Aalto University
School of Science

CS-E4530 Computational Complexity Theory

Lecture 4: Reductions

Aalto University
School of Science
Department of Computer Science

Spring 2019

Agenda

- Many-to-one reductions
- Example: Graph colouring
- Turing reductions
- Closure and completeness

Reductions

- **Recall our previous discussions about reductions**
- **Reduction R from problem L_1 to problem L_2:**
  - an algorithm that transforms an instance x of problem L_1 to an equivalent instance $R(x)$ of problem L_2
- **Relates the complexities of problems L_1 and L_2**
  - Technical requirement: *efficiency*
  - Different notions of reduction
- **This lecture: formalise these notions**

Many-to-one Reductions

- **Our basic notion of reduction:**
  - Most reductions we meet on this course are so called "many-to-one" reductions
- **Reduction between decision problems L_1 and L_2**
  - Maps instances from L_1 to L_2
  - Preserves yes-instances and no-instances
  - No postprocessing

Many-to-one Reductions: Definitions

Definition

Let $L_1, L_2 \subseteq \{0,1\}^*$ be languages. A *many-to-one reduction* (often called simply *reduction*) from L_1 to L_2 is a computable function $R\colon \{0,1\}^* \to \{0,1\}^*$ such that for every $x \in \{0,1\}^*$

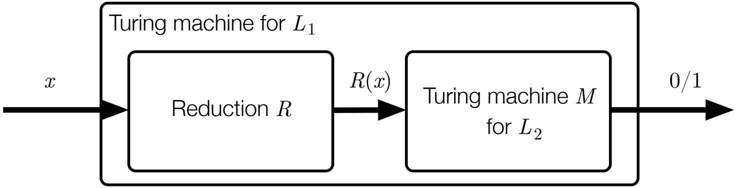
$$x \in L_1 \text{ if and only if } R(x) \in L_2 \,.$$

Definition

If there is a reduction from L_1 to L_2, we say that L_1 *reduces to* L_2, and write $L_1 \le L_2$.

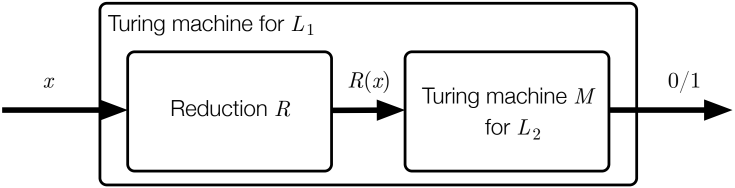
Using Reductions

- **Assume we have:**
  - A reduction from R from L_1 to L_2
  - A Turing machine M that decides L_2
- **Then we can decide L_1:**
  - Transform instance x of L_1 into an instance $R(x)$ of L_2
  - Decide $R(x)$ using M



Using Reductions

- **Assume we have:**
  - A reduction from R from L_1 to L_2 running *in time* $T_1(n)$
  - A Turing machine M that decides L_2 *in time* $T_2(n)$
- **Then we can decide L_1 *in time* $O(T_1(n) + T_2(T_1(n)))$:**
  - Transform instance x of L_1 into an instance $R(x)$ of L_2
  - Decide $R(x)$ using M
  - *Note:* $|R(x)| \leq T_1(|x|)$



Polynomial-time Reductions

Definition

Let $L_1, L_2 \subseteq \{0,1\}^*$ be languages. A *polynomial-time many-to-one reduction* or *Karp reduction*[a] from L_1 to L_2 is a polynomial-time computable function $R\colon \{0,1\}^* \to \{0,1\}^*$ such that for every $x \in \{0,1\}^*$

$$x \in L_1 \text{ if and only if } R(x) \in L_2 \,.$$

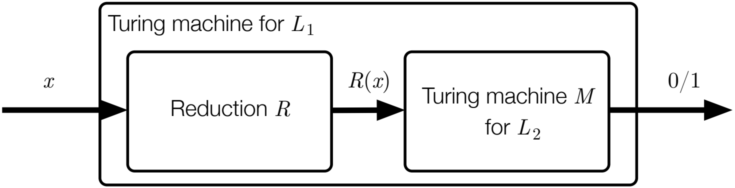
[a] In honour of Richard Karp, who first used this notion in his 1972 paper listing 21 fundamental NP-complete problems.

Definition

If there is a polynomial-time reduction from L_1 to L_2, we say that L_1 *reduces to* L_2 in polynomial time, and write $L_1 \leq_p L_2$.

Using Reductions

- **Assume we have:**
  - A *polynomial-time* reduction R from L_1 to L_2
  - A *polynomial-time* Turing machine M that decides L_2
- **Then we can decide L_1 in *polynomial time*:**
  - Transform instance x of L_1 into an instance $R(x)$ of L_2
  - Decide $R(x)$ using M
  - $q(p(n))$ is polynomial for polynomials q and p



Transitivity and Reflexivity

Theorem (Transitivity)

Let L_1, L_2 and L_3 be languages. If $L_1 \leq_p L_2$ and $L_2 \leq_p L_3$, then $L_1 \leq_p L_3$.

- **Proof:** Apply reductions sequentially.

Theorem (Transitivity)

Let L be a language. Then $L \leq_p L$.

- **Proof:** Trivial.

- **Together, these imply that $\leq_p$ is a *preorder*.**

Example: Graph Colourings

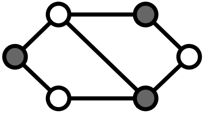
Definition

Let k be a fixed positive integer, and let $G = (V, E)$ be an undirected graph. A *k-colouring* of G is a function

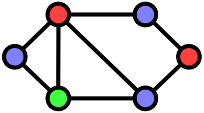
$$c\colon V \to \{1, 2, \ldots, k\}$$

such that for any two adjacent vertices v and u, $c(v) \neq c(u)$.

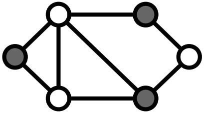
Some Colourings of a Simple Graph



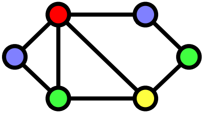
2-colouring



3-colouring



not valid
2-colouring



4-colouring

The k-colouring Problem

k-colouring problem (k-COL)

- **Instance:** Graph $G = (V, E)$.
- **Question:** Is there a k-colouring of G?

- **We shall use reductions to study relative complexity of the following k-colouring problems:**
  - *2-colouring*
  - *3-colouring*
  - *4-colouring*

2-colouring to 3-colouring

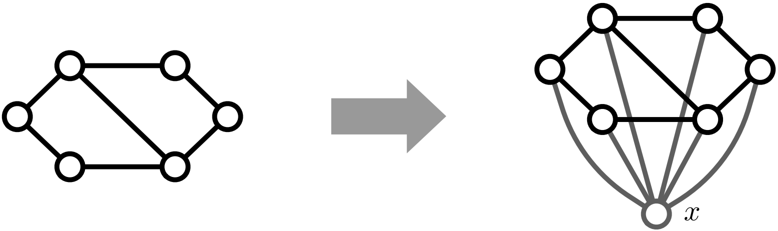
Theorem

The is a polynomial-time reduction from 2-colouring to 3-colouring.

- **We have to show that there is a polynomial time reduction R such that:**
  - R maps any graph G to a new graph $R(G)$
  - If G has a 2-colouring, then $R(G)$ has a 3-colouring
  - If $R(G)$ has a 3-colouring, then G has a 2-colouring

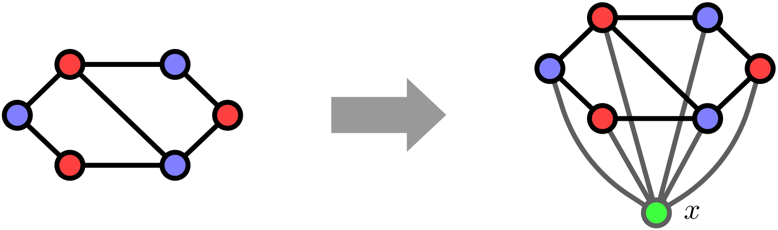
2-colouring to 3-colouring: Proof

- **Given input graph G, construct $R(G)$:**
  - Add a new vertex x to the graph
  - Connect x to all original vertices



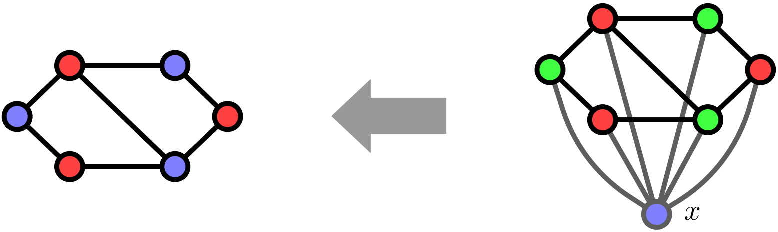
2-colouring to 3-colouring: Proof

- **If G has a 2-colouring, then $R(G)$ has 3-colouring:**
  - Colour original vertices the same way
  - Colour x with colour 3



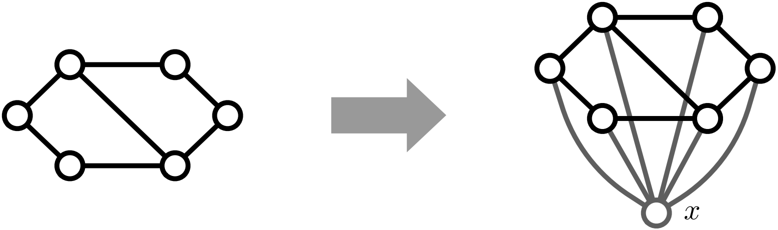
2-colouring to 3-colouring: Proof

- **If $R(G)$ has a 3-colouring, then G has 2-colouring:**
  - Original vertices cannot use the colour of x
  - Thus, original vertices are coloured with 2 colours
  - This is a 2-colouring of the original graph after renaming the colours



2-colouring to 3-colouring: Proof

- **Construction is clearly polynomial-time computable**
- **We have:** $2\text{-COL} \leq_p 3\text{-COL}$



3-colouring to 4-colouring

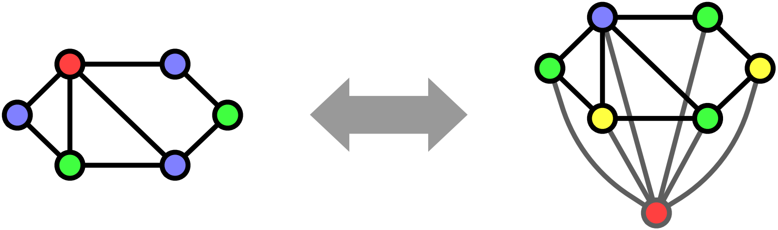
Theorem

The is a polynomial-time reduction from 3-colouring to 4-colouring.

- **We have to show that there is a polynomial time reduction R such that:**
  - R maps any graph G to a new graph $R(G)$
  - If G has a 3-colouring, then $R(G)$ has a 4-colouring
  - If $R(G)$ has a 4-colouring, then G has a 3-colouring
- *Do you immediately see why?*

3-colouring to 4-colouring

- **Same construction works for reduction from 3-colouring to 4-colouring**
  - In fact, from k-colouring to $(k+1)$-colouring
  - We have 2-COL $\leq_p$ 3-COL $\leq_p$ 4-COL $\leq_p \cdots$
  - *What about the other direction?*



4-colouring to 3-colouring

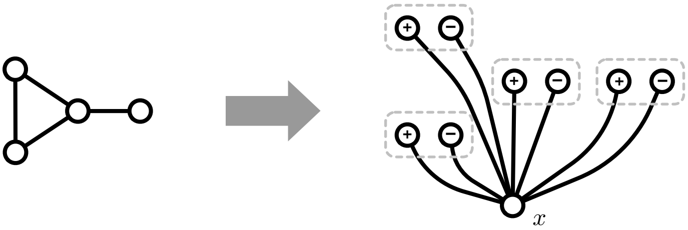
Theorem

The is a polynomial-time reduction from 4-colouring to 3-colouring.

- **We have to show that there is a polynomial time reduction R such that:**
  - R maps any graph G to a new graph $R(G)$
  - If G has a 4-colouring, then $R(G)$ has a 3-colouring
  - If $R(G)$ has a 3-colouring, then G has a 4-colouring
- **This requires considerably more work**

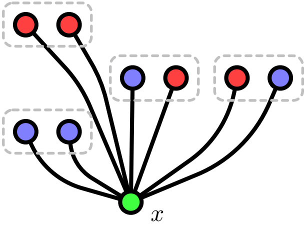
4-colouring to 3-colouring: Proof

- **Given input graph G, construct $R(G)$:**
  - We start with *a base vertex* x
  - For each original vertex v, add two new vertices v^+ and v^-
  - Connect v^+ and v^- to the vertex x



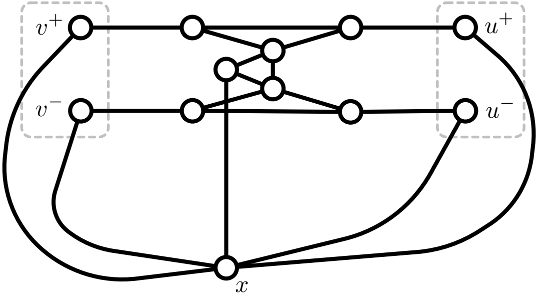
4-colouring to 3-colouring: Proof

- **The construction forces all 3-colourings of $R(G)$ to have certain form (if they exist):**
  - Vertex x has some colour (say, 3)
  - All vertices v^+, v^- have to use colours 1 and 2
  - **Idea:** use the tuple $(c(v^+), c(v^-))$ to define the colour in the original graph G



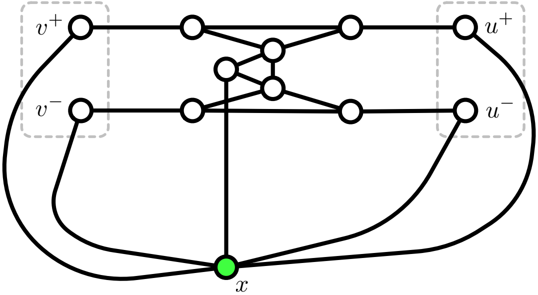
4-colouring to 3-colouring: Proof

- **For each edge $(u,v) \in E$ in the original graph G, we add a specific *gadget* to the new graph $R(G)$:**
  - Forces $(c(v^+),c(v^-)) \neq (c(u^+),c(u^-))$
  - That is, using $(c(v^+),c(v^-))$ as colour for v in the original graph gives a valid 4-colouring



4-colouring to 3-colouring: Proof

- **Construction forces certain colours in the gadget**
  - Assume $c(v^-) = c(u^-)$
  - We show this implies $c(v^+) \neq c(u^+)$



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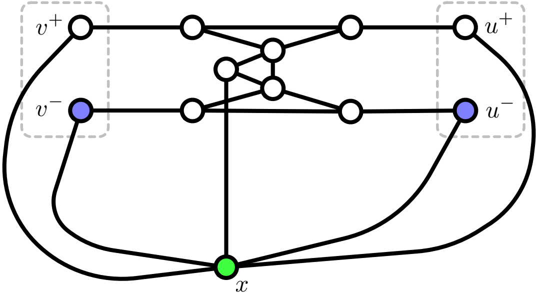

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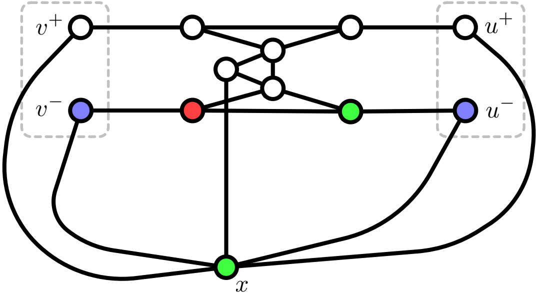

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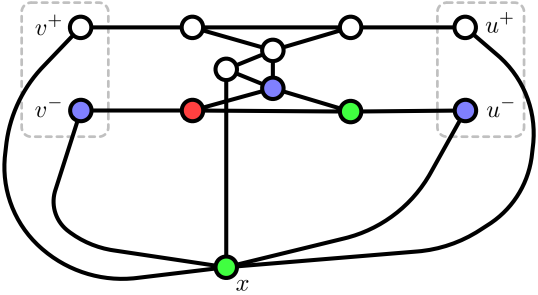

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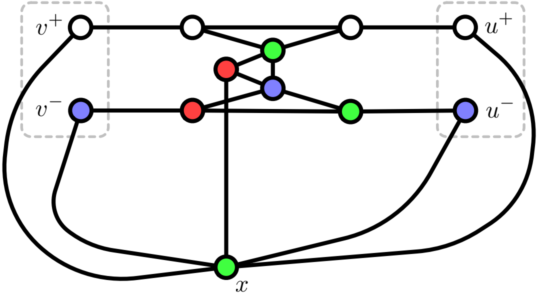

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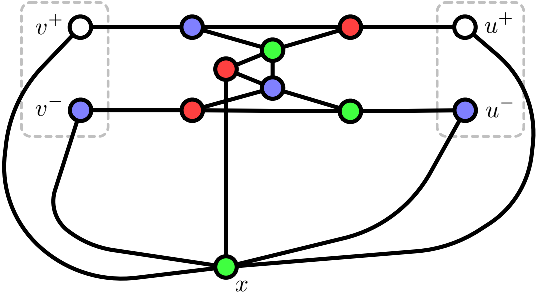

4-colouring to 3-colouring: Proof

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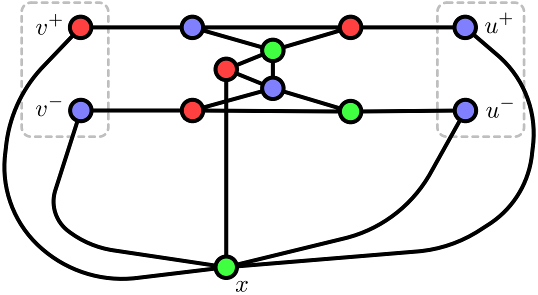

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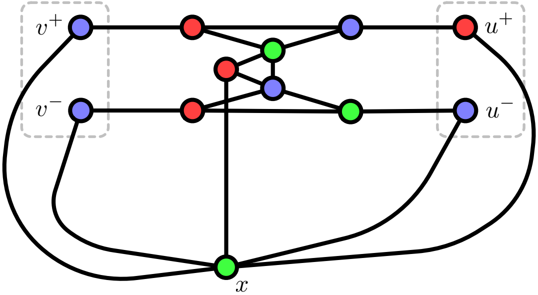

4-colouring to 3-colouring: Proof

- **Similar proof shows that if the original graph G has a 4-colouring, then we can colour $R(G)$ with 3 colours**
  - Reverse of the previous mapping
  - Use two colours for all nodes v^+ and v^-
  - Base vertex x uses the third colour
  - Each gadget can be completed in an obvious way

- **Complexity of the reduction**
  - If G has n vertices and m edges, then $R(G)$ has $1+2n+7m$ vertices
  - Clearly $R(G)$ can be computed in polynomial time

Complexity of Colouring Problems

- **We have proved that** $4\text{-COL} \leq_p 3\text{-COL}$
  - Similar reduction works for k-COL $\leq_p$ 3-COL for $k \geq 5$
  - All colouring problems are *equally hard* relative to polynomial reductions for $k \geq 3$
- **What about polynomial-time reduction from 3-colouring to 2-colouring?**

Complexity of Colouring Problems

- **We have proved that** $4\text{-COL} \leq_p 3\text{-COL}$
  - Similar reduction works for $k\text{-COL} \leq_p 3\text{-COL}$ for $k \geq 5$
  - All colouring problems are *equally hard* relative to polynomial reductions for $k \geq 3$

- **What about polynomial-time reduction from 3-colouring to 2-colouring?**
  - *2-colouring:* in class P
  - *3-colouring:* believed to be not in class P
  - This would imply that the reduction does not exist

Turing Reductions

- **More powerful notion of reduction:**
  - Corresponds to *subroutine calls*
  - Turing reduction may make multiple calls to the target language
  - Turing reduction may perform postprocessing

Oracle Turing Machines

- **Let $L \subseteq \{0,1\}^*$ be a language**
- **An *oracle Turing machine* M with oracle L is a TM:**
  - M has a special *oracle tape* (working tape)
  - M has a special state q_{query}
  - When M *enters* the state q_{query} and a string $x \in \{0,1\}^*$ is written on the oracle tape:
    - The head on the oracle tape moves to position 1
    - If $x \in L$, the oracle tape is rewritten with string 1
    - If $x \notin L$, the oracle tape is rewritten with string 0
  - The oracle call counts as *one step* in execution

Turing Reductions

Definition

Let $L_1, L_2 \subseteq \{0,1\}^*$ be languages. A *Turing reduction* from L_1 to L_2 is an oracle Turing machine with oracle L_2 that decides L_1.

Definition

Let $L_1, L_2 \subseteq \{0,1\}^*$ be languages. A *polynomial-time Turing reduction* or *Cook reduction*[a] from L_1 to L_2 is a polynomial-time oracle Turing machine with oracle L_2 that decides L_1.

[a] In honour of Stephen Cook, who introduced the notion of NP-completeness in 1971 using this reduction type.

Using Turing Reductions

- **Assume we have:**
  - A Turing reduction from R from L_1 to L_2 *in time* $T_1(n)$
  - A Turing machine M that decides L_2 *in time* $T_2(n)$
- **Then we can decide L_1 *in time* $O(T_1(n) + T_1(n) \cdot T_2(T_1(n)))$:**
  - Simulate all oracle calls with M
  - At most $T_1(n)$ calls, each instance at most $T_1(n)$ bits
- **Polynomial T_1, T_2 implies polynomial time for L_1**

Closure Under Reductions

Definition

Let R be a class of reductions, and let C be a class of decision problems. We say that C is *closed under R-reductions* if for all languages L_1 and L_2 the following holds: if $L_1 \leq_R L_2$ and $L_2 \in C$, then also $L_1 \in C$.

Hardness and Completeness

Definition

Let R be a class of reductions, and let C be a class of decision problems. We say that a language L is *C-hard under R-reductions* if for any language $L' \in C$, there is an R-reduction from L' to L.

Definition

We say that L is *C-complete under R-reductions* if L is C-hard under R-reductions and $L \in C$.

Completeness: Discussion

- **Complete problems are *the most difficult problems*:**
  - Assume that R-reductions are fast to compute compared to problems in C
  - If there is a fast algorithm for a complete problem, then there is a fast algorithm for *all* problems in C
- **Completeness allows discussion of a class in terms of a single complete problem**
  - Existence of concrete complete problems not obvious
  - We will make this idea more concrete in the next lectures

Reductions: Discussion

- **Meta-mathematical question:** ***what is the right notion of reduction to use?***
  - Why many-to-one reductions instead of Turing reductions?
  - Why specifically polynomial-time reductions?

- **General rule of thumb: reductions should be easy compared to the complexity class we are studying**
  - Weaker reductions means more fine-grained complexity picture
  - Stronger reductions are easier to work with
  - Use different reductions for studying different classes

Lecture 4: Summary

- Many-to-one reductions
- Completeness and hardness
- (Turing reductions)