf{c}^{(v)} = \begin{bmatrix} \frac{pa_1 t_0}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}^{(v)} \quad \text{and} \quad \mathbf{c}^{(v+1)} = \begin{bmatrix} \frac{pa_1 t_0}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}^{(v+1)}$$

To reduce the problem to its simplest form, let us take $t_0 = 0$ in order to get rid of the matrix **c**. Equation (8·5·11) then reduces to

$$\mathbf{A}^{(v+1)}\mathbf{t}^{(v+1)} = \mathbf{B}^{(v)}\mathbf{t}^{(v)} \tag{8·5·12}$$

The right-hand side of this equation may be computed with no more difficulty than we had in the explicit case. If we call the

result $\mathbf{r}^{(v)}$, we get

$$\mathbf{A}^{(v+1)}\mathbf{t}^{(v+1)} = \mathbf{r}^{(v)} \tag{8·5·13}$$

We now have a system of equations to solve. The trouble is that the coefficient matrix now depends upon the unknown temperatures. We must know $\mathbf{t}^{(v+1)}$ to evaluate the components of $\mathbf{A}^{(v+1)}$. We cannot solve for the temperatures by the usual gaussian elimination scheme unless we know the components of the matrix $\mathbf{A}^{(v+1)}$, and the components of this matrix are functions of the unknown temperatures that we are looking for.

To solve this problem we can use an iterative technique. The first step would be to take $\mathbf{A}^{(v+1)} \approx \mathbf{A}^{(v)}$. That is, let us assume that the components in matrix $\mathbf{A}^{(v+1)}$ are not too different than if they were evaluated using temperatures from the previous time $\mathbf{t}^{(v)}$. Under this assumption, the matrix $\mathbf{A}^{(v+1)}$ is known approximately, and the system of equations may be solved for the unknown temperatures $\mathbf{t}^{(v+1)}$. If the time steps are not too large and/or the property variations with temperature are not too strong, this result may be quite satisfactory. In practice one often reduces the size of the time step so that this result is adequate. If you want to improve your solution, you could now use the values $\mathbf{t}^{(v+1)}$ that you have just found to recompute the components in $\mathbf{A}^{(v+1)}$. An improved solution can then be found by then solving again for the unknown temperatures $\mathbf{t}^{(v+1)}$, using the improved values for the components of $\mathbf{A}^{(v+1)}$. This process could be continued.

An alternative approach for obtaining a first approximation to $\mathbf{A}^{(v+1)}$ would be to use thermal properties extrapolated from the known values at $\theta^{(v)}$ and earlier. That is, conductivity at $\theta^{(v+1)}$ could be estimated as

$$k^{(v+1)} = k^{(v)} + \left.\frac{\partial k}{\partial \theta}\right|^{(v)} \Delta\theta$$

The time derivative can be rewritten using the chain rule to give

$$k^{(v+1)} = k^{(v)} + \left.\frac{\partial k}{\partial t}\right|^{(v)} \left.\frac{\partial t}{\partial \theta}\right|^{(v)} \Delta\theta$$

The derivative of conductivity with respect to temperature is a material property. It may be almost a constant or it may be a function of temperature, in which case it might have to be evaluated numerically by the computer from a table of conductivity as a function of temperature. The derivative of temperature with respect to time might be approximated as being zero at the start of the problem, but once the first step in time had been com-

pleted, it could then be approximated as

$$\left.\frac{\partial t}{\partial \theta}\right|^{(v)} = \frac{t^{(v)} - t^{(v-1)}}{\Delta\theta}$$

Thus, for later times, conductivity could be computed as

$$k^{(v+1)} = k^{(v)} + \left.\frac{\partial k}{\partial t}\right|^{(v)} (t^{(v)} - t^{(v-1)})$$

The other properties in $\mathbf{A}^{(v+1)}$ can be computed in the same way. The purpose of this extra work in making a first estimate of $\mathbf{A}^{(v+1)}$ is that it may be a good enough approximation so that iteration can be avoided without having to take too small a time step.

The consideration of variable properties has increased the computational time necessary to solve the problem, but nevertheless we have a way to attack this complicated problem. The use of the digital computer considerably eases the burden of making these calculations.

8·5·4 Radiation boundary condition

Radiation is hard to handle by exact analytical methods because of its dependence on the fourth power of the temperature. This makes the problem nonlinear. Although the computations become more involved, the finite-difference method is quite able to handle such problems.

As an illustration of how one might handle a radiation boundary condition, let us consider the plane-wall transient once again. This time let us replace the convection boundary condition (discussed in Sec. 8·4·8) with a radiation boundary condition. The energy balance for the surface node will be the same as given by Eq. (8·4·46) except that q_{c_0} will be replaced by q_{r_0}, a radiation heat-transfer term. The conduction and storage rate equations will remain unchanged as given by Eqs. (8·4·48) and (8·4·49). From Eq. (1·1·4) the radiation rate equation can be written as

$$q_{r_0} = \epsilon A\sigma(T_\infty^4 - T_0^4) \tag{8·5·14}$$

When considering radiation, we must be careful to use the absolute temperature scale.

Combining the energy balance and the rate equation then gives

$$\epsilon A\sigma(T_\infty^4 - T_0^4) = \frac{kA}{\Delta x}(T_0 - T_1) + \rho cA\frac{\Delta x}{2}\frac{dT_0}{d\theta} \tag{8·5·15}$$

This equation replaces Eq. (8·4·50), which was derived for the convective boundary condition.

One possible way of handling Eq. (8·5·15) is to linearize the radiation term. This is done by factoring the difference in the fourth power of the temperatures as follows:

$$T_\infty^4 - T_0^4 = (T_\infty^2 + T_0^2)(T_\infty^2 - T_0^2)$$
$$= (T_\infty^2 + T_0^2)(T_\infty + T_0)(T_\infty - T_0)$$

This may now be substituted into Eq. (8·5·15) to give

$$h_r A(T_\infty - T_0) = \frac{kA}{\Delta x}(T_0 - T_1) + \rho cA \frac{\Delta x}{2}\frac{dT_0}{d\theta} \tag{8·5·16}$$

where

$$h_r = \epsilon\sigma(T_\infty^2 + T_0^2)(T_\infty + T_0) \tag{8·5·17}$$

has been defined to make Eq. (8·5·16) have the same appearance as Eq. (8·4·50). This radiation coefficient h_r is really a variable since it depends upon the surface temperature T_0, which changes.

The problem may be linearized by now assuming that the value of T_0 in Eq. (8·5·17) may be replaced by T_∞. Then

$$h_r = \epsilon\sigma(T_\infty^2 + T_\infty^2)(T_\infty + T_\infty) = 4\epsilon\sigma T_\infty^3 \tag{8·5·18}$$

The radiation heat-transfer coefficient is now a constant, and Eq. (8·5·16) may be treated exactly as we did Eq. (8·4·50) for convection at the boundary.

For cases in which T_0 is not close to T_∞, this linearization will not be adequate and the nonlinear equation (8·5·15) must be handled directly. The first thing we will do is to normalize the problem by defining $u = T/T_i$ so that the value of the normalized variable u will always be between 0 and 1.* We will also normalize x and θ in the usual manner. With these definitions, Eq. (8·5·15) becomes

$$\frac{2\epsilon\sigma L T_i^3}{k\,\Delta x}(u_\infty^4 - u_0^4) = \frac{2}{(\Delta x)^2}(u_0 - u_1) + \frac{du_0}{d\theta}$$

This may be rewritten as

$$\frac{du_0}{d\theta} = \frac{1}{(\Delta x)^2}[-2(1 + R\,\Delta x\,u_0^3)u_0 + 2u_1] + \frac{2R}{\Delta x}u_\infty^4 \tag{8·5·19}$$

where $R = \epsilon\sigma L T_i^3/k$ has been defined for convenience.

Equation (8·5·19) takes the place of the first of Eqs. (8·4·51).

*We are assuming that T_i is greater than T_∞.

Insofar as possible it has been written in a comparable form. The most serious difference is that u_0 appears to the fourth power in Eq. (8·5·19). This nonlinearity will complicate the solution of the problem. It should also be observed that, in this nonlinear equation, it is impossible to absorb the ambient temperature into the nondimensional temperature as we usually did in the linear cases we have studied. Thus we will have to be content to carry along an additive constant in this equation. The remaining equations for the interior nodes are the same as those given in Eqs. (8·4·51).

To see what is involved in the numerical solution of this set of differential equations, let us first consider the Euler method for moving ahead in time. The matrix representation of the numerical problem is given by

$$\mathbf{u}^{(v+1)} = \mathbf{B}^{(v)}\mathbf{u}^{(v)} + \mathbf{c} \tag{8·5·20}$$

where, for $\Delta x = \frac{1}{4}$,

$$\mathbf{B}^{(v)} = \begin{bmatrix} 1 - 2p(1 + R\,\Delta x\,u_0^3) & 2p & & & \\ p & 1-2p & p & & \\ & p & 1-2p & p & \\ & & p & 1-2p & p \\ & & & 2p & 1-2p \end{bmatrix}^{(v)}$$

and

$$\mathbf{c} = \begin{bmatrix} 2pR\,\Delta x\,u_\infty^4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The major difference between Eq. (8·5·20) and the corresponding equation for the convection boundary condition, Eq. (8·4·52), is that the matrix $\mathbf{B}$ now depends upon time. This is denoted by writing it as $\mathbf{B}^{(v)}$. This does not cause much difficulty, however. It simply means that the one component of $\mathbf{B}^{(v)}$ which contains the temperature will have to be recomputed at each time step before going on to the next step. Since the value of u_0 that is needed in this computation is known, $u_0^{(v)}$, there is no difficulty. The constant matrix $\mathbf{c}$ will have to be added to the product $\mathbf{B}^{(v)}\mathbf{u}^{(v)}$ at each step. This is no problem either since $\mathbf{c}$ is known

and constant. The numerical solution can be carried out on the computer with no difficulty.

It should also be mentioned in passing that the stability situation will be different than it was in previous examples. It will be more restrictive than the original case we considered of a step change in surface temperature where we found $p_{crit} = 0.5$. This is the same thing that happened to us when we considered the convective boundary condition in Sec. 8·4·8.

The Crank-Nicolson method of solution may also be used to solve this problem. This will significantly complicate the problem, however. The matrix equation that results is

$$\mathbf{A}^{(v+1)}\mathbf{u}^{(v+1)} = \mathbf{B}^{(v)}\mathbf{u}^{(v)} + \mathbf{c} \tag{8·5·21}$$

where

$$\mathbf{A}^{(v+1)} = \begin{bmatrix} 1 + p + pR\,\Delta x\,u_0^3 & -p & & & \\ -p/2 & 1 + p & -p/2 & & \\ & -p/2 & 1 + p & -p/2 & \\ & & -p/2 & 1 + p & -p/2 \\ & & & -p & 1 + p \end{bmatrix}^{(v+1)}$$

$$\mathbf{B}^{(v)} = \begin{bmatrix} 1 - p - pR\,\Delta x\,u_0^3 & p & & & \\ p/2 & 1 - p & p/2 & & \\ & p/2 & 1 - p & p/2 & \\ & & p/2 & 1 - p & p/2 \\ & & & p & 1 - p \end{bmatrix}^{(v)}$$

$$\mathbf{c} = \begin{bmatrix} 2pR\,\Delta x\,u_\infty^4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice that the right-hand side of Eq. (8·5·21) has the same general form as in the Euler method (8·5·20). When the matrix operations on the right-hand side are carried out, the following matrix equation is obtained:

$$\mathbf{A}^{(v+1)}\mathbf{u}^{(v+1)} = \mathbf{r}^{(v)} \tag{8·5·22}$$

The right-hand matrix $\mathbf{r}^{(v)}$ is completely known.

A major difficulty arises on the left-hand side of the equation because $\mathbf{A}^{(v+1)}$ is a function of one of the unknown temperatures at the next time, $u_0^{(v+1)}$. In most of the previous examples we have discussed, the matrix **A** was a constant, independent of temperature. In these cases we had a simple, linear set of algebraic equations to solve for the new temperatures. We could use gaussian elimination on the matrix **A** since it was entirely known.

In the variable thermal-property case every component of **A** depended upon temperature, and we were forced to do an iterative solution of the entire system of equations. We could also do this in the present problem. The first step in the iterative process would be to assume that $\mathbf{A}^{(v+1)} \approx \mathbf{A}^{(v)}$. If we only take the first step in this process and replace Eq. (8·5·22) by $\mathbf{A}^{(v)}\mathbf{u}^{(v+1)} = \mathbf{r}^{(v)}$, we are effectively linearizing the radiation term once again but in a slightly different way. We are replacing $T_0^{(v+1)}$ by $T_0^{(v)}$ rather than by T_∞.* Since only one row of the matrix contains an unknown temperature, a simpler scheme can be employed to avoid linearization or iteration of the entire matrix. This scheme makes use of gaussian elimination for most of the matrix and an iterative solution only for the row containing the unknown temperature.

As an illustration of how this gaussian elimination–iteration scheme works, let us consider a numerical example. Suppose we take $R\,\Delta x = 1.0$, $p = 0.5$, and $u_\infty = 0.7$. The matrices then reduce to

$$\mathbf{A}^{(v+1)} = \begin{bmatrix} 1.50 + 0.50u_0^3 & -0.50 & & & \\ -0.25 & 1.50 & -0.25 & & \\ & -0.25 & 1.50 & -0.25 & \\ & & -0.25 & 1.50 & -0.25 \\ & & & -0.50 & 1.50 \end{bmatrix}^{(v+1)}$$

$$\mathbf{B}^{(v)} = \begin{bmatrix} 0.50 - 0.50u_0^3 & 0.50 & & & \\ 0.25 & 0.50 & 0.25 & & \\ & 0.25 & 0.50 & 0.25 & \\ & & 0.25 & 0.50 & 0.25 \\ & & & 0.50 & 0.50 \end{bmatrix}^{(v)}$$

*Still another method of linearization is suggested in Exercise 8·23.

$$\mathbf{c} = \begin{bmatrix} 0.240 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The value of u_0 that appears in each of the first two matrices will change as time goes on. The value that appears in $\mathbf{B}^{(v)}$ will always be known, but the value in $\mathbf{A}^{(v+1)}$ will be unknown. If, for example, we start out from the uniform initial condition $\mathbf{u}^{(0)} = 1.0$, the first component of $\mathbf{B}^{(0)}$ becomes zero. The matrix operations on the right-hand side may now be carried out as follows:

$$\mathbf{r}^{(0)} = \mathbf{B}^{(0)}\mathbf{u}^{(0)} + \mathbf{c}$$

$$= \begin{bmatrix} 0.00 & 0.50 & & & \\ 0.25 & 0.50 & 0.25 & & \\ & 0.25 & 0.50 & 0.25 & \\ & & 0.25 & 0.50 & 0.25 \\ & & & 0.50 & 0.50 \end{bmatrix} \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \end{bmatrix} + \begin{bmatrix} 0.240 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.50 \\ 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \end{bmatrix} + \begin{bmatrix} 0.240 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.740 \\ 1.000 \\ 1.000 \\ 1.000 \\ 1.000 \end{bmatrix}$$

We have now arrived at the following matrix equation:

$$\begin{bmatrix} 1.50 + 0.50u_0^3 & -0.50 & & & \\ -0.25 & 1.50 & -0.25 & & \\ & -0.25 & 1.50 & -0.25 & \\ & & -0.25 & 1.50 & -0.25 \\ & & & -0.50 & 1.50 \end{bmatrix}^{(1)} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(1)} = \begin{bmatrix} 0.740 \\ 1.000 \\ 1.000 \\ 1.000 \\ 1.000 \end{bmatrix}$$

At this point we have a set of algebraic equations that must now be solved. Since the first row has an unknown component, we will begin in the lower right corner of the coefficient matrix and eliminate all the components above the diagonal. The last row can be multiplied by 0.25/1.50 and added to the next-to-last

row to give

$$\begin{bmatrix} 1.50 + 0.50u_0^3 & -0.50 & & & \\ -0.25 & 1.50 & -0.25 & & \\ & -0.25 & 1.50 & -0.25 & \\ & & -0.25 & 1.417 & \\ & & & -0.50 & 1.50 \end{bmatrix}^{(1)} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(1)} = \begin{bmatrix} 0.740 \\ 1.000 \\ 1.000 \\ 1.167 \\ 1.000 \end{bmatrix}$$

Now the next-to-last equation may be multiplied by 0.25/1.417 and added to the middle equation. This process can be continued until all the terms above the main diagonal have been eliminated. At that point we have the following situation:

$$\begin{bmatrix} 1.414 + 0.500u_0^3 & & & & \\ -0.250 & 1.457 & & & \\ & -0.250 & 1.456 & & \\ & & -0.250 & 1.417 & \\ & & & -0.500 & 1.500 \end{bmatrix}^{(1)} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}^{(1)} = \begin{bmatrix} 1.154 \\ 1.207 \\ 1.206 \\ 1.167 \\ 1.000 \end{bmatrix} \quad (8\cdot5\cdot23)$$

If we knew the value of the first component in the coefficient matrix, we could now begin the back-substitution process to solve for the unknowns in the usual gaussian elimination manner. In this case, however, the unknown u_0 appears in this component, and we are forced to handle this equation differently.

Let us now examine the first equation by itself. It may be written as

$$1.414u_0^{(1)} + 0.500u_0^{(1)^4} = 1.154 \quad (8\cdot5\cdot24)$$

The only thing that changes in this equation at the succeeding time steps would be the constants 1.414 and 1.154 and the superscripts denoting the time step. We have the same general problem to solve at each time step.

To consider the solution of Eq. (8·5·24) in a more general way, let us rewrite it as follows:

$$f(\xi) = 0.500\xi^4 + a\xi - b \quad (8\cdot5\cdot25)$$

In this expression we have let $1.414 = a$ and $1.154 = b$ and replaced the unknown $u_0^{(1)}$ by ξ. A qualitative picture of $f(\xi)$ is shown in Fig. 8·39. We are looking for the point at which $f(\xi) = 0$.

The Newton-Raphson technique of finding the zero of this function is a straightforward one to use. The first step is to assume a value of ξ, call it ξ_k in general, and then evaluate $f(\xi_k)$ and the derivative of the function, $f'(\xi_k)$, at this assumed value of ξ. An improved value of ξ, call it ξ_{k+1}, is then computed by "sliding down the tangent" at ξ_k to the point where it crosses the

ξ axis as shown in Fig. 8·39. A mathematical relation among these quantities may be obtained from geometry by writing the slope as

$$f'(\xi_k) = \frac{f(\xi_k) - 0}{\xi_k - \xi_{k+1}}$$

Solving for ξ_{k+1} gives

$$\xi_{k+1} = \xi_k - \frac{f(\xi_k)}{f'(\xi_k)} \tag{8·5·26}$$

This improved value ξ_{k+1} can then be used to begin the process again.

Equation (8·5·26) may be simplified by substituting Eq. (8·5·25) to give

$$\xi_{k+1} = \xi_k - \frac{0.500\xi_k^4 + a\xi_k - b}{2.000\xi_k^3 + a}$$

These terms may then be combined to give the following result:

$$\xi_{k+1} = \frac{1.500\xi_k^4 + b}{2.000\xi_k^3 + a} \tag{8·5·27}$$

The values of a and b (1.414 and 1.154 in this first time step) come from the gaussian elimination procedure. The unknown that we are solving for ($u_0^{(1)}$ in this first time step) is the surface temperature at the new time. The easiest first guess for the temperature at the new time is the temperature at the old time (in this case, $u_0^{(0)} = 1.0$). The calculations then proceed as follows:

ξ_1 = (guessed from last-known value of u_0) = 1.000

$$\xi_2 = \frac{1.500(1.0)^4 + 1.154}{2.000(1.0)^3 + 1.414} = 0.777$$

$$\xi_3 = \frac{1.500(0.777)^4 + 1.154}{2.000(0.777)^3 + 1.414} = 0.723$$

$$\xi_4 = \frac{1.500(0.723)^4 + 1.154}{2.000(0.723)^3 + 1.414} = 0.721$$

$$\xi_5 = \frac{1.500(0.721)^4 + 1.154}{2.000(0.721)^3 + 1.414} = 0.721$$

It should be observed that fairly rapid convergence to the solution has been obtained. This computation is easily programmed for the computer.

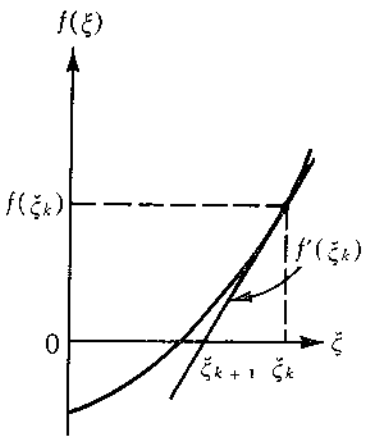


FIG. 8·39 *Newton-Raphson technique for finding the zero of a function.*

We have just found $u_0^{(1)} = 0.721$. This value can then be used in the second row of Eq. (8·5·23) to begin the usual back-substitution process in the gaussian elimination method. This results in the following solution:

$$\mathbf{u}^{(1)} = \begin{bmatrix} 0.721 \\ 0.952 \\ 0.992 \\ 0.999 \\ 1.000 \end{bmatrix}$$

We are now ready to move on to the next time step.

We have just seen how a single radiation boundary condition can be handled in a one-dimensional, transient problem. The basic solution technique was still the gaussian elimination method, but we had to do a special iterative solution to handle the single nonlinear equation. If we had had a radiation boundary condition at each surface of the wall, the problem would be more involved. We would not be able to even start the gaussian elimination process because we would have unknown components in each corner of the coefficient matrix. If one of the two radiation boundary conditions was not too critical, it could be linearized. This would allow the procedure we have just discussed to be used. If neither one of the boundary conditions can be linearized, you would then probably either iterate the entire matrix to find a solution at each time step[1] or else return to an explicit scheme for moving ahead in time.

8·5·5 Irregular regions

One of the most common situations in heat-transfer calculations is to be required to obtain a temperature distribution in an irregularly shaped region. This also happens to be a situation that can rarely be handled by exact analytical methods. Finite-difference methods can be adapted to these problems, however.

To show how finite-difference methods can be applied to a problem with an irregular boundary, let us consider a square mesh of grid points near a boundary that does not coincide with any of the nodal points as shown in Fig. 8·40. Observe that the

[1] You could guess a value for the least-sensitive surface temperature, carry out the elimination-iteration procedure to find the temperature at the other surface, back substitute to find the internal temperatures, and iterate the final equation to obtain a better value of the surface temperature that was guessed initially. The entire process could then be continued to convergence.

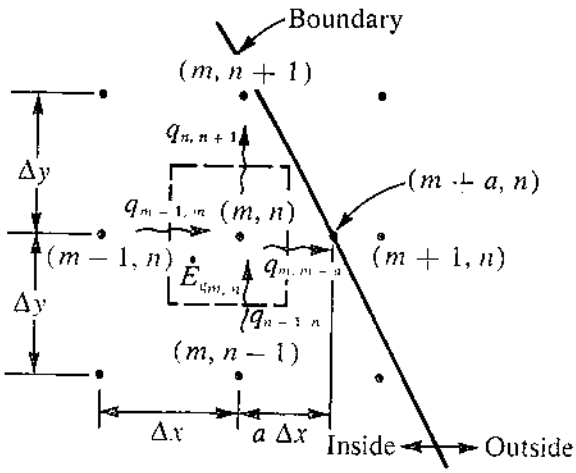


FIG. 8·40 *Typical two-dimensional nodal system near an irregular boundary.*

node $(m + 1, n)$ lies outside the boundary of the region of interest. When this occurs we will introduce a new point $(m + a, n)$ that is on the boundary and use it in the formulation of the problem in place of the point outside the boundary.[1]

If we assume that this is a steady-state problem with uniform generation, the important energy terms are those shown in Fig. 8·40. The energy balance may be written as

$$q_{m-1,m} + q_{n-1,n} + \dot{E}_{g_{m,n}} = q_{m,m+a} + q_{n,n+1}$$

The rate equations may then be written as follows:

$$q_{m-1,m} = k\,\Delta y \frac{t_{m-1,n} - t_{m,n}}{\Delta x}$$

$$q_{n-1,n} = k\left(\frac{\Delta x}{2} + \frac{a\,\Delta x}{2}\right)\frac{t_{m,n-1} - t_{m,n}}{\Delta y}$$

$$\dot{E}_{g_{m,n}} = g'''\left(\frac{\Delta x}{2} + \frac{a\,\Delta x}{2}\right)\Delta y$$

$$q_{m,m+a} = k\,\Delta y \frac{t_{m,n} - t_{m+a,n}}{a\,\Delta x}$$

$$q_{n,n+1} = k\left(\frac{\Delta x}{2} + \frac{a\,\Delta x}{2}\right)\frac{t_{m,n} - t_{m,n+1}}{\Delta y}$$

The only difference in these rate equations from what was done in Sec. 8·3·1 is that some of the distances have been modified to reflect the nonuniform spacing between the nodes.

These rate equations can now be substituted into the energy

[1] If point $(m, n + 1)$ was also outside the boundary, we could then introduce a second new point $(m, n + b)$ on the boundary.

balance to give (for $\Delta x = \Delta y$)

$$k(t_{m-1,n} - t_{m,n}) + k\left(\frac{1+a}{2}\right)(t_{m,n-1} - t_{m,n}) + g'''\left(\frac{1+a}{2}\right)(\Delta x)^2 = k\left(\frac{1}{a}\right)(t_{m,n} - t_{m+a,n}) + k\left(\frac{1+a}{2}\right)(t_{m,n} - t_{m,n+1})$$

The terms in this equation can now be combined and rearranged to give the following result:

$$-at_{m-1,n} - \frac{a(1+a)}{2}t_{m,n-1} + (1+a)^2 t_{m,n} - \frac{a(1+a)}{2}t_{m,n+1} - t_{m+a,n} = \frac{a(1+a)}{2}\frac{g'''(\Delta x)^2}{k} \quad (8\cdot5\cdot28)$$

The only change that has occurred in this development is that the coefficients in Eq. (8·5·28) have different numerical values than the corresponding equation (8·3·1) that was developed for the case in which all the nodal points were inside the boundary. It should be observed that both equations become identical for $a = 1$ as you would expect.

The finite-difference formulation still reduces to a set of simultaneous algebraic equations to solve which have the same general character as Eq. (8·3·7). Some of the numerical values of the components in the coefficient matrix will be changed as will some of the components in the column matrix on the right-hand side of Eq. (8·3·7). The computer solution of the equations would be carried out in exactly the same manner as described in Sec. 8·3.

An irregularly shaped boundary may influence the spacing of the nodal points. The mesh size should be small enough so that a reasonable approximation of the boundary may be made. Figure 8·41*a* shows a nodal spacing that would be too large for

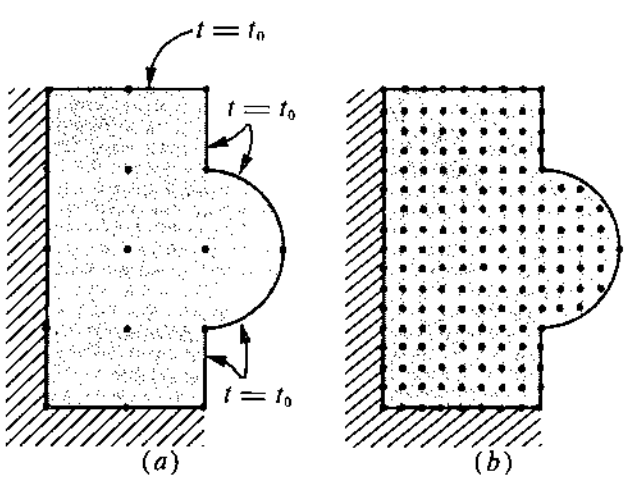


FIG. 8·41 *Coarse and fine nodal-point arrangements for an irregular region.*

the region of interest. The temperature distribution in a semicircular region would not be very well determined by the node at the center of the circle. A smaller mesh size is called for. Figure 8·41*b* would be much better since the circular region has many more points than before. Notice that as more points are added the curved boundary can be closely approximated by the nodal points themselves. This suggests another way of handling a curved boundary. Let us approximate the boundary by a set of line segments connecting the nodes nearest the boundary as shown in Fig. 8·42. This would avoid having to modify the equations near the boundaries as we did in arriving at Eq. (8·5·28). The temperatures at the nodal points on this approximate boundary would then be taken to be t_0 as given by the boundary condition.

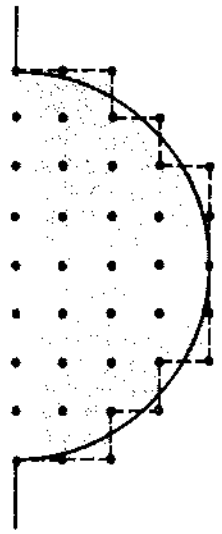


FIG. 8·42 *Approximation of an irregular region with a uniform nodal-point arrangement.*

To get a better solution within the circular region, we had to make the mesh size smaller. By decreasing Δx by a factor of 4, we have increased the number of unknown nodal temperatures from 9 to 154. This means that we would have to do a lot more computational work to find a solution. In a practical case it may not be necessary to increase the number of nodes over the entire region to improve the accuracy near a particular portion of the boundary. This leads us into the idea of a graded network as discussed in the next section.

8·5·6 Graded networks

The selection of the number and placement of the nodes in a problem is governed by two considerations. First, the more nodes you have the more accurate you expect the solution to be. Second, the fewer nodes you have, the smaller the work and time required to solve the problem. The notion of graded networks can help to give you the advantage of accuracy where needed without unduly complicating portions of the problem where accuracy is not as critical.

In the discussion in the previous section, Fig. 8·41*b* showed a network of nodal points that might be about right for the circular region but which might be too fine a spacing for the rectangular region.[1] To save on the number of nodes we might like to use a graded network as shown in Fig. 8·43. This pattern contains 58 unknown nodal temperatures, which is quite a saving compared to the 154 nodes shown in Fig. 8·41*b*.

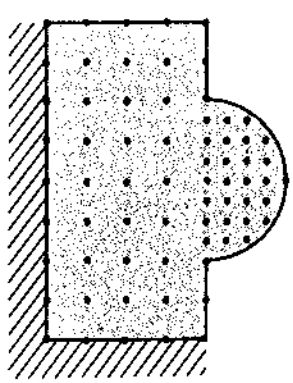


FIG. 8·43 *Graded network for an irregular region.*

Arriving at the nodal equations for nodes at the interface

[1]This assumes that the temperatures in the rectangular region are not as critical to us.

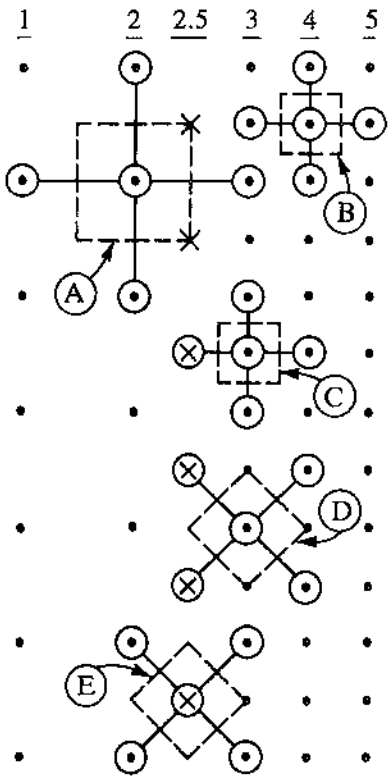


FIG. 8·44 *Interface region of a graded network.*

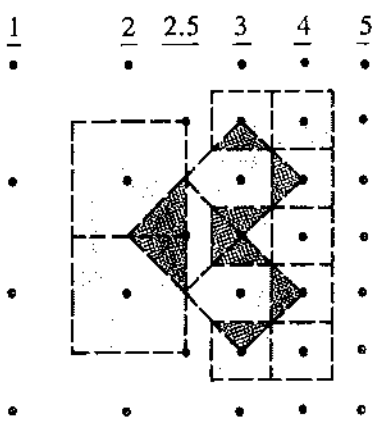


FIG. 8·45 *Overlapping in the interface region of a graded network.*

between regions of two different mesh sizes requires special consideration. Figure 8·44 considers this interface region in more detail. Two columns (1 and 2) of the coarse network and three columns (3, 4, and 5) of the fine network are shown. The nodal equation for a typical node in column 2 is found by making an energy balance on system (A). The resulting equation would involve the central temperature and the four surrounding ones as indicated in the figure. A node in column 4 would be handled by considering system (B).

To get nodal equations for the nodes in column 3 we must introduce some intermediate nodal points between columns 2 and 3. These are the nodes shown in column 2.5. Half the nodal equations for column 3 may then be found by considering a system like (C), and the other half may be found by considering a system like (D). Notice that system (D) is at 45° relative to the coarse and fine networks. Its size is in between the coarse and fine sizes. Nodal equations for the nodes in column 2.5 are determined by considering systems like (E).

When using three different network sizes as in Fig. 8·44, you must be especially careful to use the correct equations for each node. Conceptually, however, there is no additional difficulty. A set of simultaneous algebraic equations will be obtained—one equation for each node whose temperature is unknown.

One of the difficulties that arises in using a graded network is shown in Fig. 8·45. This figure shows the systems that are used in making energy balances on each node. Each system would have a resistance and a capacitance associated with it. If the nodal pattern was regular, the sum of these nodal resistances and capacitances would approximate the total resistance and capacitance of the region. Notice, however, in Fig. 8·45 that for a graded network these systems overlap. Thus some of the regions are counted twice. This will give too much resistance and too much capacitance. If this interface region is relatively small compared to the entire region, not too much error will be introduced. In some cases, however, the entire region may be graded and significant errors can be introduced. In this case the nodal resistances and capacitances would have to be modified in some manner to reduce this problem.

8·6 SUMMARY REMARKS ON FINITE DIFFERENCES

This chapter has developed the fundamental ideas of the finite-difference method of finding approximate solutions to heat-conduction problems. With the aid of the digital computer the

finite-difference method is capable of handling almost any problem (if you are willing to do the computations). The basic ideas were presented in Secs. 8·2 to 8·4. These were then extended in Sec. 8·5 to incorporate some of the complications that arise in practice. The extension of this chapter to three-dimensional, steady-state problems and to two- and three-dimensional transient problems would follow the same basic ideas as those we have developed in this chapter.

One of the headaches that must be faced in handling practical problems is that computational times can become quite long (even on the computer). This is especially true in two- and three-dimensional transient problems. Bandwidths can be quite large so that implicit methods become too costly. At the same time, however, the boundary conditions and irregular nodal spacing can cause severe restrictions in the allowable time-step size in the Euler method because of stability considerations. There are many alterations which people are forced to make to save on computational time or to improve stability limits. The particular modifications that must be made for a specific problem are usually determined by personal experience and trial and error. Textbooks and research papers can only serve as guides to suggest possible alternatives. The references at the end of this chapter contain introductions to some of these techniques.

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1. Forsythe, G. E., and C. B. Moler: *Computer Solution of Linear Algebraic Systems*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967.

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4. Ketter, R. L., and S. P. Prawel: *Modern Methods of Engineering Computation*, McGraw-Hill Book Company, New York, 1969.

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EXERCISES

8·1 Use gaussian elimination to solve the following set of algebraic equations:

$$\begin{aligned} 4u_1 - u_2 &= 2 \\ -u_1 + 4u_2 - u_3 &= 4 \\ -2u_2 + 4u_3 &= 8 \end{aligned}$$

8·2 Carry out the following matrix operations:

(*a*) $\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(*b*)
$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

(*c*)
$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(*d*)
$$\det \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{bmatrix}$$

8·3 The convective heat loss from the nodal system in the thin-rod problem is given by

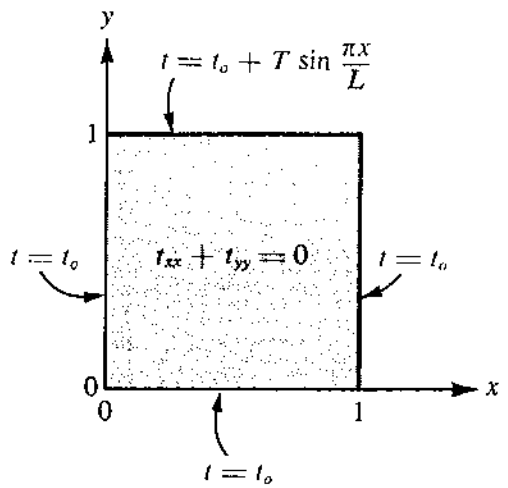
$$q_{c_m} = \int_{x_m - \Delta x/2}^{x_m + \Delta x/2} hpt(x)\, dx$$

In Sec. 8·2·1 it was assumed that t_m was a good average value to use for the temperature. Thus the above integral was replaced by $q_{c_m} = hpt_m\, \Delta x$. A better approximation is to actually integrate an assumed piecewise-linear profile. Derive the finite-difference equations using this better approximation of the convection loss. Compare your result to Sec. 8·2·1.

8·4 Using finite differences, determine the steady-state temperature distribution in a plane wall with uniform energy generation inside it. Take the thickness of the wall to be $2L$ and each surface to be held at temperature t_0. Compare your result to the exact solution.

8·5 Solve Exercise 1·17 using finite-difference techniques. Derive all expressions you need. Take node 0 at one end of the rod and node 4 midway between the two ends of the rod.

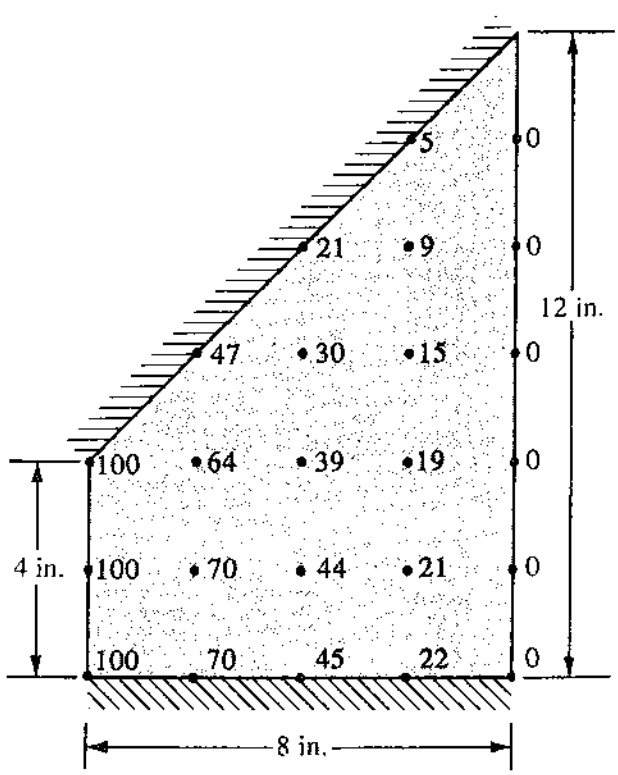
8·6 Using finite differences, obtain a solution for the temperature distribution in the square plate shown. Compare your result to the exact solution obtained in Exercise 3·36.



8·7 Using the finite-difference results in Table 8·2, verify that the heat loss through the side $x = L$ is equal to $0.5000\, g'''L^2$ as given by the exact result.

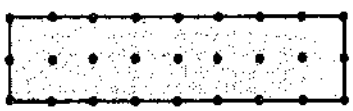
8·8 The temperatures obtained by a finite-difference solution for a two-dimensional conduction problem are as shown. For those results,

(*a*) Derive the finite-difference equation for the node whose temperature is 39°F and verify that the above solution approximately satisfies this equation.



(*b*) Evaluate the heat-transfer rate per foot depth (in Btu/hr-ft) between the surface whose temperature is 100°F and the surface whose temperature is 0°F. Take $k = 1.0$ Btu/hr-ft-°F.

8·9 The 27 nodal positions for a two-dimensional, steady-state conduction problem in a rectangular region are arranged as shown. Number the nodal points to obtain the minimum bandwidth in the resulting system of 27 algebraic equations. Determine the upper bandwidth.



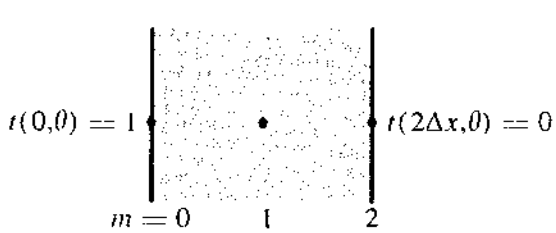
8·10 Determine an algebraic expression for λ as a function of p for the Crank-Nicolson curve in Fig. 8·34.

8·11 Reduce Eqs. (8·4·10) to a system of two equations corresponding to nodal points at $x = 0.5$ and $x = 1$. Solve these equations by the Euler method, the Crank-Nicolson method, and the pure implicit method for values of $p = \Delta\theta/(\Delta x)^2$ of 0.25, 0.5, and 1.0. Compute your solution out to $\theta = 1$. Discuss your results with regard to the stability curves in Fig. 8·34.

8·12 Consider a plane wall whose width is $L = 2\Delta x$. The temperature on one face of the wall is $t(0,\theta) = 1$, while on the other wall it is $t(2\Delta x,\theta) = 0$. The initial condition is $t(x,0) = 0$. The governing partial differential equation is

$$\frac{\partial^2 t}{\partial x^2} = \frac{1}{\alpha}\frac{\partial t}{\partial \theta}$$

Thus, for only one interior node,



Taking $t_0 = 1$ and $t_2 = 0$ for all times, show that the finite-difference solution for $t = t_1$ is given by

$$t^{(\nu+1)} = (1 - 2p)^{\nu+1} t^{(0)} + p \sum_{j=0}^{\nu} (1 - 2p)^j$$

where $t_1 = t^{(0)} = 0$ at zero time and $p = \alpha\Delta\theta/(\Delta x)^2$.

(*a*) What does the temperature-time response look like for the following increments p?

$p = 0, \frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$, and 2

(*b*) Plot the exact solution, as given by Eq. (3·2·12), on the results of part *a*. Points for $\alpha\theta/(\Delta x)^2 = 0.25$, 0.50, and 1.00 should be enough.

(*c*) What can you conclude regarding the solution for various values of p?

8·13 A 6-in.-thick plane wall has an initial temperature distribution given by

$$t(x,0) = 1000 \sin \frac{\pi x}{L}$$

where $x = 0$ and $x = L$ are the two faces of the wall. The temperature of each face is held at zero, and the wall is allowed to approach equilibrium as time increases. Determine the temperature-time response throughout the wall using finite-difference techniques. Assume that $\alpha = 0.01$ ft²/hr. Plot a temperature-time curve for the center point in the wall and compare it with the analytical solution.

8·14 Using finite differences, solve the partial differential equation

$$u_{xx} + 8 = u_\theta$$

with the following boundary and initial conditions:

$u(0,\theta) = 1$

$u(1,\theta) = 0$

$u(x,0) = 0$

Compare your result to the exact solution obtained in Exercise 3·30.

8·15 In deriving Eq. (8·4·6), we assumed that t_m was an acceptable average temperature to use to represent the thermal storage in the system. A better average to use might be the true average of a piecewise-linear profile between nodes $m - 1$, m, and $m + 1$.

(*a*) Modify Eq. (8·4·6) to incorporate this new assumption.
(*b*) Obtain the pair of ordinary differential equations for the plane-wall problem discussed in Sec. 8·4 with nodes at $x = L/2$ and $x = L$.
(*c*) Obtain the analytical solution to these equations.
(*d*) Derive expressions for the λ-p curves for the three numerical methods of solution.

Compare these results to the exact solution and to the finite-difference method as discussed in the text.

8·16 For only two nodes the system of equations given by Eqs. (8·4·51) reduces to the following pair of equations:

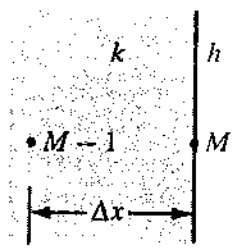
$$\frac{du_0}{d0} = \frac{1}{(\Delta x)^2}[-2(1 + H\,\Delta x)u_0 + 2u_1]$$

$$\frac{du_1}{d0} = \frac{1}{(\Delta x)^2}[2u_0 - 2u_1]$$

(*a*) Derive the exact analytical solution of this pair of equations and compare it to the exact solution of the original problem which may be obtained from the solution to Exercise 3·17. Use values of $H = 1$ and 10 for the comparison.
(*b*) Derive analytical expressions for the λ-p stability curves for the Euler, Crank-Nicolson, and pure implicit numerical methods of solution. Discuss the influence of H on the oscillatory behavior of the numerical solutions.

8·17 Reduce Eq. (8·4·55) to a two-node case and then determine its stability curve. How does it compare to the result for the pure Euler method in which the surface node was not implicitized?

8·18 Derive a stability condition for a plane-wall transient with a convection boundary condition. Use the physical approach discussed in Sec. 8·4·7 and consider only the surface node as shown.



8·19 A plane wall (conductivity = k), initially at a uniform temperature t_i, is suddenly exposed to convection with an ambient at t_∞ on the wall surface at $x = 0$. The wall surface at $x = L$ is insulated. The surface heat-transfer coefficient is h. Using finite differences, determine the temperature within the wall as a function of position and time for $hL/k = 1.0$.

8·20 A plane wall (conductivity = k), initially at a uniform temperature t_i, is suddenly subjected to a constant heat flux per unit area, q_0'', on the surface located at $x = 0$. The surface at $x = L$ is perfectly insulated. Using finite differences, determine the temperature within the wall as a function of position and time.

8·21 Derive a stability condition for a two-dimensional transient conduction problem. Consider only a typical interior node and use the physical approach discussed in Sec. 8·4·7.

8·22 Using finite differences, evaluate the tip temperature and the heat loss from the fin described in Exercise 2·26.

8·23 A thin circular fin (inner radius = r_i, outer radius = r_o, thickness = δ, conductivity = k) has a temperature of t_0 at its inner radius and may be assumed to be adiabatic at its outer radius. The surface heat-transfer coefficient is h. Using finite differences, determine the temperature distribution within the fin and the heat loss through the fin. Use cylindrical coordinates and compare your result to the exact solution for $hr_o^2/k\delta = 1.0$.

8·24 Using the finite-difference method obtain a solution to

$$\frac{d^2t}{dr^2} + \frac{1}{r}\frac{dt}{dr} + \frac{g'''}{k} = 0$$

for $t = 0$ at $r = r_o$. This is most readily done using cylindrical coordinates. Special attention must be given to the node at $r = 0$ since the system containing this node is a solid cylinder, whereas for the other nodes the system is a hollow cylinder. Compare your result to the exact solution.

8·25 An alternative approach to the linearization of Eq. (8·5·21) would be to expand the $u_0^{(\nu+1)4}$ that appears in the first row in a Taylor series in time as

$$u_0^{(\nu+1)4} = u_0^{(\nu)4} + \left.\frac{\partial u_0^4}{\partial \theta}\right|^{(\nu)} \Delta\theta$$

Upon taking the indicated derivative and replacing it with a difference approximation, the following expression is obtained:

$$u_0^{(\nu+1)4} = u_0^{(\nu)4} + 4u_0^{(\nu)3}(u_0^{(\nu+1)} - u_0^{(\nu)})$$

Continue this development and compare the final computation scheme with the one discussed in Sec. 8·5·4.

8·26 A small metallic object, initially at 500°R, is suddenly placed in a region at 2000°R. The only mechanism for heat transfer is radiation. The governing differential equation for the transient is given as

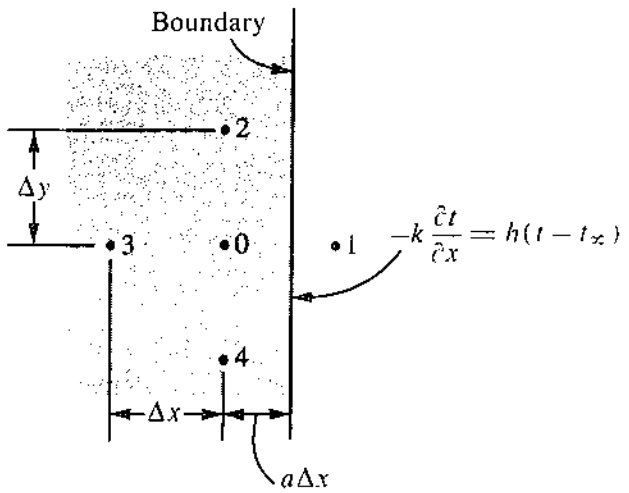
$$\frac{dT}{d\theta} + \frac{\sigma\epsilon A}{\rho V c}(T^4 - T_\infty^4) = 0$$

with $T(0) = 500$°R and $T_\infty = 2000$°R. Determine the solution using finite-difference methods.

8·27 Consider the solution to the equation

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0$$

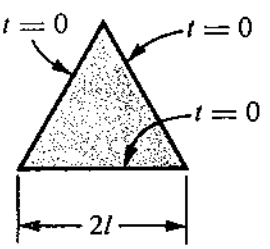
Derive the finite-difference equation for node 0 which is near (but not at) a boundary with convection. Take $\Delta x = \Delta y$.



8·28 Obtain a finite-difference solution for energy generation in an equilateral triangle. Use a square network of nodes and make use of symmetry to reduce the number of equations. The problem is described by

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{g'''}{k} = 0$$

What is the maximum temperature?



8·29 Derive the nodal finite-difference equation for a node (m,n) located near enough to a specified temperature boundary so that two adjacent nodes $(m + 1, n)$ and $(m, n + 1)$ lie outside the region of interest as shown.