# 1. Metric measure spaces

1.1. Metric. Let $X$ be a set. A function $d: X \times X \to [0,\infty)$ is a metric (or a distance function) if

(i) $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$,

(ii) $d(x,y) = d(y,x)$ $\forall\, x,y \in X$ (symmetry),

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ $\forall\, x,y,z \in X$. (the triangle inequality)

$(X,d)$ is a metric space.

1.2. Measure. A function $\mu : \{A : A \subset X\} \to [0,\infty]$ is an outer measure on $X$ if

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(A) \leq \mu(B)$ if $A \subset B \subset X$, (monotonicity)

(iii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$, $A_i \subset X$, $i = 1,2,\dots$ (countable subadditivity)

We call such a $\mu$ a measure. We assume that $\mu$ is a Borel regular (outer) measure, that is, for every $A \subset X$ there exists a Borel set $B \supset A$ such that $\mu(A) = \mu(B)$.

Recall that the collection of Borel sets in $X$ is the smallest $\sigma$-algebra containing the open (or equivalently closed) subsets of $X$.

**1.3. Balls.** We denote an open ball in $X$ with a center $x \in X$ and a radius $0<r<\infty$ by

$$B(x,r)=\{y\in X : d(y,x)<r\}.$$

The corresponding closed ball is

$$\overline{B}(x,r)=\{y\in X : d(y,x)\le r\}.$$

Note that a ball in a metric space does not necessarily have a unique center and a radius. Moreover, $\overline{B}(x,r)$ may be a larger set than $\overline{B(x,r)}$, the topological closure of $B(x,r)$.

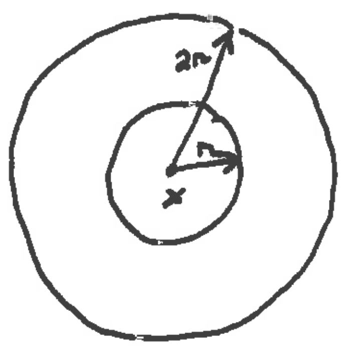
If $B=B(x,r)$ is a ball in $X$ and $0<t<\infty$, then $tB$ is the ball with the same center as $B$ and radius $tr$, that is,

$$tB = tB(x,r) = B(x,tr).$$

**1.4. Doubling measure.** A measure $\mu$ is doubling, if there exists a constant $c_d$ such that

$$\mu(B(x,2r)) \le c_d\, \mu(B(x,r))$$

for every $x\in X$ and $0<r<\infty$. We call $c_d$ a doubling constant of $\mu$.



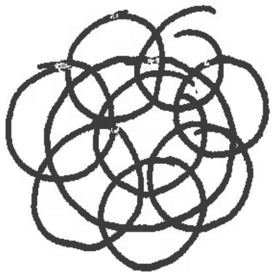
We assume that $\mu$ is doubling and that it is nontrivial in the sense that there exists a ball $B\subset X$ such that

$$0<\mu(B)<\infty.$$

The doubling condition implies that the measure of every nonempty open set is positive and the measure of every bounded set is finite. In particular, this implies that $\mu$ is a Radon measure. Such a metric measure space can be written as a union of a countable collection of open sets with finite measure, for example, if $x\in X$, then

$$X=\bigcup_{i=1}^{\infty} B(x,i).$$

1.5. <u>Doubling space</u>. A metric space is doubling, if there exists a constant $C$ such that every ball $B(x,r)$ can be covered by at most $C$ balls of radius $\frac{r}{2}$. In other words, in every ball $B(x,r)$ there are at most $C$ points whose mutual distance is at least $\frac{r}{2}$.



1.6. <u>Lemma</u>. Assume that $X$ is a doubling space with a constant $C$. Then every ball $B(x,r)$ can be covered at most $C^k$ balls of radius $\frac{r}{2^k}$, $k\in\mathbb{N}$.

Proof: The case $k=1$ follows directly from the doubling condition. Assume that the claim holds true for some $k \in \mathbb{N}$. Then every ball of radius $\frac{r}{2^k}$ can be covered by at most $C$ balls of radius $\frac{r}{2^{k+1}}$. This implies, that the original ball is covered by $C^{k+1}$ balls of radius $\frac{r}{2^{k+1}}$. $\square$

1.7. Total boundedness. A subset $A$ of a metric space $X$ is totally bounded if for every $\varepsilon > 0$ there is a finite number of balls ~~that~~ $B(x_i, \varepsilon)$, $x_i \in X$, that cover $A$. This is equivalent to the existence of a finite $\varepsilon$-net.

A metric space is proper, if every closed and bounded set is compact. Lemma 1.6 shows that bounded sets are totally bounded in a doubling space. This together with standard results on metric spaces implies the following result.

1.8. Lemma. A doubling metric space is proper if and only if it is complete.

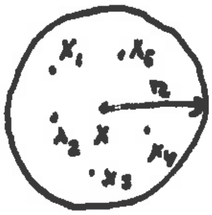
Proof: $\boxed{\text{"}\Rightarrow\text{"}}$ Assume that $X$ is proper. Let $(x_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $X$. As a Cauchy sequence, $(x_i)_{i\in\mathbb{N}}$ is bounded. Thus there exists $r > 0$ such that $x_i \in B(x_1, r)$ for every $i \in \mathbb{N}$. Since $X$ is proper, $\overline{B(x_1, r)}$ is compact and the sequence $(x_i)_{i\in\mathbb{N}}$ has a limit.

$\boxed{\text{"}\Leftarrow\text{"}}$: Assume that $X$ is complete. Let $F \subset X$ be closed and bounded. ~~As a subset of~~ A closed subset of a complete metric space is compact if and only if it is totally bounded, see Theorem 3.5.6 in [Friedman]. As a subset of a totally bounded space, $F$ is totally bounded. Thus $F$ is compact. $\square$

1.9. <u>Theorem</u>. If $X$ is a metric space with a doubling measure $\mu$, then $X$ is doubling as a metric space.

<u>Proof</u>: Let $B(x,r)$, $x \in X$, $0<r<\infty$, be a ball in $X$. Assume that there are $C$ points $\{x_i\}_{i=1}^{C} \subset B(x,r)$ such that

$$d(x_i, x_j) \geq \frac{r}{2} \quad \text{for every } i \neq j.$$




We observe that

$$B(x_i, \tfrac{r}{4}) \cap B(x_j, \tfrac{r}{4}) = \emptyset, \quad i \neq j,$$

and $B(x_i, \frac{r}{4}) \subset B(x, 2r)$ for every $i$. Thus

$$\begin{aligned}\sum_{i=1}^{C} \mu(B(x_i, \tfrac{r}{4})) &= \mu\Big(\bigcup_{i=1}^{C} B(x_i, \tfrac{r}{4})\Big) \\ &\leq \mu(B(x, 2r)) \\ &\leq c_d\, \mu(B(x,r)).\end{aligned}$$

On the other hand, $B(x,2r) \subset B(x_i, 8r)$ for every $i$ and thus

$$\mu(B(x,2r)) \leq \mu(B(x_i, 8r)) \leq c_d^{5} \mu(B(x_i, \tfrac{r}{4}))$$

for every $i$.

This implies

$$\mu(B(x,r)) \geq \frac{1}{C_d} \sum_{i=1}^{C} \mu\left(B\left(x_i, \tfrac{r}{4}\right)\right) \geq \frac{C}{C_d^6} \mu(B(x, 2r)) \geq \frac{C}{C_d^6} \mu(B(x,r))$$

and thus $C \leq C_d^6$. This shows that the ball $B(x,r)$ can be covered by at most $C_d^6$ balls with radius $\frac{r}{2}$, since the maximal collection of balls of radius $\frac{r}{2}$ whose centers have distance at least $\frac{r}{2}$ covers the ball $B(x,r)$.
The maximal collection can be constructed by starting with the center and adding points as long as possible. $\square$

<u>Remark</u>. The existence of a doubling measure and the doubling property of a space are almost equivalent, since if the space is a complete doubling metric space, then there exits a doubling measure on it, see [Heinonen].

1.10. <u>Takeaways</u>. Assume that $X$ is a (complete) metric space with a doubling measure $\mu$.

(i) ~~If X is complete, then~~ $X$ is proper, in particular, $\overline{B}(x,r)$, $x \in X$, $0 < r < \infty$, is compact.

(ii) Since every compact subset of a metric space is separable, $X$ is separable.

(iii) The local compactness and separability imply that every open set is $\sigma$-compact. In particular, $X$ is $\sigma$-finite.

(iv) $X$ is a locally compact Hausdorff space with the property that every open set is $\sigma$-compact. If $\mu$ is a Borel (regular) measure on $X$ such that the measure of every compact set is finite, then

$$\mu(U) = \sup\{\mu(K) : K \subset U \text{ compact}\}$$ (This holds also for $\mu$-measurable sets.)

for open sets $U \subset X$ and

$$\mu(A) = \inf\{\mu(V) : A \subset V,\ V \text{ open}\}$$

for every $A \subset X$. In other words, every locally finite Borel measure is a Radon measure, see [Rudin].

(v) Covering theorems and the Lebesgue differentiation theorem holds.

The following result gives a concept of dimension related to the measure.

1.11. <u>Lemma</u>. Let $B(x,R)$ be a ball in $X$. If $y \in B(x,R)$, $0 < r \le R$ and $\delta \ge \log_2 c_d$, then

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge 4^{-\delta}\left(\frac{r}{R}\right)^{\delta}.$$


<u>Proof</u>: Let $k \in \mathbb{N}$ such that $2^{k-1} r \le R \le 2^k r$. Then $B(x,R) \subset B(y, 2^{k+1} r)$ (and ~~thus~~ $B(y,r) \subset B(x,2R)$). This implies

$$\mu(B(x,R)) \le c_d^{k+1} \mu(B(y,r)) \left(\le c_d^{k+2} \mu(B(x,R))\right).$$

Thus

$$\begin{aligned}
\mu(B(y,r)) &\ge c_d^{-(k+1)} \mu(B(x,R)) = \left(2^{\log_2 c_d}\right)^{-(k+1)} \mu(B(x,R)) \\
&\underset{\delta \ge \log_2 c_d}{\ge} 2^{-\delta(k+1)} \mu(B(x,R)) = 2^{-\delta(1+1)}\, 2^{-\delta(k-1)} \mu(B(x,R)) \\
&\underset{\frac{r}{R} \le 2^{-(k-1)}}{\ge} 4^{-\delta}\left(\frac{r}{R}\right)^{+\delta} \mu(B(x,R)). \quad \square
\end{aligned}$$

1.12. <u>Remark</u>. The Lebesgue measure on the Euclidean space $\mathbb{R}^n$ is doubling with $c_\mu = 2^n$. In this case the best exponent $\delta$ in Lemma 1.11 is $\log_2 c_\mu = n$.

Next we consider an inequality to the reverse direction, which is true in a connected space. More generally, we may consider a uniformly perfect metric space which satisfies the property that there exists a constant $C \ge 1$ such that, for every $x \in X$ and $0 < r < \infty$, the set $B(x,r) \setminus B(x,\frac{r}{C})$ is nonempty whenever the set $X \setminus B(x,r)$ is nonempty. This condition excludes isolated components in a uniform manner. It is clear that connected spaces are uniformly perfect.

1.13. <u>Lemma</u>. Assume that $X$ is connected. For $0<k<1$, there exists a constant $c_1 = c_1(c_\mu, k)$ with $0 < c_1 < 1$ such that

$$\mu(B(x,kr)) \le c_1 \mu(B(x,r))$$

for every $x \in X$ and $0 < r < \frac{1}{2}\operatorname{diam} X$. Moreover, there exist constants $c_2 = c_2(c_\mu)$ and $\sigma = \sigma(c_\mu) > 0$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \le c_2 \left(\frac{r}{R}\right)^{\sigma}$$

for every ball $B(x,R)$, $y \in B(x,R)$ and $0 < r \le R$.

<u>Proof</u>: Let $0<k<1$ and $B(x,r)$ be a ball. Let $y \in X$ be such that

$$d(y,x) = \tfrac{1}{2}(1+k)r < r.$$

Such a point exists by the connectedness of $X$ and the assumption that $0<r<\frac{1}{2}\operatorname{diam} X$.

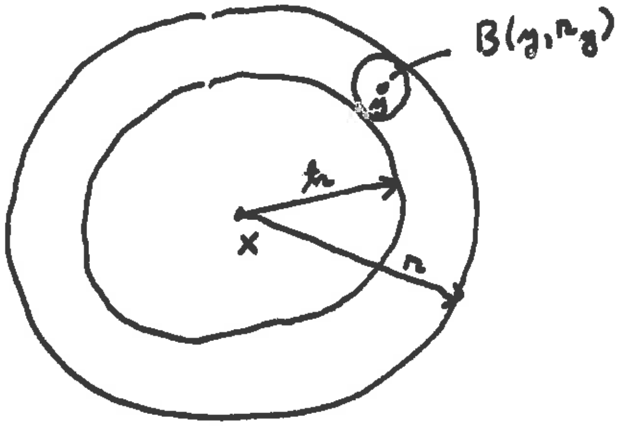
Let $r_y = \frac{1}{2}(1-t)r$. Then

$$B(y, r_y) \subset B(x,r) \setminus B(x, tr)$$

and

$$B(x,r) \subset B(y, 2r) = B(y, s\tfrac{1}{2}(1-t)r) = B(y, s r_y),$$

where $s = \dfrac{4}{1-t}$.




By Lemma 1.11, we have

$$\mu(B(x,r)) \le \mu(B(y, s r_y)) \\ \le 4^{\delta} s^{\delta} \mu(B(y, r_y)),$$

where $\delta = \log_2 c_d$. Thus

$$\mu(B(x,tr)) \le \mu(B(x,r)) - \mu(B(y,r_y)) \\ \le \left(1 - \frac{1}{4^{\delta} s^{\delta}}\right) \mu(B(x,r)).$$

This proves the first claim with $c_1(c_d, t) = 1 - \dfrac{1}{4^{\delta} s^{\delta}}$.

To prove the second claim, let $B(x,R)$ be a ball, $y \in B(x,R)$ and $0 < r \leq R$. Then $B(y,R) \subset B(x,2R)$ and

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \leq c_d \frac{\mu(B(y,r))}{\mu(B(y,R))}.$$

Let $k \in \mathbb{N}$ such that $2^{k-1} r \leq R < 2^k r$. Then the first claim with $t \geq \frac{1}{2}$ gives $c_1$ with $0 < c_1 < 1$ and

$$\begin{aligned} \mu(B(y,r)) &\leq c_1^{k-1} \mu(B(y, 2^{k-1} r)) \\ &\leq c_1^{k-1} \mu(B(y,R)), \end{aligned}$$

where

$$\begin{aligned} c_1^{k-1} &= (2^{\log_2 c_1})^{k-1} = (2^{k-1})^{\log_2 c_1} \\ &= 2^{-\log_2 c_1} (2^k)^{\log_2 c_1} \\ &\leq \frac{1}{c_1} \left(\frac{R}{r}\right)^{\log_2 c_1} = \frac{1}{c_1} \left(\frac{r}{R}\right)^{-\log_2 c_1}. \end{aligned}$$

Thus

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \leq c_d c_1^{k-1} \frac{\mu(B(y,R))}{\mu(B(y,R))} \leq c_2 \left(\frac{r}{R}\right)^{-\log_2 c_1}. \qquad \square$$