Lecture 1: Overview of Bayesian Modeling of Time-Varying Systems

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Contents

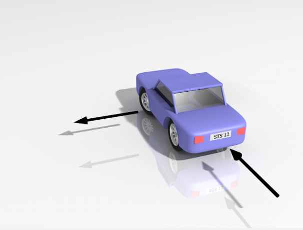
Contents of Course

- Modeling with stochastic state space models.
- Bayesian theory of optimal filtering.
- Gaussian approximations: Derivation of Kalman, extended Kalman and unscented Kalman filters, Gauss-Hermite and cubature Kalman filters from the general theory.
- Monte Carlo methods: Particle filtering, Rao-Blackwellized filtering.
- Bayesian theory of optimal smoothing and related Kalman (=Gaussian) and particle type methods.
- Various illustrative applications to backup the theory.
- Various exercises to practice modeling and estimation.

Some History

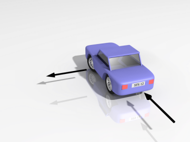
- In 40's, Wiener's work on stochastic analysis and optimal filtering (and "cybernetics")
- In late 50's, state space models, Bellman's dynamic programming, Swerling's filter, Stratonovich's conditional Markov processes.
- In early 60's, Kalman filter and Kalman-Bucy filter, stability analysis of linear state space models (mostly by Kalman).
- In mid 60's, Rauch-Tung-Striebel smoother, extended Kalman filters (EKF).
- In late 60's, Bayesian approach to optimal filtering, first practical applications (e.g. Apollo program).
- In 70's and 80's, first particle filters, square root Kalman filters, new algorithms and applications.
- In 90's, rebirth of particle filters, sigma-point and unscented Kalman filters (UKF), new applications.
- In 2000–, new algorithm variations and applications.

Recursive Estimation of Dynamic Processes



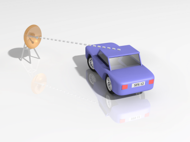
- Dynamic, that is, time varying phenomenon - e.g., the motion state of a car or smart phone.
- The phenomenon is measured - for example by a radar or by acceleration and angular velocity sensors.
- The purpose is to compute the state of the phenomenon when only the measurements are observed.
- The solution should be recursive, where the information in new measurements is used for updating the old information.

Bayesian Modeling of Dynamics

- The laws of physics, biology, epidemiology etc. are typically differential equations.
- Uncertainties and unknown sub-phenomena are modeled as stochastic processes:
  - Physical phenomena: differential equations + uncertainty $\Rightarrow$ stochastic differential equations.
  - Discretized physical phenomena: Stochastic differential equations $\Rightarrow$ stochastic difference equations.
  - Naturally discrete-time phenomena: Systems jumping from step to another.
- Stochastic differential and difference equations can be represented in stochastic state space form.

Bayesian Modeling of Measurements




- The relationship between measurements and phenomenon is mathematically modeled as a probability distribution.
- The measurements could be (in ideal world) computed if the phenomenon was known (forward model).
- The uncertainties in measurements and model are modeled as random processes.
- The measurement model is the conditional distribution of measurements given the state of the phenomenon.

Why Bayesian Approach?

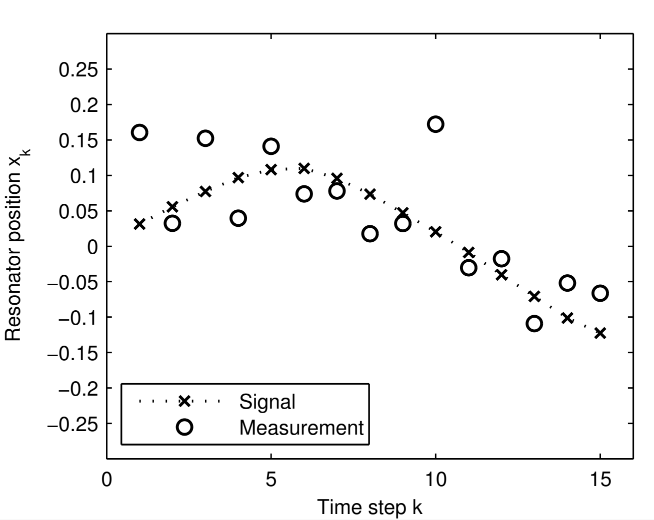
- Theory of optimal filtering can be formulated in many ways:

  1. Least squares optimization framework $\Rightarrow$ hard to extend recursive estimation beyond linear models, uncertainties cannot be modeled.
  2. Maximum likelihood framework $\Rightarrow$ the theoretical basis of dynamic models is somewhat heuristic, uncertainties cannot be modeled.
  3. Bayesian framework $\Rightarrow$ theory is quite complete, but the computational complexity can be unbounded.
  4. Other approaches $\Rightarrow$ typically applicable to restricted special cases.

- For practical “engineering” reasons, Bayesian approach is used here (because it works!).
- Kalman filter (1960) was originally derived in least squares framework
- Non-linear filtering theory has been Bayesian from the beginning (about 1964).

Bayesian Estimation of Dynamic Process

Time-varying process x_k and noisy measurements y_k from it:



Mathematical Model of Dynamic Process

- Generally, Markov model for the state:

$$\mathbf{x}_k \sim p(\mathbf{x}_k \,|\, \mathbf{x}_{k-1}).$$

- Likelihood distribution of the measurement:

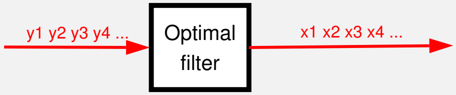
$$\mathbf{y}_k \sim p(\mathbf{y}_k \,|\, \mathbf{x}_k).$$

- In principle, we could simply use the Bayes' rule

$$p(\mathbf{x}_1, \ldots, \mathbf{x}_T \,|\, \mathbf{y}_1, \ldots, \mathbf{y}_T) = \frac{p(\mathbf{y}_1, \ldots, \mathbf{y}_T \,|\, \mathbf{x}_1, \ldots, \mathbf{x}_T)\, p(\mathbf{x}_1, \ldots, \mathbf{x}_T)}{p(\mathbf{y}_1, \ldots, \mathbf{y}_T)}.$$

- Curse of computational complexity: complexity grows more than linearly with number of measurements.

Optimal Filter



- The classical recursive (efficient) solution to the dynamic estimation problem is called an optimal filter.
- The Bayesian optimal filter computes the (marginal) posterior distribution of the state given the measurements:

$$p(\mathbf{x}(t_k) \mid \mathbf{y}_1, \ldots, \mathbf{y}_k).$$

- The "filtered" state $\hat{\mathbf{x}}(t_k)$ typically is the posterior mean

$$\hat{\mathbf{x}}(t_k) = E(\mathbf{x}(t_k) \mid \mathbf{y}_1, \ldots, \mathbf{y}_k).$$

- The solution is called filter for historical reasons.

Bayesian Filtering, Prediction and Smoothing

- Recursively computable marginal distributions:
  - Filtering distributions:

$$p(\mathbf{x}_k \,|\, \mathbf{y}_1, \ldots, \mathbf{y}_k), \qquad k = 1, \ldots, T.$$

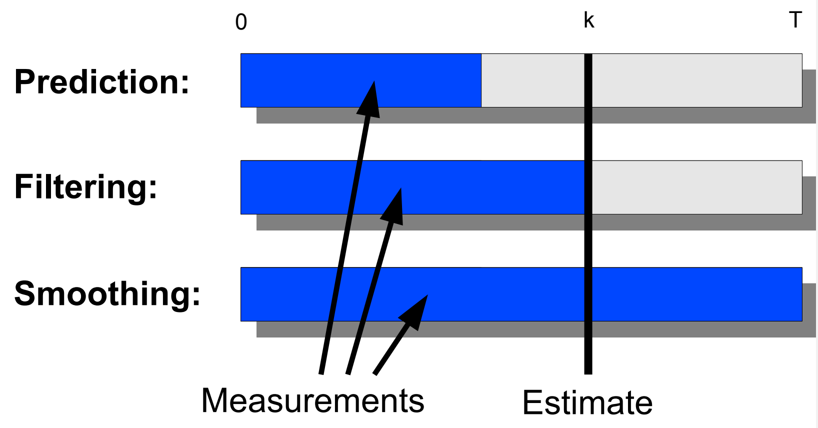
  - Prediction distributions:

$$p(\mathbf{x}_{k+n} \,|\, \mathbf{y}_1, \ldots, \mathbf{y}_k), \qquad k = 1, \ldots, T, \quad n = 1, 2, \ldots,$$

  - Smoothing distributions:

$$p(\mathbf{x}_k \,|\, \mathbf{y}_1, \ldots, \mathbf{y}_T), \qquad k = 1, \ldots, T.$$

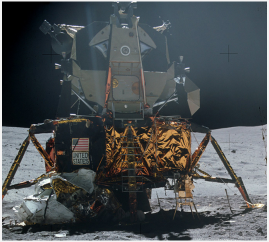
Bayesian Filtering, Prediction and Smoothing (cont.)



Algorithms for Computing the Solutions

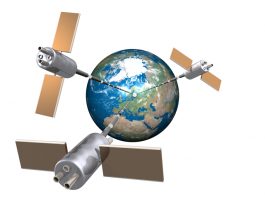
- Kalman filter is the classical optimal (Bayesian) filter for linear-Gaussian models.
- Extended Kalman filter (EKF) is linearization based extension of Kalman filter to non-linear models.
- Unscented Kalman filter (UKF) is sigma-point transformation based extension of Kalman filter.
- Gauss-Hermite and Cubature Kalman filters (GHKF/CKF) are numerical integration based extensions of Kalman filter.
- Particle filter forms a Monte Carlo representation (particle set) to the distribution of the state estimate.
- Grid based filters approximate the probability distributions by a finite grid.
- Mixture Gaussian approximations are used, for example, in multiple model Kalman filters and Rao-Blackwellized Particle filters.

Navigation of Lunar Module



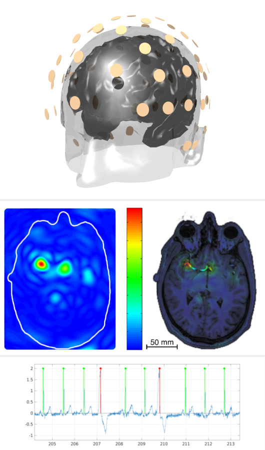
- The navigation system of Eagle lunar module AGC was based on an optimal filter - this was in the year 1969.
- The dynamic model was Newton's gravitation law.
- The measurements at lunar landing were the radar readings.
- On rendezvous with the command ship the orientation was computed with gyroscopes and their biases were also compensated with the radar.
- The optimal filter was an extended Kalman filter.

Satellite Navigation (GPS)



- The dynamic model in GPS receivers is often the Newton's second law where the force is completely random, that is, the Wiener velocity model.
- The measurements are time delays of satellite signals.
- The optimal filter computes the position and the accurate time.
- Also the errors caused by multi path can be modeled and compensated.
- Acceleration and angular velocity measurements are sometimes used as extra measurements.

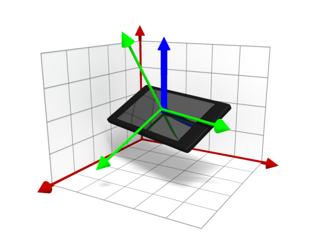
Health and Medical Applications



- Many brain imaging methods (e.g. MEG & EEG) be recasted as Kalman filtering.
- The Kalman filter solves the inverse problem recursively.
- Bayesian filters can also be used for post-processing brain imaging data.
- Biomedical signal processing (e.g. ECG and BCG) also require e.g. noise reduction which can be done with Kalman filters.
- ECG signal analysis can also be done with extended Kalman filter (EKF).

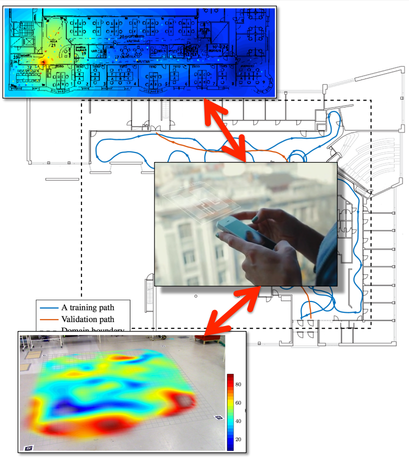
Mobile phone sensor fusion

- Acceleration and angular velocity can be integrated to give position and orientation.
- Unknown initial conditions and sensors drifts cause problems.
- The known gravitation direction helps in orientation tracking.
- Accelerometer can also be used to detect steps – gives a measurement of speed/distance.
- Barometer can be used to for local height tracking.
- Can be combined with radio and magnetic field fingerprinting.

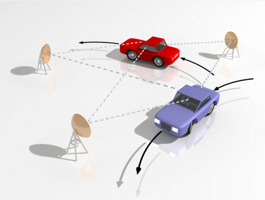


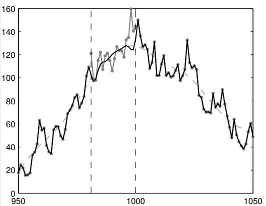
Simultaneous localization and mapping (SLAM)

- In simultaneous localization and mapping (SLAM) radio/magnetic map is created while positioning.
- Considerably harder than separate mapping and positioning.
- Typically detect a return to known location:
  - Loop closure to confirm the traveled path.
  - Inertial navigation can be used to map a small unknown area at a time.
  - Known wall locations provide constraints.



Other Applications





- Autonomous cars with multitude of sensors – sensor fusion.
  Target tracking, where one or many targets are tracked with many passive sensors - air surveillance.
- Machine learning in time series data – Gaussian process regression is related to Kalman filtering.
- Analysis/restoration of audio signals.
- Telecommunication systems - optimal receivers, signal detectors.
- State estimation of control systems - chemical processes, auto pilots, control systems of cars.

Generic Probabilistic State Space Model

- General form of probabilistic state space models:

$$
\begin{aligned}
\mathbf{x}_k &\sim p(\mathbf{x}_k \,|\, \mathbf{x}_{k-1}) \\
\mathbf{y}_k &\sim p(\mathbf{y}_k \,|\, \mathbf{x}_k) \\
\mathbf{x}_0 &\sim p(\mathbf{x}_0).
\end{aligned}
$$

- $\mathbf{x}_k$ is the generalized state at time step k, including all physical state variables and parameters.
- $\mathbf{y}_k$ is the vector of measurements obtained at time step k.
- The dynamic model $p(\mathbf{x}_k \,|\, \mathbf{x}_{k-1})$ models the dynamics of the state.
- The measurement model $p(\mathbf{y}_k \,|\, \mathbf{x}_k)$ models the measurements and their uncertainties.
- The prior distribution $p(\mathbf{x}_0)$ models the information known about the state before obtaining any measurements.

Linear Gaussian State Space Models

- General form of linear Gaussian state space models:

$$\begin{aligned}
\mathbf{x}_k &= \mathbf{A}\,\mathbf{x}_{k-1} + \mathbf{q}_{k-1}, & \mathbf{q}_{k-1} &\sim \mathrm{N}(0, \mathbf{Q}) \\
\mathbf{y}_k &= \mathbf{H}\,\mathbf{x}_k + \mathbf{r}_k, & \mathbf{r}_k &\sim \mathrm{N}(0, \mathbf{R}) \\
\mathbf{x}_0 &\sim \mathrm{N}(\mathbf{m}_0, \mathbf{P}_0).
\end{aligned}$$

- In probabilistic notation the model is:

$$\begin{aligned}
p(\mathbf{y}_k \,|\, \mathbf{x}_k) &= \mathrm{N}(\mathbf{y}_k \,|\, \mathbf{H}\,\mathbf{x}_k, \mathbf{R}) \\
p(\mathbf{x}_k \,|\, \mathbf{x}_{k-1}) &= \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{A}\,\mathbf{x}_{k-1}, \mathbf{Q}).
\end{aligned}$$

- Surprisingly general class of models – linearity is from measurements to estimates, not from time to outputs.

Non-Linear State Space Models

- General form of non-linear Gaussian state space models:

$$\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{q}_{k-1})$$
$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k, \mathbf{r}_k).$$

- $\mathbf{q}_k$ and $\mathbf{r}_k$ are typically assumed Gaussian.
- Functions $\mathbf{f}(\cdot)$ and $\mathbf{h}(\cdot)$ are non-linear functions modeling the dynamics and measurements of the system.
- Equivalent to the generic probabilistic models of the form

$$\mathbf{x}_k \sim p(\mathbf{x}_k \,|\, \mathbf{x}_{k-1})$$
$$\mathbf{y}_k \sim p(\mathbf{y}_k \,|\, \mathbf{x}_k).$$

Modeling with State Space Models

- Probabilistic state space models are very general – every finite dimensional Bayesian estimation problem has a state space representation.
- The most difficult task is figure out how to formulate an estimation problem in state space form.
- Formulating state space representations of physical problems is engineering in its basic form.
- Best way to learn this engineering is by examples and practical work – in this lecture we shall give examples.

Linear and Linear in Parameters Models

- Basic linear regression model with noise ϵ_k:

$$y_k = a_0 + a_1\, x_k + \epsilon_k, \qquad k = 1, \ldots, N.$$

- First rename x_k to e.g. s_k to avoid confusion:

$$y_k = a_0 + a_1\, s_k + \epsilon_k, \qquad k = 1, \ldots, N.$$

- Define matrix $\mathbf{H}_k = (1 \; s_k)$ and state $\mathbf{x} = (a_0 \; a_1)^T$:

$$y_k = \mathbf{H}_k\, \mathbf{x} + e_k, \qquad k = 1, \ldots, N.$$

- For notation sake we can also define $\mathbf{x}_k = \mathbf{x}$ such that $\mathbf{x}_k = \mathbf{x}_{k-1}$:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{x}_{k-1} \\ y_k &= \mathbf{H}_k\, \mathbf{x}_k + e_k. \end{aligned}$$

- Thus we have a linear Gaussian state space model, solvable with the basic Kalman filter.

Linear and Linear in Parameters Models (cont.)

- More general linear regression models:

$$y_k = a_0 + a_1\, s_{k,1} + \cdots + a_d\, s_{k,d} + \epsilon_k, \qquad k = 1, \ldots, N.$$

- Defining matrix $\mathbf{H}_k = (1 \;\, s_{k,1} \;\, \cdots \;\, s_{k,d})$ and state $\mathbf{x}_k = \mathbf{x} = (a_0 \;\, a_1 \;\, \cdots \;\, a_d)^T$ gives linear Gaussian state space model:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{x}_{k-1} \\ y_k &= \mathbf{H}_k\, \mathbf{x}_k + \epsilon_k. \end{aligned}$$

- Linear in parameters models:

$$y_k = a_0 + a_1\, f_1(s_k) + \cdots + a_d\, f_d(s_k) + \epsilon_k.$$

- Definitions $\mathbf{H}_k = (1 \;\, f_1(s_k) \;\, \cdots \;\, f_d(s_k))$ and $\mathbf{x}_k = \mathbf{x} = (a_0 \;\, a_1 \;\, \cdots \;\, a_d)^T$ again give linear Gaussian state space model.

Non-Linear and Neural Network Models

- Non-linearity in measurements models arises in generalized linear models, e.g.

$$y_k = g^{-1}(a_0 + a_1\, s_k) + \epsilon_k.$$

- The measurement model is now non-linear and if we define $\mathbf{x} = (a_0\ a_1)^T$ and $h(\mathbf{x}) = g^{-1}(x_1 + x_2\, s_k)$ we get non-linear Gaussian state space model:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{x}_{k-1} \\ y_k &= h(\mathbf{x}_k) + \epsilon_k. \end{aligned}$$

- Neural network models such as multi-layer perceptron (MLP) models can be also transformed into the above form.
- Instead of basic Kalman filter we need extended Kalman filter or unscented Kalman filter to cope with the non-linearity.

Adaptive Filtering Models

- In digital signal processing, a commonly used signal model is the autoregressive model

$$y_k = w_1 \, y_{k-1} + \cdots + w_d \, y_{k-d} + \epsilon_k,$$

- In adaptive filtering the weights w_i are estimated from data.
- If we define matrix $\mathbf{H}_k = (y_{k-1} \;\; \cdots \;\; y_{k-d})$ and state as $\mathbf{x}_k = (w_1 \;\; \cdots \;\; w_d)^T$, we get linear Gaussian state space model:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{x}_{k-1} \\ y_k &= \mathbf{H}_k \, \mathbf{x}_k + \epsilon_k. \end{aligned}$$

- The estimation problem can be solved with Kalman filter.
- The LMS algorithm can be interpreted as approximate version of this Kalman filter.

Adaptive Filtering Models (cont.)

- In time varying autoregressive models (TVAR) models the weights are time-varying:

$$y_k = w_{1,k}\, y_{k-1} + \cdots + w_{d,k}\, y_{k-d} + \epsilon_k,$$

- Typical model for the time dependence of weights:

$$w_{i,k} = w_{i,k-1} + q_{k-1,i}, \quad q_{k-1,i} \sim \mathrm{N}(0, \sigma^2), \quad i = 1, \ldots, d.$$

- Can be written as linear Gaussian state space model with process noise $\mathbf{q}_{k-1} = \begin{pmatrix} q_{k-1,1} & \cdots & q_{k-1,d} \end{pmatrix}^T$:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{x}_{k-1} + \mathbf{q}_{k-1} \\ y_k &= \mathbf{H}_k\, \mathbf{x}_k + \epsilon_k. \end{aligned}$$

- More general (TV)ARMA models can be handled similarly.

Spectral and Covariance Models

- Time series can be often modeled in terms of spectral density

$$S(\omega) = \{\text{some function of angular velocity } \omega\}.$$

- Or in terms of mean and covariance function:

$$\begin{aligned}\mathbf{m}(t) &= \mathrm{E}[\mathbf{x}(t)] \\ \mathbf{C}(t, t') &= \mathrm{E}[(\mathbf{x}(t) - \mathbf{m}(t))\,(\mathbf{x}(t') - \mathbf{m}(t'))^T]\end{aligned}$$

- Such Gaussian processes have representations as outputs of linear Gaussian systems driven by white noise.
- We often can construct a linear Gaussian state space model with a given spectral density or covariance function.
- If spectral density is a rational function, this is possible.

Stochastic Differential Equation Models

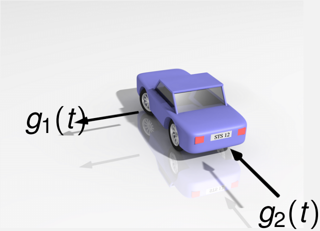
- Physical systems can be often modeled as differential equations with random terms such as

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(t)\,\mathbf{w}(t),$$

  where $\mathbf{w}(t)$ is a continuous-time white noise process.
- The noise process can be used for modeling the deviation from the ideal solution $d\mathbf{x}(t)/dt = \mathbf{f}(\mathbf{x}, t)$.
- For example, locally (short term) linear functions, almost periodic functions, etc.
- The dynamic model has to be dicretized somehow in computations.
- Typically, measurements are assumed to be obtained at discrete instances of time:

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}(t_k)) + \mathbf{r}_k,$$

Dynamic Model for a Car [1/3]



- The dynamics of the car in 2d (x_1, x_2) are given by the Newton's law:
$$\mathbf{g}(t) = m\,\mathbf{a}(t),$$
where $\mathbf{a}(t)$ is the acceleration, m is the mass of the car, and $\mathbf{g}(t)$ is a vector of (unknown) forces acting the car.
- We shall now model $\mathbf{g}(t)/m$ as a 2-dimensional white noise process:

$$\begin{aligned} \mathrm{d}^2 x_1 / \mathrm{d}t^2 &= w_1(t) \\ \mathrm{d}^2 x_2 / \mathrm{d}t^2 &= w_2(t). \end{aligned}$$

Dynamic Model for a Car [2/3]

- If we define $x_3(t) = \mathrm{d}x_1/\mathrm{d}t$, $x_4(t) = \mathrm{d}x_2/\mathrm{d}t$, then the model can be written as a first order system of differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\mathbf{F}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{L}} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

- In shorter matrix form:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{F}\,\mathbf{x} + \mathbf{L}\,\mathbf{w}.$$

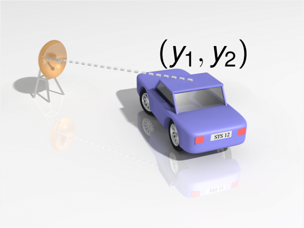
Dynamic Model for a Car [3/3]

- If the state of the car is measured (sampled) with sampling period Δt it suffices to consider the state of the car only at the time instances $t \in \{0, \Delta t, 2\Delta t, \ldots\}$.
- The dynamic model can be discretized, which leads to the linear difference equation model

$$\mathbf{x}_k = \mathbf{A}\,\mathbf{x}_{k-1} + \mathbf{q}_{k-1},$$

where $\mathbf{x}_k = \mathbf{x}(t_k)$, $\mathbf{A}$ is the transition matrix and $\mathbf{q}_k$ is a discrete-time Gaussian noise process.

Measurement Model for a Car



- Assume that the position of the car (x_1, x_2) is measured and the measurements are corrupted by Gaussian measurement noise $e_{1,k}, e_{2,k}$:

$$
\begin{aligned}
y_{1,k} &= x_{1,k} + e_{1,k} \\
y_{2,k} &= x_{2,k} + e_{2,k}.
\end{aligned}
$$

- The measurement model can be now written as

$$
\mathbf{y}_k = \mathbf{H}\,\mathbf{x}_k + \mathbf{e}_k, \qquad \mathbf{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
$$

Model for Car Tracking

- The dynamic and measurement models of the car now form a linear Gaussian filtering model:

$$\mathbf{x}_k = \mathbf{A}\,\mathbf{x}_{k-1} + \mathbf{q}_{k-1}$$
$$\mathbf{y}_k = \mathbf{H}\,\mathbf{x}_k + \mathbf{r}_k,$$

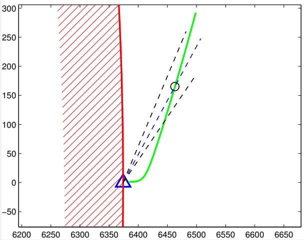
  where $\mathbf{q}_{k-1} \sim \mathrm{N}(\mathbf{0}, \mathbf{Q})$ and $\mathbf{r}_k \sim \mathrm{N}(\mathbf{0}, \mathbf{R})$.

- The posterior distribution is Gaussian

$$p(\mathbf{x}_k \mid \mathbf{y}_1, \ldots, \mathbf{y}_k) = \mathrm{N}(\mathbf{x}_k \mid \mathbf{m}_k, \mathbf{P}_k).$$

- The mean $\mathbf{m}_k$ and covariance $\mathbf{P}_k$ of the posterior distribution can be computed by the Kalman filter.

Re-Entry Vehicle Model [1/3]



- Gravitation law:

$$\mathbf{F} = m\,\mathbf{a}(t) = -\frac{G\,m\,M\,\mathbf{r}(t)}{|\mathbf{r}(t)|^3}.$$

- If we also model the friction and uncertainties:

$$\mathbf{a}(t) = -\frac{G\,M\,\mathbf{r}(t)}{|\mathbf{r}(t)|^3} - D(\mathbf{r}(t))\,|\mathbf{v}(t)|\,\mathbf{v}(t) + \mathbf{w}(t).$$

Re-Entry Vehicle Model [2/3]

- If we define $\mathbf{x} = (x_1 \; x_2 \; \frac{\mathrm{d}x_1}{\mathrm{d}t} \; \frac{\mathrm{d}x_2}{\mathrm{d}t} \;)^T$, the model is of the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}) + \mathbf{L}\,\mathbf{w}(t).$$

  where $\mathbf{f}(\cdot)$ is non-linear.

- The radar measurement:

$$r = \sqrt{(x_1 - x_r)^2 + (x_2 - y_r)^2} + e_r$$
$$\theta = \tan^{-1}\left(\frac{x_2 - y_r}{x_1 - x_r}\right) + e_\theta,$$

  where $e_r \sim \mathrm{N}(0, \sigma_r^2)$ and $e_\theta \sim \mathrm{N}(0, \sigma_\theta^2)$.

Re-Entry Vehicle Model [3/3]

- By suitable numerical integration scheme the model can be approximately written as discrete-time state space model:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{q}_{k-1}) \\ \mathbf{y}_k &= \mathbf{h}(\mathbf{x}_k, \mathbf{r}_k), \end{aligned}$$

  where $\mathbf{y}_k$ is the vector of measurements, and $\mathbf{q}_{k-1} \sim \mathrm{N}(\mathbf{0}, \mathbf{Q})$ and $\mathbf{r}_k \sim \mathrm{N}(\mathbf{0}, \mathbf{R})$.
- The tracking of the space vehicle can be now implemented by, e.g., extended Kalman filter (EKF), unscented Kalman filter (UKF) or particle filter.

Summary

- The purpose of is to estimate the state of a time-varying system from noisy measurements obtained from it.
- The linear theory dates back to 50's, non-linear Bayesian theory was founded in 60's.
- The efficient computational solutions can be divided into prediction, filtering and smoothing.
- Applications: tracking, navigation, telecommunications, audio processing, control systems, etc.
- The formal Bayesian estimation equations can be approximated by e.g. Gaussian approximations, Monte Carlo or Gaussian mixtures.
- Formulating physical systems as state space models is a challenging engineering topic as such.

Matlab Demo: EKF/UKF Toolbox

http://becs.aalto.fi/en/research/bayes/ekfukf/