1. **Debye length**
   As mentioned in the lectures, shielding of the bare charges in the plasma is of fundamental importance to the interactions and, thus, to the behaviour of the plasma as a whole. The shielding is characterized by the Debye length and you shall derive the expression for it in two different ways in this exercise.

   a) **Shielding of an individual charge.** Solve the expression for the perturbation potential due to a point charge $q\delta(\boldsymbol{r})$ in an electron-proton plasma that is in a thermal equilibrium. The densities of the plasma species are
   $$n_a(\boldsymbol{r}) = n_0 \mathrm{e}^{-q_a\Phi(\boldsymbol{r})/kT},$$
   First, write down the Poisson's equation, then perturb the system with an additional charge, $q\delta(\boldsymbol{r})$, and use Taylor expansion for the densities around the equilibrium. You should obtain an differential equation for the perturbation potential with a delta-function source at the origin. Solve this equation using spherical coordinates to get the result
   $$\delta\Phi = \frac{q}{4\pi\epsilon_0 r}\mathrm{e}^{-r/\lambda_D}$$

   b) **Shielding at material surfaces: width of the *sheath region* in front of a probe.** Let us consider a 1-dimensional case, where a potential $\Phi_0$ is imposed at $x = 0$, representing an infinitely small probe (in 1D, a plate). We have to find how this potential decays as one moves away from the plate, i.e., $\Phi(x)$. You can assume ions fixed, and you know that the electron velocities in an electrostatic field given by potential $\Phi(x)$ are distributed according to $f(v) = A\exp[-(\frac{1}{2}mv^2 - e\Phi)/T])$. Infinitely far from the plate the potential vanishes and $n_0 = \int A\exp[-(\frac{1}{2}mv^2/T)] = n_i$. Using Poisson equation, show that the potential decays as $\Phi(x) = \Phi_0\exp(-|x|/\lambda_D)$, where $\lambda_D = \epsilon_0 T/ne^2$. Do the analysis 'sufficiently' far from the plate so that you can assume $|e\Phi/T| << 1$.

2. **Plasma oscillations.**
   Take a neutral plasma and perturb it by moving a small slab of electrons by $\delta x$, i.e., consider the situation in 1D only. Then, at the faces of the slab a surface charge $\pm\sigma$ will appear: $\sigma = en\delta x$. Use the Gauss law to find the restoring force, apply it to the electron equation of motion and show that the electrons will start oscillating with the *plasma frequency* $\omega_p = \sqrt{e^2 n/\epsilon_0}$

3. **The plasma parameter $\Lambda$ and weakly/strongly coupled plasmas**
   In the first lecture we gave the plasma parameter as the number of particles in the Debye sphere, $\Lambda = \frac{4\pi}{3}n\lambda^3$. Your job is now to show that it can also be written as $\Lambda = \frac{1}{\sqrt{4\pi}}(\frac{r_d}{r_c})^{3/2}$, where $r_d = n^{-1/3}$ is the inter-particle distance and $r_c$ is the 'distance of closest approach' under Coulomb interaction, discussed in the first lecture: $\frac{1}{2}mv^2 = \frac{q_1 q_2}{4\pi\epsilon_0 r_c}$. Thus our requirement for a collection of charged particles to be called a plasma, $\Lambda >> 1$ not only means that there are a lot of particles in a Debye sphere, but also that $r_d >> r_c$. i.e., that the average distance between particles is much greater than the distance at which particle energy equals the Coulomb energy. In this regime, loosely speaking, the particles have some 'freedom' between interactions and the plasma is called *weakly interacting*, while at the other extreme the dynamics is entirely dominated by Coulomb interaction and the plasma is called *weakly interacting.* As mentioned in the lecture, generally only weakly interacting plasmas are considered plasmas.

4. **Maxwellian distribution in many ways...**
   (a) Find the normalization coefficient $A$ for a 1-dimensional Maxwellian distribution, $f(v) = A_1 e^{-mv^2/2T}$.
   (b) Using the result from (a), find the normalization coefficients for a 2-dimensional Maxwellian $f(\mathbf{v}) = A_2 e^{-mv^2/2T}$, $\mathbf{v} = v_x\hat{x} + v_y\hat{y}$, and a 3-dimensional Maxwellian $f(\mathbf{v}) = A_3 e^{-mv^2/2T}$, $\mathbf{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ distribution.
   (c) Use spherical coordinates and recalculate the normalization coefficient for the 3-dimensional Maxwellian distribution.

5. (**More on Maxwellian distribution ...**)
   (a) Derive the *speed* distribution $g(v)$, and
   (b) obtain the most probable speed,
   (c) obtain the average speed.
   (d) Using the speed distribution $g(v)$, derive the *energy* distribution $h(E)$. (Make sure the normalization is right)

(e) Calculate the average energy for a Maxwellian distribution. You can choose which presentation of the Maxwellian you use.

6. (**Some warm-ups for velocity moments plus a slightly more challenging one**).
(a) Calculate the first 3 velocity moments of the distribution function when the velocity space dependence is given by the Maxwellian normalized to particle number.
(b) Derive the continuity equation

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \boldsymbol{V_s}) = \frac{dn_s}{dt} + n_s \nabla \cdot \boldsymbol{V_s} = 0$$

starting from the Vlasov equation

$$\frac{\partial f_s}{\partial t} + \dot{\boldsymbol{r}} \cdot \nabla f_s + \dot{\boldsymbol{v}} \cdot \nabla_v f_s = 0.$$

Recipe: integrate Vlasov equation over the velocity (=take the zeroth velocity moment). (c) Derive the momentum equation

$$m_s n_s \frac{d\boldsymbol{V_s}}{dt} + \nabla \cdot \boldsymbol{p}_s - q_s n_s \left(\boldsymbol{E} + \boldsymbol{V_s} \times \boldsymbol{B}\right) = 0$$

starting from the Vlasov equation

$$\frac{\partial f_s}{\partial t} + \dot{\boldsymbol{r}} \cdot \nabla f_s + \dot{\boldsymbol{v}} \cdot \nabla_v f_s = C_s\left(f\right).$$

Hints: Multiply the Vlasov equation by $m_s \boldsymbol{v}$ and integrate over velocity, i.e. take the first velocity moment. For the first term you can interchange the derivation and integration. For the second term use equation

$$\nabla \cdot (f\boldsymbol{A}) = f\nabla \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla f$$

to simplify the term $\boldsymbol{v} \cdot \nabla f$. For the force term you need to work a little bit more. Start as you did for the second term, then show that for the Lorenz force

$$\nabla_v \cdot \boldsymbol{a} = \nabla_v \cdot \left(\frac{q}{m}\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)\right) = 0.$$

The rest is vector algebra. NRL's will be provided. :)

7. (**Plasma as a dielectric medium – do it yourself!**)
Revisit the high-frequency electrostatic perturbation for another useful result: Linearize the electron equation of motion and continuity equation, and use them together with the linearized Gauss' law to show that the latter can be expressed in terms of the *plasma dielectric coefficient* $\epsilon_p$ as $\nabla \cdot (\epsilon_p \mathbf{E}) = 0$, where $\epsilon_p = 1 - \frac{\omega_p^2}{\omega^2}$.

8. (**Magnetic mirror in a tokamak**).
In a tokamak the plasma column has been bent into a *torus* and the field lines twist around it helically. The magnetic field strength thus increases towards the symmetry axis, a particle moving along the field line sees a non-uniform field and, when moving to higher field strength, can get reflected if its parallel velocity is not sufficiently high. I.e., the particle encounters a magnetic mirror.
(a) Assuming that the field strength has the simple $1/R$ dependence, where $R$ (called the *major radius*) measures the distance from the symmetry axis, find the expression for $R_b$, the distance at which a particle bounces, i.e., gets reflected.
(b) Show that the condition for a particle to get reflected is given by $\frac{\mu B_0}{E} > 1 - a/R_0$, where $\mu$ is magnetic moment, $B_0$ is the magnetic field strength at the center of the plasma column (called *magnetic axis*), $E$ is particle energy, $a$ is the minor radius of the plasma (i.e., distance from the magnetic axis to the edge of the plasma column), and $R_0$ is the major radius of the magnetic axis.
(Hints: Use the conservation of energy and magnetic moment. You can also assume large aspect ratio, i.e., $R/a >> 1$)