2. Upper gradients

2.1. Paths and line integrals. Let (X, d) be a metric space. A path in X is a continuous mapping $\gamma : [a,b] \to X$. The length of a path $\gamma : [a,b] \to X$ is

$$\ell(\gamma) = \sup \sum_{i=1}^{m} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all subdivisions

$$a = t_1 \le t_2 \le \dots \le t_{m+1} = b.$$

A path γ is rectifiable, if $\ell(\gamma) < \infty$. ~~A subpath is a restriction~~ of ~~a path to a subinterval. A path is locally rectifiable, if so.~~

Remark. A path $\gamma : [a,b] \to \mathbb{R}^n$ is rectifiable if and only if the coordinate functions are continuous and of bounded variation.

The length function associated with a rectifiable path $\gamma : [a,b] \to X$ is $s_\gamma : [a,b] \to [0, \ell(\gamma)]$, $s_\gamma(t) = \ell(\gamma|_{[a,t]})$. If $\gamma : [a,b] \to X$ is a rectifiable path, there exits a unique path $\tilde{\gamma} : [0, \ell(\gamma)] \to X$ such that

$$\gamma = \tilde{\gamma} \circ s_\gamma.$$

Moreover, $\ell(\tilde{\gamma}|_{[0,t]}) = t$ for every $t \in [0, \ell(\gamma)]$. The path $\tilde{\gamma}$ is called the parametrization of γ by the arc length. Note that $\tilde{\gamma}$ is 1-Lipschitz continuous and thus absolutely continuous.

If $\gamma : [a,b] \to X$ is a path, the set

$$|\gamma| = \{\gamma(t) : t \in [a,b]\}$$

is a curve. We shall not distinguish paths and curves although this is dangerous. See [Heinonen, Koskela, Shanmugalingam and Tyson] for more.

For a rectifiable path γ and a Borel function $\rho : X \to [0,\infty]$, the line integral of ρ over γ is

$$\int_\gamma \rho \, ds = \int_0^{\ell(\gamma)} \rho(\tilde{\gamma}(t))\, dt.$$

Remark. Since $\rho \circ \tilde{\gamma}$ is a nonnegative Borel function on $[0, \ell(\gamma)]$ it is measurable and the integral exists with value in $[0,\infty]$.

Remark. If $\gamma : [a,b] \to \mathbb{R}^n$ is absolutely continuous, that is, the coordinate functions are absolutely continuous, then

$$\int_\gamma \rho \, ds = \int_a^b \rho(\gamma(t))\,|\gamma'(t)|\, dt,$$

where $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$ and $\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t))$.

2.2. Modulus of a path family. Let μ be a Borel regular measure on a metric space X. Let Γ be a family of paths in X and $p \geq 1$. The p-modulus of Γ is

$$\mathrm{Mod}_p(\Gamma) = \inf \int_X \rho^p \, d\mu,$$

where the infimum is taken over all Borel functions $\rho : X \to [0,\infty]$ satisfying

$$\int_\gamma \rho \, ds \geq 1$$

for every locally rectifiable path $\gamma \in \Gamma$. Functions ρ above are called admissible densities for Γ. Note that the modulus has value in $[0,\infty]$. If there are no admissible densities, we set $\mathrm{Mod}_p(\Gamma) = \infty$.

By definition, the modulus of the family of all paths in X that are not locally rectifiable is zero, and the modulus of every family containing a constant path is infinity.

Next we show that the modulus is an outer measure on the set of all ~~(locally rectifiable)~~ paths.

2.3. <u>Lemma</u>. (i) $\operatorname{Mod}_p(\emptyset) = 0$,

(ii) $\operatorname{Mod}_p(\Gamma_1) \le \operatorname{Mod}_p(\Gamma_2)$ if $\Gamma_1 \subset \Gamma_2$ and

(iii) $\operatorname{Mod}_p\left(\bigcup_{i=1}^{\infty} \Gamma_i\right) \le \sum_{i=1}^{\infty} \operatorname{Mod}_p(\Gamma_i)$.

<u>Proof</u>: $\boxed{\text{(i)}}$: If $\Gamma = \emptyset$, then $\rho = 0$ is admissible for Γ and thus $\operatorname{Mod}_p(\Gamma) = 0$.

$\boxed{\text{(ii)}}$: If $\Gamma_1 \subset \Gamma_2$, then every ρ that is admissible for Γ_2 is also admissible for Γ_1 and thus $\operatorname{Mod}_p(\Gamma_1) \le \operatorname{Mod}_p(\Gamma_2)$.

$\boxed{\text{(iii)}}$: Without loss of generality, we may assume that $\operatorname{Mod}_p(\Gamma_i) < \infty$ for every i. Let $\varepsilon > 0$. For every i there exists an admissible density ρ_i for Γ_i such that

$$\int_X \rho_i^p \, d\mu \le \operatorname{Mod}_p(\Gamma_i) + \frac{\varepsilon}{2^i}.$$

Then the function $\rho = \left(\sum_{i=1}^{\infty} \rho_i^p\right)^{\frac{1}{p}}$ is admissible for $\bigcup_{i=1}^{\infty} \Gamma_i$, since $\rho \ge \rho_i$ for every $i \in \mathbb{N}$ and thus

$$\int_\gamma \rho \, ds \ge \int_\gamma \rho_i \, ds \ge 1$$

for every $\gamma \in \bigcup_{i=1}^{\infty} \Gamma_i$. This implies

$$\operatorname{Mod}_p\left(\bigcup_{i=1}^{\infty} \Gamma_i\right) \le \int_X \rho^p \, d\mu \le \sum_{i=1}^{\infty} \int_X \rho_i^p \, d\mu \le \sum_{i=1}^{\infty} \operatorname{Mod}_p(\Gamma_i) + \varepsilon. \qquad \square$$

2.4. Remark. If Γ_0 and Γ are path families in X such that for every path $\gamma \in \Gamma$ there is a subpath $\gamma_0 \in \Gamma_0$, then

$$\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\Gamma_0).$$

In this case we say that Γ majorizes Γ_0. To prove the inequality above, we observe that every ρ that is admissible for Γ_0 is also admissible for Γ. Roughly speaking, $\mathrm{Mod}_p(\Gamma)$ is large if there are many curves in Γ or if the curves in Γ are short.

2.5. Exceptional path families. A family of paths is called p-exceptional if it has p-modulus zero. We say that a property holds for p-almost every path, if the collection of paths for which the property fails to hold is p-exceptional.

We will use, without further mention, the following simple observation: if Γ' is a p-exceptional subfamily of Γ, then

$$\mathrm{Mod}_p(\Gamma) = \mathrm{Mod}_p(\Gamma \setminus \Gamma').$$

$\boxed{\geq}$: $\Gamma \setminus \Gamma' \subset \Gamma$ implies $\mathrm{Mod}_p(\Gamma \setminus \Gamma') \leq \mathrm{Mod}_p(\Gamma)$.

$\boxed{\leq}$:
$$\begin{aligned}\mathrm{Mod}_p(\Gamma) &= \mathrm{Mod}_p\big((\Gamma \setminus \Gamma') \cup \Gamma'\big) \\ &\leq \mathrm{Mod}_p(\Gamma \setminus \Gamma') + \underbrace{\mathrm{Mod}_p(\Gamma')}_{=0} \\ &= \mathrm{Mod}_p(\Gamma \setminus \Gamma').\end{aligned}$$

2.6. Lemma. Let Γ be a family of paths in X. Then $\mathrm{Mod}_p(\Gamma) = 0$ if and only if there exists a p-integrable Borel function $\rho : X \to [0,\infty]$ such that

$$\int_\gamma \rho \, ds = \infty$$

for every path $\gamma \in \Gamma$.

Proof: $\boxed{\Leftarrow}$: Assume that there exists a Borel function $\rho \geq 0$, $\rho \in L^p(X)$ such that $\int_\gamma \rho \, ds = \infty$ for every $\gamma \in \Gamma$. Then $\rho_i = 2^{-i}\rho$ is admissible for Γ for every $i \in \mathbb{N}$ and

$$\int_X \rho_i^p \, d\mu = 2^{-ip} \int_X \rho^p \, d\mu \longrightarrow 0 \text{ as } i \to \infty.$$

Thus $\mathrm{Mod}_p(\Gamma) = 0$.

$\boxed{\Rightarrow}$: Assume then that $\mathrm{Mod}_p(\Gamma) = 0$. For every $i \in \mathbb{N}$, there exists an admissible function ρ_i for Γ with

$$\int_X \rho_i^p \, d\mu \leq 2^{-ip}.$$

Then $\rho = \sum_{i=1}^{\infty} \rho_i$ is a Borel function and $\rho \in L^p(X)$. Since

$$\int_\gamma \rho_i \, ds \geq 1 \quad \text{for every } \gamma \in \Gamma$$

we have

$$\int_\gamma \rho \, ds = \sum_{i=1}^{\infty} \int_\gamma \rho_i \, ds = \infty \quad \text{for every } \gamma \in \Gamma. \qquad \square$$

2.7. Remark. If $f \geq 0$ is a Borel function with $f \in L^p(X)$, then

$$\int_\gamma f \, ds < \infty$$

for p-almost every ~~?~~ (nonconstant rectifiable) path γ in X.

A standard result in real analysis states that a convergent sequence in L^p has a subsequence that converges pointwise almost everywhere. The following lemma of Fuglede shows that there is an analogous result for p-exceptional families of paths.

2.8. Fuglede's lemma. Let $1 \le p < \infty$ and assume that f, f_i, $i \in \mathbb{N}$, are Borel functions such that $f_i \to f$ in $L^p(X)$ as $i \to \infty$. Then there exists a subsequence (f_{i_k}) such that

$$\lim_{k\to\infty} \int_\gamma |f_{i_k} - f| \, ds = 0$$

for p-almost every path γ in X.

Proof: There exists a subsequence (f_{i_k}) such that

$$\int_X |f_{i_k} - f|^p \, d\mu \le 2^{-(p+1)k}$$

for every $k \in \mathbb{N}$. Let

$$g_k = |f_{i_k} - f|$$

for every $k \in \mathbb{N}$,

$$\Gamma = \{ \gamma : \limsup_{k\to\infty} \int_\gamma g_k \, ds > 0 \}$$

and

$$\Gamma_k = \{ \gamma : \int_\gamma g_k \, ds > 2^{-k} \}, \quad k \in \mathbb{N}.$$

Then $\Gamma \subset \bigcup_{k=j}^{\infty} \Gamma_k$ for every $j \in \mathbb{N}$.

For every $k \in \mathbb{N}$, the function $2^k g_k$ is admissible for Γ_k. Thus

$$\operatorname{Mod}_p(\Gamma_k) \le 2^{kp} \int_X g_k^p \, d\mu = 2^{kp} \int_X |f_{i_k} - f|^p \, d\mu \le 2^{-k}$$

for every $k \in \mathbb{N}$. The subadditivity of the p-modulus gives

$$\operatorname{Mod}_p(\Gamma) \le \sum_{k=j}^{\infty} \operatorname{Mod}_p(\Gamma_k) \le 2^{-j+1}$$

for every $j \in \mathbb{N}$. This implies $\operatorname{Mod}_p(\Gamma) = 0$. $\square$

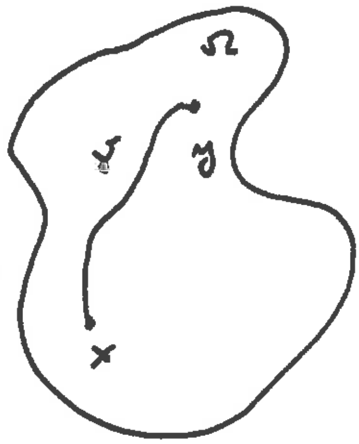
2.9. Remark. Let μ be a σ-finite Borel regular ~~measure on~~ X. Assume that $f: X \to [-\infty,\infty]$ is ~~measurable~~.
Then there exists a Borel function $g: X \to [-\infty,\infty]$ such that ~~$g(x) \ge f(x)$~~ ~~for every $x \in X$ and~~ $g(x) = f(x)$ for μ-almost every $x \in X$.
Moreover, if $f \ge 0$, then we can choose g such that $g(x) \ge f(x)$ for every $x \in X$. Thus the previous lemma holds true for the Borel representatives of L^p-functions.

2.10. Upper gradient. A Borel function $g: X \to [0,\infty]$ is an upper gradient of a function $u: X \to [-\infty,\infty]$, if

$$|u(x) - u(y)| \le \int_\gamma g \, ds$$

for every $x, y \in X$ and rectifiable path γ in X that joins the points x and y whenever $u(x)$ and $u(y)$ are finite, otherwise $\int_\gamma g \, ds = \infty$. If the inequality above holds for p-almost every rectifiable path γ, then g is a p-weak upper gradient of u.

2.11. <u>Remark</u>. Let Ω be an open and connected set in $\mathbb{R}^n$ and assume that $u \in C^1(\Omega)$. Let $\gamma : [a,b] \to \Omega$ be a path with $\gamma(a) = x$ and $\gamma(b) = y$. Let $\tilde{\gamma} : [0, \ell(\gamma)] \to \Omega$ be the parametrization of γ by the arc length. Then



$$\begin{aligned}
|u(x)-u(y)| &= |u(\gamma(a)) - u(\gamma(b))| \\
&= |u(\tilde{\gamma}(0)) - u(\tilde{\gamma}(\ell(\gamma)))| \\
&\underset{\substack{\uparrow \\ \text{abs. cont.}}}{=} \left| \int_0^{\ell(\gamma)} \frac{\partial}{\partial t}\big(u(\tilde{\gamma}(t))\big)\, dt \right| \\
&\underset{\substack{\uparrow \\ \text{chain rule}}}{=} \left| \int_0^{\ell(\gamma)} \nabla u(\tilde{\gamma}(t)) \cdot \tilde{\gamma}'(t)\, dt \right| \\
&\le \int_0^{\ell(\gamma)} |\nabla u(\tilde{\gamma}(t))| \underbrace{|\tilde{\gamma}'(t)|}_{=1 \text{ for a.e. } t \in [0,\ell(\gamma)]} dt \\
&= \int_\gamma |\nabla u|\, ds.
\end{aligned}$$

Thus $|\nabla u|$ is an upper gradient of u in Ω. In this sense an upper gradient is a generalization of $|\nabla u|$ to metric spaces.

2.12. <u>Examples</u>. (1) $g = \infty$ is an upper gradient for every function, but we are interested in upper gradients for which $\int_\gamma g\, ds$ is finite.

(2) An upper gradient is not unique: if g_u is an upper gradient of u and $g \ge 0$ is any Borel function, then $g_u + g$ is an upper gradient of u.

(3) If X has no nonconstant rectifiable paths, then $g=0$ is an upper gradient of every function u.

(4) If $u: X \to \mathbb{R}$ is L-Lipschitz, then $g=L$ is an upper gradient of u.

(5) If g_u and g_v are upper gradients of u and v, respectively, then $g_u + g_v$ is an upper gradient of $u+v$ and $|a| g_u$ is an upper gradient of au for every $a \in \mathbb{R}$.

(6) If g_u and g_v are upper gradients of u and v, respectively, then $g_u - g_v$ is not an upper gradient of $u-v$, in general. Let $u(x)=x$ and $v(x)=-x$. Then $g_u(x)=g_v(x)=1$ will do as an upper gradient in $\mathbb{R}$. However $g_u(x)-g_v(x)=0$ is not an upper gradient of $u(x)-v(x)=2x$.

The concept of an upper gradient is purely metric, but the concept of a p-weak upper gradient depends on the underlying measure structure and p. The pure upper gradient theory does not seem to be very fruitful, since this concept is not stable under limits. Moreover, weak upper gradients can be modified on a set of measure zero.

2.13. <u>Lemma</u>. Let g be a p-weak upper gradient of u. If $f: X \to [0,\infty]$ is a Borel function such that $f=g$ μ-almost everywhere, then f is a p-weak upper gradient of u.

<u>Proof</u>: Let $g_i = |g-f|$ for every $i \in \mathbb{N}$. Then the sequence $(g_i)_{i\in\mathbb{N}}$ trivially converges to 0 in $L^p(X)$, and by Fuglede's lemma 2.8,

$$\int_\gamma |g-f|\, ds = 0$$

for p-almost every path γ in X. Let γ be a rectifiable path with endpoints x and y such that

$$|u(x)-u(y)| \le \int_\gamma g\, ds \quad \text{and} \quad \int_\gamma |g-f|\, ds = 0.$$

Then

$$|u(x)-u(y)| \le \int_\gamma g\, ds = \int_\gamma g\, ds - \int_\gamma |g-f|\, ds \le \int_\gamma f\, ds. \qquad \square$$

2.14. <u>Remark</u>. The previous lemma implies that if E is a Borel set with $\mu(E)=0$ and g is a p-weak upper gradient of u, then $g\chi_{X\setminus E}$ is a p-weak upper gradient of u. The corresponding modification on a set of measure zero cannot be done for upper gradients. For example, let $u: \mathbb{R}^2 \to \mathbb{R}$, $u(x) = x$. Then $g=1$ is an upper gradient of u, but $g\chi_{\mathbb{R}^2\setminus\mathbb{R}}$ is not.

The next result asserts that every p-weak upper gradient can be approximated by upper gradients.

2.15. <u>Lemma</u>. If a Borel function ~~and $g \in L^p(X)$~~ g is a p-weak upper gradient of a function u, then there exists a sequence $(g_i)_{i\in\mathbb{N}}$ of upper gradients of u such that ~~$g_i \le g_{i+1}$~~ $g \le g_{i+1} \le g_i$ and

$$\| g_i - g \|_{L^p(X)} \to 0 \quad \text{as } i \to \infty.$$

Proof: Let Γ be the path family with $\operatorname{Mod}_p(\Gamma)=0$ for which

$$|u(x)-u(y)| \le \int_\gamma g\,ds$$

for every $x,y\in X$ does not hold true. By Lemma 2.6, there exists a Borel function $\rho \ge 0$ such that $\rho\in L^p(X)$ and

$$\int_\gamma \rho\,ds = \infty$$

for every $\gamma\in\Gamma$. Let $g_i : X \to [0,\infty]$, $g_i = g + 2^{-i}\rho$. Then every g_i is an upper gradient of u, (g_i) is decreasing and

$$\|g_i - g\|_{L^p(X)} = 2^{-i}\|\rho\|_{L^p(X)} \to 0 \text{ as } i\to\infty. \qquad \square$$

The following result gives nontrivial examples of upper gradients.

2.16. *Theorem*. If $f: X\to\mathbb{R}$ is locally Lipschitz, then the pointwise lower dilation

$$\operatorname{lip} f(x) = \liminf_{r\to 0} \sup_{y\in B(x,r)} \frac{|f(y)-f(x)|}{r}$$

is an upper gradient of f.

Proof: Let $\gamma : [0, \ell(\gamma)] \to X$ be a path parametrized by the arc length. Then γ is 1-Lipschitz and $f\circ\gamma$ is Lipschitz and thus absolutely continuous on $[0,\ell(\gamma)]$. This gives

$$
\begin{aligned}
|f(\gamma(0)) - f(\gamma(\ell(\gamma)))| &\underset{\text{Remark 2.11}}{\le} \int_0^{\ell(\gamma)} |(f\circ\gamma)'(t)|\,dt \\
&\overset{\text{See the next page}}{\le} \int_0^{\ell(\gamma)} \operatorname{lip} f(\gamma(t))\,dt \\
&= \int_\gamma \operatorname{lip} f\,ds. \qquad \boxtimes
\end{aligned}
$$

~~As in Remark 2.11~~

Observe, that for every $\varepsilon > 0$ we have (since γ is 1-lipschitz)

$$\cancel{|\gamma(t+r)-\gamma(t)| \le |r|}. \quad d(\gamma(t+r),\gamma(t)) \le |r|.$$

Thus $\gamma(t+r) \in B(\gamma(t),(1+\varepsilon)|r|)$. As a lipschitz function $(f\circ\gamma)'(t)$ exists for almost every $t \in (0, \ell(\gamma))$ and

$$|(f\circ\gamma)'(t)| = \lim_{r\to 0} \frac{|f(\gamma(t+r)) - f(\gamma(t))|}{|r|}$$

$$\le \liminf_{r\to 0} \sup_{y\in B(\gamma(t),(1+\varepsilon)r)} \frac{|f(y)-f(\gamma(t))|}{|r|} = (1+\varepsilon)\,\mathrm{lip}\, f(\gamma(t)).$$

Thus $|(f\circ\gamma)'(t)| \le \mathrm{lip}\, f(\gamma(t))$ for a.e. $t \in (0,\ell(\gamma))$. To see that $\mathrm{lip}\, f$ is a Borel function, we consider

$$f_r(x) = \sup_{y\in B(x,r)} \frac{|f(y)-f(x)|}{r}.$$

The set $\{x \in X : f_r(x) > t\}$ is open for every $t > 0$ and thus f_r is lower semicontinuous. It follows that

$$\mathrm{lip}\, f = \liminf_{r\to 0} f_r(x)$$

is Borel, since the infimum of (countably many) Borel functions is Borel and a limit of Borel functions is Borel. $\square$

2.17. <u>Sobolev spaces</u>. The definition of a Sobolev space on a metric measure space can be based on upper gradients. Let $\tilde{N}^{1,p}(X)$ with $1 \le p < \infty$ be the collection of functions $u : X \to [-\infty,\infty]$ with $u \in L^p(X)$ which have a p-weak upper gradient $g \in L^p(X)$. For $u \in \tilde{N}^{1,p}(X)$, let

$$\|u\|_{\tilde{N}^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all p-weak upper gradients g of u. Note that here we consider functions that are defined at every point instead of equivalence classes of functions that coincide almost everywhere. $\tilde{N}^{1,p}(X)$ is a vector space with a seminorm $\|\cdot\|_{\tilde{N}^{1,p}(X)}$. In particular, we have the triangle inequality

$$\|u+v\|_{\tilde{N}^{1,p}(X)} \le \|u\|_{\tilde{N}^{1,p}(X)} + \|v\|_{\tilde{N}^{1,p}(X)}.$$

We define an equivalence relation in $\tilde{N}^{1,p}(X)$ by setting

$$u \sim v \iff \|u-v\|_{\tilde{N}^{1,p}(X)} = 0.$$

The Newtonian space (Sobolev space) is the quotient space

$$N^{1,p}(X) = \tilde{N}^{1,p}(X)/\sim$$

with the norm $\|u\|_{N^{1,p}(X)} = \|u\|_{\tilde{N}^{1,p}(X)}$.

2.18. <u>Absolute continuity on paths</u>. A function $u: X \to \mathbb{R}$ is absolutely continuous on p-almost every path in X, denoted by $u \in ACC_p(X)$, if $u \circ \tilde{\gamma}$ is absolutely continuous on $[0, \ell(\gamma)]$ for p-almost every rectifiable path γ. Recall that $\tilde{\gamma}$ is the arc length parametrization of γ.

Recall that a function $f: [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{m} |\overset{f(b_i)-f(a_i)}{\cancel{f(x_{i+1})-f(x_i)}}| < \varepsilon,$$

where $\overset{a \le a_1 < b_1 \le a_2 < b_2 \le \dots \le a_m < b_m \le b}{\cancel{a \le x_1 \le x_2 \le \dots \le x_{m+1} \le b}}$ is a subdivision of $[a,b]$ such that

$$\sum_{i=1}^{m} \underset{(b_i-a_i)}{\cancel{(x_{i+1}-x_i)}} < \delta.$$

Recall that f is absolutely continuous on $[a,b]$ if and only if $f' \in L^1([a,b])$ and

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

for every $x \in [a,b]$. This is a general form of the fundamental theorem of calculus.

2.17. Lemma. If $u \in \tilde{N}^{1,p}(X)$, then $u \in ACC_p(X)$.

Proof: Since $u \in \tilde{N}^{1,p}(X)$, it has ~~an~~ a p-weak upper gradient $g \in L^p(X)$.

Then

$$|u(x) - u(y)| \le \int_\gamma g\,ds \qquad (*)$$

for p-almost every rectifiable path γ that joins x and y. Let

$$\Gamma_0 = \{\gamma : (*) \text{ does not hold for } u \text{ and } g\},$$

$$\Gamma_\infty = \{\gamma : \int_\gamma g\,ds = \infty\} \text{ and}$$

$$\Gamma = \{\gamma : \gamma \text{ has a subpath in } \Gamma_0 \cup \Gamma_\infty\}.$$

By the definition of the p-weak upper gradient $\operatorname{Mod}_p(\Gamma_0) = 0$. Since $g \in L^p(X)$, Lemma 2.6 implies $\operatorname{Mod}_p(\Gamma_\infty) = 0$. By the definition of the p-modulus $\operatorname{Mod}_p(\Gamma) \le \operatorname{Mod}_p(\Gamma_0 \cup \Gamma_\infty) = 0$. If γ is a rectifiable path that does not belong to Γ, then

$$|u(x) - u(y)| \le \int_{\gamma_{xy}} g\,ds < \infty$$

for every $x, y \in |\gamma|$.

$0 \le a_1 < b_1 \le a_2 < b_2 \le \dots \le a_m < b_m \le \ell(\gamma)$

Let ~~$0 \le t_1 \le t_2 \le \dots \le t_{m+1} \le \ell(\gamma)$~~ be a subdivision of $[0,\ell(\gamma)]$.

Then

$$\sum_{i=1}^{m} \left| u(\tilde{\gamma}(\overset{b_i}{\cancel{t_{i+1}}})) - u(\tilde{\gamma}(\overset{a_i}{\cancel{t_i}})) \right| \le \int_{\bigcup_{i=1}^{m} \overset{\cancel{[t_i, t_{i+1}]}}{[a_i, b_i]}} g(\tilde{\gamma}(t))\, dt$$

which implies that $u \circ \gamma$ is absolutely continuous on $[0,\ell(\gamma)]$ by the absolute continuity of the integral. $\square$

2.20. <u>Remark</u>. By Lemma 2.19, the fundamental theorem of calculus holds for ~~$\tilde{\gamma}$~~ $u \circ \gamma$ on p-almost every path γ. This will be a very useful tool when we prove the Leibniz or chain rule for p-weak upper gradients. Note that Lemma 2.19 is valid if u has a p-integrable p-weak upper gradient without itself being p-integrable.

2.21. <u>Lemma</u>. Assume that $u \in ACC_p(X)$.

(i) If $g \in L^p(X)$ is a p-weak upper gradient of u, then

$$|(u \circ \tilde{\gamma})'(t)| \le g(\tilde{\gamma}(t)) \quad \text{for almost every } t \in [0,\ell(\gamma)] \quad (*)$$

holds for p-almost every rectifiable path γ in X.

(ii) If $g : X \to [0,\infty]$ is a measurable function and $(*)$ holds for p-almost every rectifiable path γ in X, then g is a p-weak upper gradient of u.

<u>Proof</u>: $\boxed{(i)}$ By the proof of lemma 2.19, p-almost every rectifiable path γ satisfies the following properties: u is absolutely continuous on γ, the upper gradient inequality

$$|u(x) - u(y)| \le \int_\gamma g\, ds$$

holds for γ and every subpath of γ and $\int_\gamma g\, ds < \infty$.

Let γ be one of those paths. As an absolutely continuous function, $u \circ \tilde{\gamma}$ is differentiable at almost every $t \in [0, \ell(\gamma)]$. Moreover, almost every $t \in [0, \ell(\gamma)]$ is a Lebesgue point of $g \circ \tilde{\gamma}$. For such t, we have

$$\begin{aligned} |(u \circ \tilde{\gamma})'(t)| &= \lim_{h \to 0+} \left| \frac{u(\tilde{\gamma}(t+h)) - u(\tilde{\gamma}(t))}{h} \right| \\ &\le \lim_{h \to 0+} \frac{1}{h} \int_t^{t+h} g(\tilde{\gamma}(s))\, ds \\ &= g(\tilde{\gamma}(t)). \end{aligned}$$

$\boxed{\text{(ii)}}$ Let γ be a path on which u is absolutely continuous, $(*)$ holds and $\int_\gamma g\, ds$ is well-defined. These properties hold for p-almost every rectifiable path, see [Björn, Lemma 1.43] for the last property. Then

$$\begin{aligned} |u(x) - u(y)| &= |u(\tilde{\gamma}(0)) - u(\tilde{\gamma}(\ell(\gamma)))| \\ &\le \int_0^{\ell(\gamma)} |(u \circ \tilde{\gamma})'(t)|\, dt \\ &\le \int_0^{\ell(\gamma)} g(\tilde{\gamma}(t))\, dt \\ &= \int_\gamma g\, ds. \qquad \square \end{aligned}$$

2.22. <u>Leibniz rule</u>. Let $u, v : X \to \mathbb{R}$ be measurable functions with p-weak upper gradients $g_u, g_v \in L^p(X)$. Then $|u| g_v + |v| g_u$ is a p-weak upper gradient of uv.

<u>Proof</u>: By the assumptions, the function $|u|g_v + |v|g_u$ is measurable and nonnegative. By Lemma 2.17, $u, v \in ACC_p(X)$ and thus $uv \in ACC_p(X)$. p-almost every rectifiable ~~curve~~ path γ in X satisfies the following properties: u and v are absolutely continuous on γ and the upper gradient inequality holds for pairs u, g_u and v, g_v.

Let γ be one of these paths. By the absolute continuity of u, v and uv on γ and Lemma 2.21 (i), the derivatives

$$(u\circ\tilde{\gamma})'(t),\ (v\circ\tilde{\gamma})'(t) \text{ and } ((uv)\circ\tilde{\gamma})'(t)$$

exists and

$$|(u\circ\tilde{\gamma})'(t)| \le g_u(\tilde{\gamma}(t)) \text{ and } |(v\circ\tilde{\gamma})'(t)| \le g_v(\tilde{\gamma}(t))$$

for almost every $t \in [0, \ell(\gamma)]$. Thus

$$\begin{aligned} |((uv)\circ\tilde{\gamma})'(t)| &= |u(\tilde{\gamma}(t))(v\circ\tilde{\gamma})'(t) + v(\tilde{\gamma}(t))(u\circ\tilde{\gamma})'(t)| \\ &\le |u(\tilde{\gamma}(t))|\, g_v(\tilde{\gamma}(t)) + |v(\tilde{\gamma}(t))|\, g_u(\tilde{\gamma}(t)) \end{aligned}$$

and by Lemma 2.21, the function $|u|g_v + |v|g_u$ is a p-weak upper gradient of uv. $\square$

2.23. <u>Remark</u>. The existence of a Borel representative guarantees that there is no real problem in extending the notion of an upper gradient to measurable functions. More precisely, if g and $\tilde{g}$ are nonnegative measurable functions on X such that $g = \tilde{g}$ μ-almost everywhere, then

$$\int_\gamma g\, ds = \int_\gamma \tilde{g}\, ds$$

for p-almost every path γ, see [Björn, Lemma 1.43].

2.24. Minimal p-weak upper gradient. Let $u : X \to \mathbb{R}$ and $1 \le p < \infty$. If u has a p-weak upper gradient $g \in L^p(X)$, then it has a minimal p-weak upper gradient $g_u \in L^p(X)$ with $g_u \le g$ μ-almost everywhere in X for every p-weak upper gradient $g \in L^p(X)$ of u. Moreover, the minimal upper gradient is unique up to sets of measure zero.

Proof: Let

$$I = \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all p-weak upper gradients g of u. Since u has a p-weak upper gradient $g \in L^p(X)$, we have $0 \le I < \infty$. Let $(g_i)_{i \in \mathbb{N}}$ be a minimizing sequence of p-weak upper gradients of u such that

$$\lim_{i \to \infty} \|g_i\|_{L^p(X)} = I.$$

We may assume that the sequence $(g_i)_{i \in \mathbb{N}}$ is decreasing. If not, we may consider a new sequence

$$\tilde{g}_i = \min_{1 \le j \le i} g_j .$$

By Lemma 2.21 it can be shown that $\tilde{g}_i$ is a p-weak upper gradient of u for every $i \in \mathbb{N}$, see [Lemma 2.6, Björn]. The sequence $(\tilde{g}_i)$ is decreasing and

$$\lim_{i \to \infty} \|\tilde{g}_i\|_{L^p(X)} = I .$$

To see this, note that

$$I \le \liminf_{i \to \infty} \|\tilde{g}_i\|_{L^p(X)} \le \limsup_{i \to \infty} \|\tilde{g}_i\|_{L^p(X)} \le \lim_{i \to \infty} \|g_i\|_{L^p(X)} = I.$$

Since the sequence $(g_i)_{i\in\mathbb{N}}$ is nonnegative and decreasing, it converges pointwise to a Borel function g_u. By the dominated convergence theorem for the sequence $(f_i)_{i\in\mathbb{N}}$, with $f_i = (g_i - g_u)^p$, we have

$$\lim_{i\to\infty} \int_X |g_i - g_u|^p \, d\mu = \lim_{i\to\infty} \int_X (g_i - g_u)^p \, d\mu$$
$$= \int_X \lim_{i\to\infty} (g_i - g_u)^p \, d\mu = 0.$$

Thus $g_i \to g_u$ in $L^p(X)$ as $i\to\infty$ and

$$\|g_u\|_{L^p(X)} = \lim_{i\to\infty} \|g_i\|_{L^p(X)} = I.$$

Fuglede's lemma 2.8 implies that g_u is a p-weak upper gradient of u. This shows that there exists a p-weak upper gradient g_u of u such that it minimizes the L^p-norm.

To show the pointwise minimizing property, assume that there exists a p-weak upper gradient h of u such that $\mu(E)>0$ for the set

$$E = \{x\in X : g_u(x) > h(x)\}.$$

By Lemma 2.21 it can be shown that

$$g = h\chi_E + g_u \chi_{X\setminus E} = \min\{h, g\}$$

is a p-weak upper gradient of u. Then

$$\|g\|_{L^p(X)} \le \|h\|_{L^p(E)} + \|g_u\|_{L^p(X\setminus E)}$$
$$\underset{h<g_u \text{ on } E,\ \mu(E)>0}{<} \|g_u\|_{L^p(X)} = I,$$

which is impossible by the definition of I.

As for the uniqueness, if g is another minimal p-weak upper gradient of u, then $g \le g_u$ and $g_u \le g$ almost everywhere. Thus $g = g_u$ almost everywhere. $\square$

2.25. <u>Remark</u>. If $u, v : X \to \mathbb{R}$ are functions with minimal p-weak upper gradients g_u and g_v, respectively. Then

(i) $g_u = g_{-u}$,

(ii) $g_{u+v} \le g_u + g_v$,

(iii) $g_{|u|} \le g_u$ (in fact "=" holds), and

(iv) $g_{\lambda u} = |\lambda| g_u$, $\lambda \in \mathbb{R}$.

Moreover, if $f : \mathbb{R} \to \mathbb{R}$ is L-Lipschitz, then $g_{f\circ u} \le L g_u$.

Note that the minimal p-weak upper gradient has the smallest L^p-norm amongst all p-weak upper gradients of u. By Lemma 2.15, we could consider the minimum ~~of~~ over all upper gradients as well, but also in this case the minimizing function is a p-weak upper gradient.

2.26. <u>Corollary</u>. If $u \in N^{1,p}(X)$, then $\|u\|_{N^{1,p}(X)} = \|u\|_{L^p(X)} + \|g_u\|_{L^p(X)}$.