3 STABILITY ANALYSIS

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on the topics of week 4:

- ☐ Stability of structure, principle of virtual work for large displacements, Green-Lagrange strain measure

- ☐ Principle of virtual work for stability analysis and stability analysis by FEM

- ☐ Non-linear stability term element contributions for beam and plate elements.

3.1 NON-LINEAR ELASTICITY

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. ←

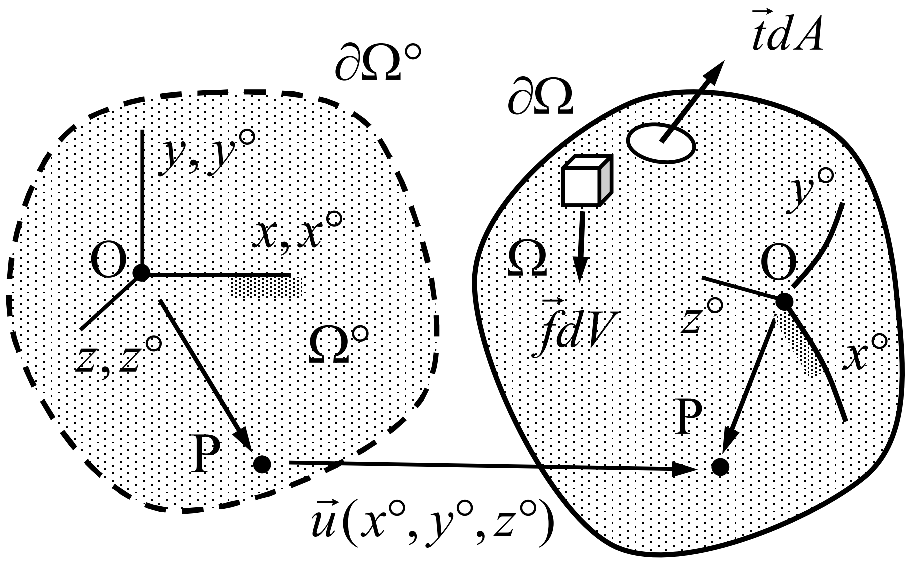
Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. ←

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

DISPLACEMENT OF A SOLID BODY

Assuming equilibrium on the initial domain Ω°, the aim is to find a new equilibrium on the deformed domain Ω, when e.g. external forces acting on the structure are changed.



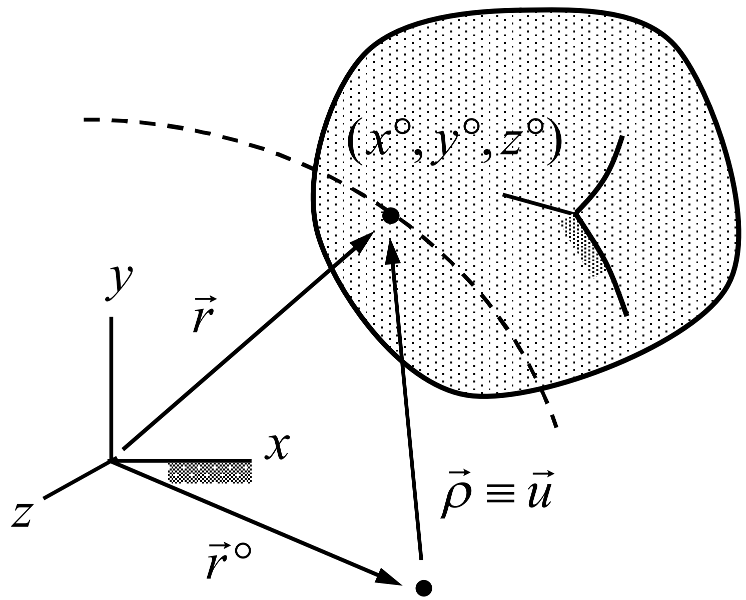
The local forms of the balance laws are concerned with the deformed domain which depends on the displacement! This brings a severe non-linearity into the boundary value problem for the displacement components.

DESCRIPTION OF MOTION

In solid mechanics, displacement with respect to the initial geometry $(t = 0)$ is used in description of motion. Particles are identified by the material coordinates $(x^\circ, y^\circ, z^\circ)$.

Generic $$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} \aleph_x(x^\circ, y^\circ, z^\circ, t) \\ \aleph_y(x^\circ, y^\circ, z^\circ, t) \\ \aleph_z(x^\circ, y^\circ, z^\circ, t) \end{Bmatrix},$$

Solid $$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} x^\circ \\ y^\circ \\ z^\circ \end{Bmatrix} + \begin{Bmatrix} u_x(x^\circ, y^\circ, z^\circ, t) \\ u_y(x^\circ, y^\circ, z^\circ, t) \\ u_z(x^\circ, y^\circ, z^\circ, t) \end{Bmatrix}$$



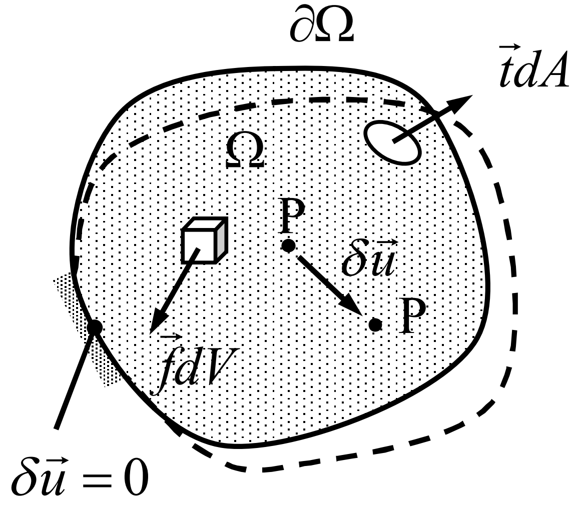
Displacement vanishes at the initial geometry $t = 0$ so that $x = x^\circ$, $y = y^\circ$, and $z = z^\circ$. Therefore, particles of the body can also be identified by (x, y, z) of the initial geometry so that the motion is described by $u_x(x, y, z, t)$, $u_y(x, y, z, t)$, and $u_z(x, y, z, t)$.

PRINCIPLE OF VIRTUAL WORK

Principle of virtual work $\delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \;\; \forall \delta\vec{u}$ is concerned with the deformed domain Ω. To avoid the complications due to a non-constant domain, the description of motion (a is used to express all quantities in the Cartesian $(x, y, z)-$system of the initial geometry:

$$\delta W^{\text{int}} = \int_{\Omega^\circ} -(\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ \delta E_{zz} \end{Bmatrix} + 2 \begin{Bmatrix} S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta E_{xy} \\ \delta E_{yz} \\ \delta E_{zx} \end{Bmatrix}) dV^\circ$$

$$\delta W^{\text{ext}} = \int_{\Omega^\circ} (\begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}) | J | dV^\circ + \ldots$$



The Green-Lagrange strain measure $\vec{\vec{E}}$ is non-linear. Also, the PK2 stress $\vec{\vec{S}}$ differs from the Cauchy (true) stress $\vec{\vec{\sigma}}$.

GREEN-LAGRANGE STRAIN MEASURE

The Green-Lagrange strain has the components (in the basis of the initial geometry)

$$\begin{Bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} (\partial u_x / \partial x)^2 + (\partial u_y / \partial x)^2 + (\partial u_z / \partial x)^2 \\ (\partial u_x / \partial y)^2 + (\partial u_y / \partial y)^2 + (\partial u_z / \partial y)^2 \\ (\partial u_x / \partial z)^2 + (\partial u_y / \partial z)^2 + (\partial u_z / \partial z)^2 \end{Bmatrix},$$

$$\begin{Bmatrix} E_{xy} \\ E_{yz} \\ E_{zx} \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} (\partial u_x / \partial x)(\partial u_x / \partial y) + (\partial u_y / \partial x)(\partial u_y / \partial y) + (\partial u_z / \partial x)(\partial u_z / \partial y) \\ (\partial u_x / \partial y)(\partial u_x / \partial z) + (\partial u_y / \partial y)(\partial u_y / \partial z) + (\partial u_z / \partial y)(\partial u_z / \partial z) \\ (\partial u_x / \partial z)(\partial u_x / \partial x) + (\partial u_y / \partial z)(\partial u_y / \partial x) + (\partial u_z / \partial z)(\partial u_z / \partial x) \end{Bmatrix}.$$

Green-Lagrange $\overleftrightarrow{E}$ gives zero strain in a rigid body motion, whereas linear strain $\overleftrightarrow{\varepsilon}$ does not. Linear strain $\overleftrightarrow{\varepsilon}$ can be taken as an approximation to $\overleftrightarrow{E}$ valid when strains and rotations of material elements are small!

ELASTIC MATERIAL

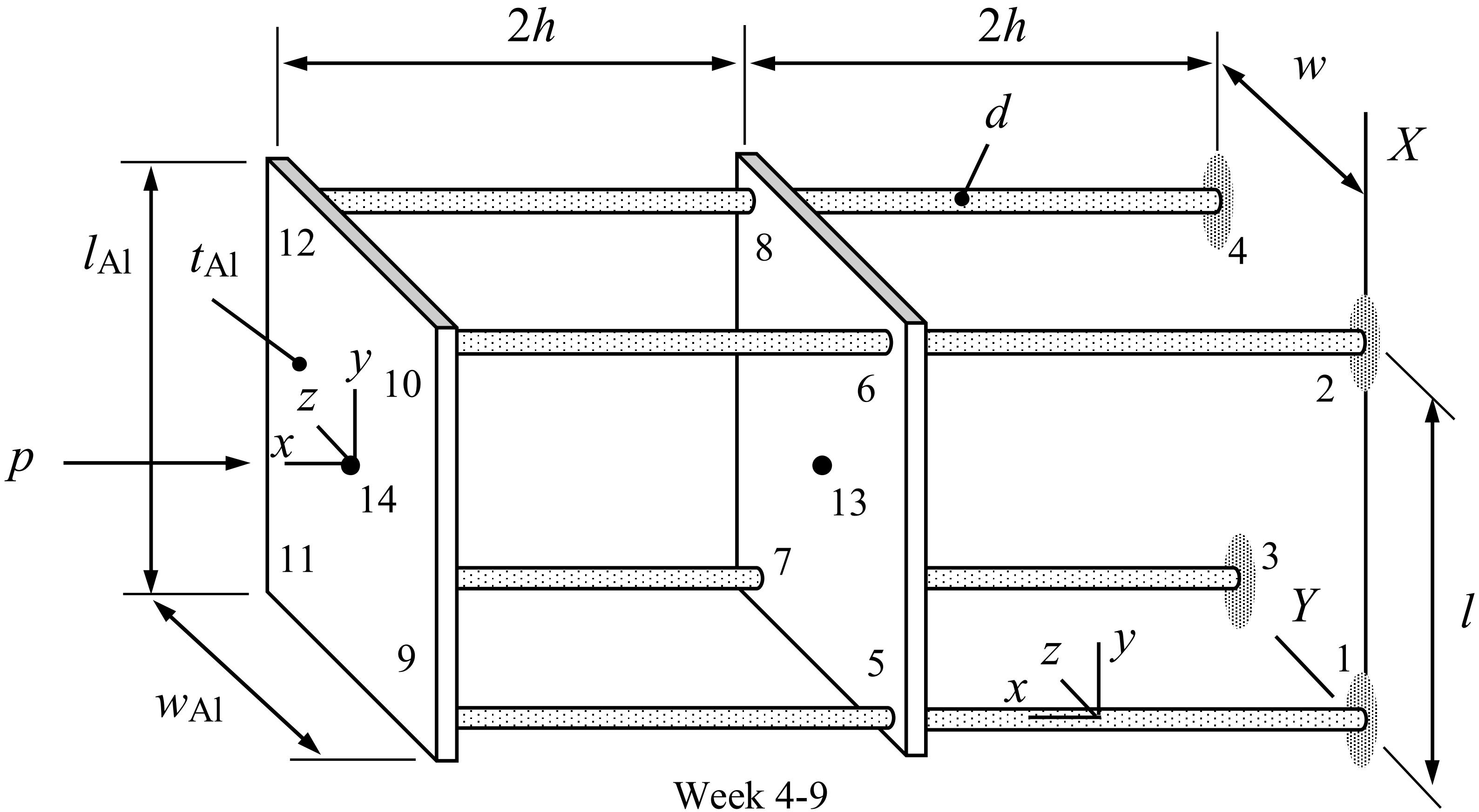
Under the assumption of large displacement and small strains the Green-Lagrange strain measure does not differ much from the linear setting with small displacements and small strains. Constitutive equations

$$\begin{Bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \end{Bmatrix} = \frac{1}{C}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix} \text{ and } \begin{Bmatrix} 2E_{xy} \\ 2E_{yz} \\ 2E_{zx} \end{Bmatrix} = \frac{1}{G}\begin{Bmatrix} S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix},$$

with material parameters C (which replaces E), ν, and $G = C/(2+2\nu)$ are same as those of the linear case, are assumed to simplify the setting. Also, the uni-axial and two-axial (plane) stress and strain relationships follows just by using strains instead of engineering strains and C instead of E.

3.2 STABILITY OF STRUCTURE

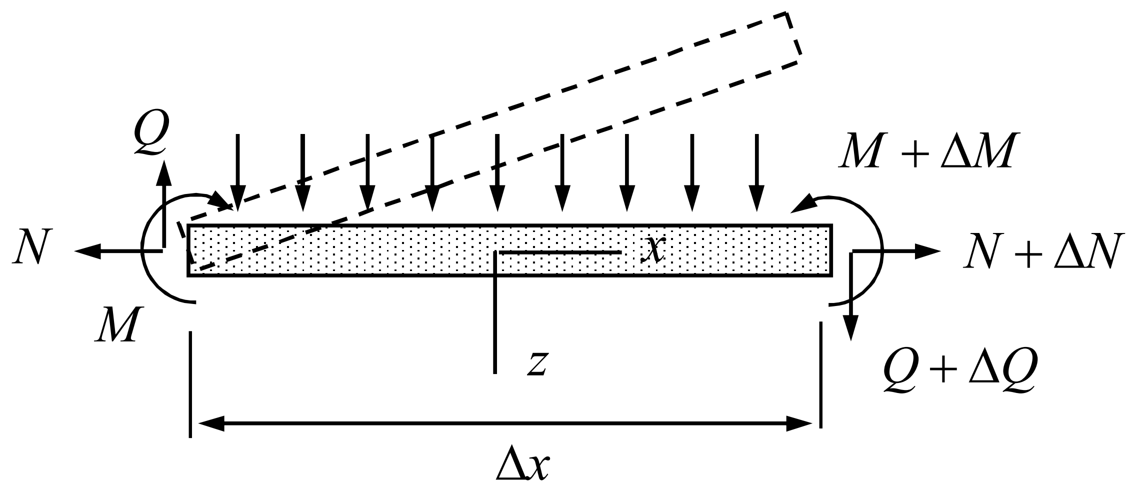
In stability analysis, the goal is to find the critical value p_{cr} of parameter p (force, load, displacement etc.) which makes the solution non-unique so that several equilibrium positions may coexist.



BEAM BUCKLING

In the simplified stability analysis, only the most significant non-linear terms are retained. By taking into account the xz−plane bending moment due to the axial force, equilibrium of a beam element

$$\frac{dM}{dx} - Q + N\frac{dw}{dx} = 0 \quad x \in]0, L[,$$



$$\frac{dQ}{dx} + f_z = 0 \quad x \in]0, L[,$$

where $M = -EI\dfrac{d^2w}{dx^2}$.

The more precise equilibrium equation takes into account coupling of the bar and bending modes of beam (bending is affected by the bar mode but not the other way around).

- The table by George William Herbert - *Own work, after Table C.1.8.1 in Steel Construction Manual, 8th edition, 2nd revised printing, American Institute of Steel Construction, 1987, CC BY-SA 2.5*, is based on the equilibrium equation

$$EI\frac{d^4w}{dx^4}+p\frac{d^2w}{dx^2}=0 \quad x\in]0,L[,$$

for the $xz-$plane bending with a compressive $N=-p$. The different values in the table are due to different boundary and symmetry conditions imposed on the generic solution

$$w=a+bx+c\sin(\sqrt{\frac{p}{EI}}x)+d\cos(\sqrt{\frac{p}{EI}}x).$$

BUCKLING LOAD OF BEAM $p_{\text{cr}} = \pi^2 \dfrac{EI}{(KL)^2}$

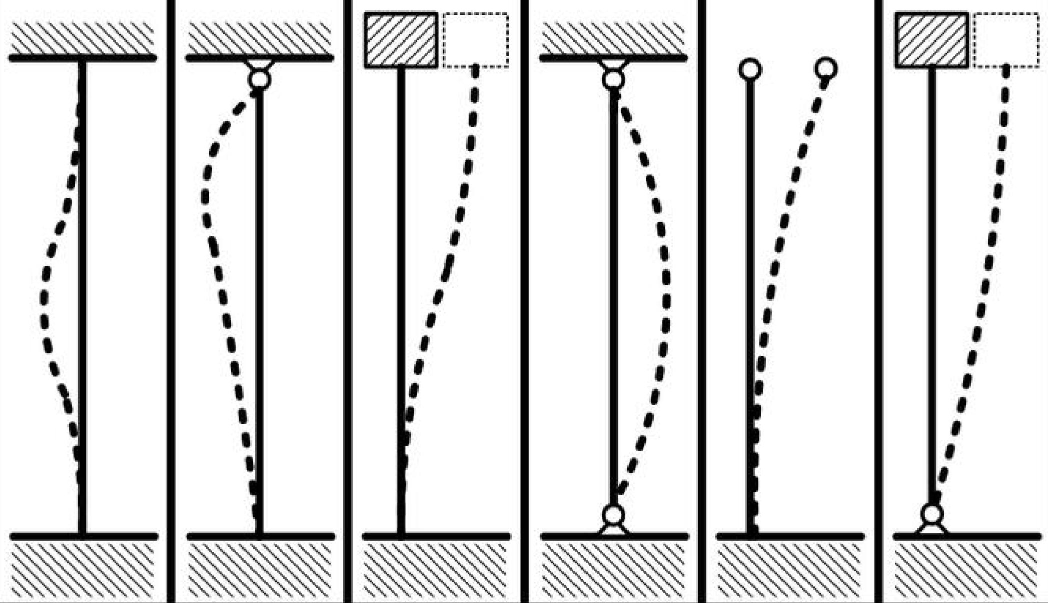
Buckled shape of column shown by dashed line						
Theoretical K value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended design value K	0.65	0.80	1.2	1.0	2.10	2.0

PLATE BUCKLING

In the simplified stability analysis, only the effect of the thin-slab mode on bending is accounted for (much in the same manner as with the beam model). Assuming that the material coordinate system is placed at the mid-plane, material is homogeneous, transverse distributed external loading vanishes, and that the in-plane stress resultants are constants, the outcome is the bending equation

$$D(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4})-N_{xx}\frac{\partial^2 w}{\partial x^2}-(N_{xy}+N_{yx})\frac{\partial^2 w}{\partial x \partial y}-N_{yy}\frac{\partial^2 w}{\partial y^2}=0 \quad (x,y)\in\Omega,$$

in which $D=Et^3/(12-12\nu^2)$. In the model, bending mode is affected by the thin slab mode but not the other way around. Therefore, the thin slab equations can be solved independently for the in-plane stress resultants.

REFINED VIRTUAL WORK DENSITIES

In the simplified stability analysis, displacement is assumed to be small so that the difference between the initial and deformed geometry can be omitted. The refined virtual work density expressions contain the additional coupling terms

Beam: $\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} d\delta v / dx \\ d\delta w / dx \end{Bmatrix}^{\mathrm{T}} N \begin{Bmatrix} dv / dx \\ dw / dx \end{Bmatrix}$ where $N = EA\dfrac{du}{dx}$,

Plate: $\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} \partial\delta w / \partial x \\ \partial\delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix}$, $\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}$.

The coupling affects only the bending mode as the variations are concerned with the bending modes.

3.3 STABILITY ANALYSIS

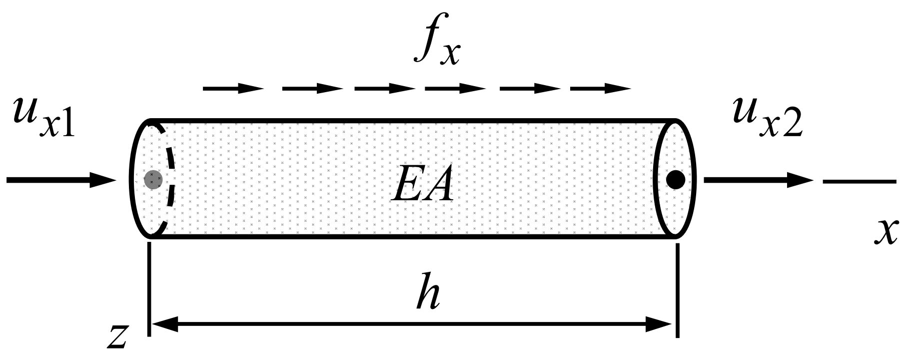
- Model the structure as a collection of beam, plate, etc. elements. Derive the element contributions δW^e and express the nodal displacement and rotation components of the material coordinate system in terms of those in the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get $\delta W = -\delta \mathbf{a}^{\mathrm{T}}[\mathbf{R}(\mathbf{a}) - \mathbf{F}]$.

- Use the principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$, fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the equilibrium equations $\mathbf{R}(\mathbf{a}) - \mathbf{F} = 0$. Solve for the bar/thin slab modes from the linear part and substitute into the non-linear part to get $(\mathbf{K} + p\mathbf{K}_\sigma)\mathbf{a} = 0$. Finally, find the values of p making the solution non-unique. The smallest of the values for p is p_{cr}.

BAR MODE

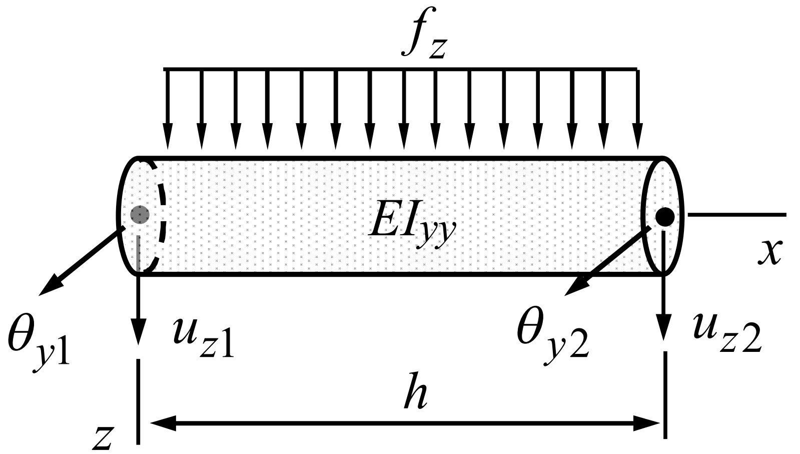
Assuming that $v=0$, $w=0$, $\phi=0$ and a linear approximation to the axial displacement $u(x)$ in terms of the nodal displacements u_{x1}, u_{x2}, virtual work expressions of the internal and external forces take the forms



$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

BENDING MODE (*xz*-plane)

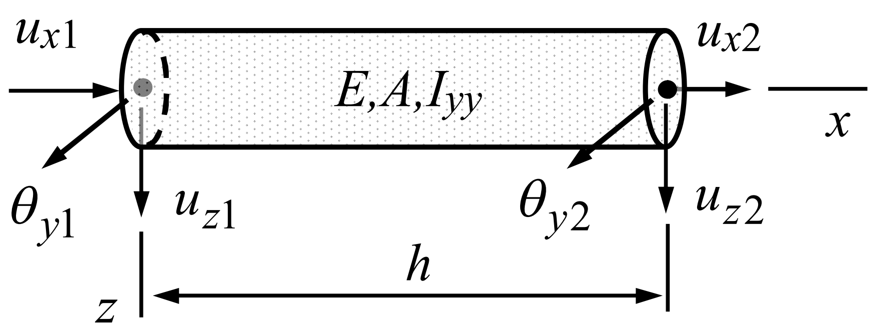
Assuming that $u = 0$, $v = 0$, $\phi = 0$ and a cubic approximation to the transverse displacement $w(x)$ in terms of point displacements u_{z1}, u_{z2} and rotations θ_{y1}, θ_{y2}:



$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$

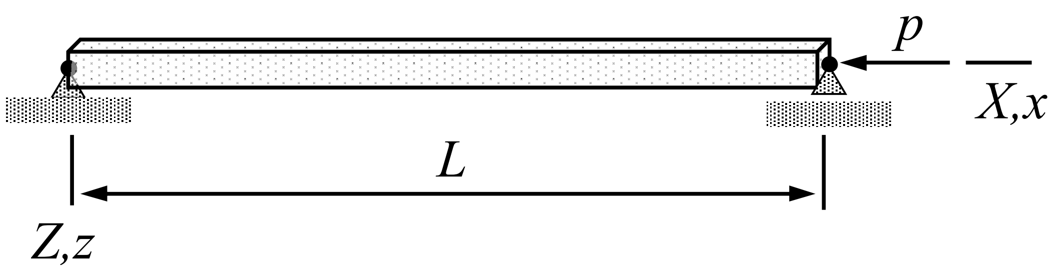
BENDING-BAR COUPLING (*xz*-plane)

Assuming that $v = 0$, $\phi = 0$, a cubic approximation to $w(x)$ in terms of nodal displacements/rotations u_{z1}, u_{z2}, θ_{y1}, and θ_{y2}, and a linear approximation to $u(x)$ in terms of the nodal displacements u_{x1}, u_{x2}



$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \text{ where } N = EA\frac{u_{x2} - u_{x1}}{h}$$

EXAMPLE 3.1. Consider a simply supported beam loaded by a compressive axial force p acting on the right end. Assuming that displacement is confined to the $xz-$plane, use a single beam element to determine the buckling force p_{cr}. Cross-section properties A, I and Young's modulus E are constants.



Answer $p_{\mathrm{cr}} = 12\dfrac{E}{L^2}$ (exact to the model $p_{\mathrm{cr}} = \pi^2 \dfrac{EI}{L^2}$)

- The non-zero nodal displacements/rotations are θ_{Y1}, θ_{Y2}, and u_{X2}. Virtual work expression for the beam $\delta W^1 = \delta W^{\text{int}} + \delta W^{\text{sta}}$ and the point force δW^2 are (here $N = EA(u_{x2} - u_{x1})/h = EAu_{X2}/L$)

$$\delta W^1 = -\delta u_{X2}\frac{EA}{L}u_{X2} - \begin{Bmatrix}\delta\theta_{Y1}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}} \frac{EI}{L}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix} - \begin{Bmatrix}\delta\theta_{Y1}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}} \frac{EAu_{X2}}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix},$$

$$\delta W^2 = -p\delta u_{X1}.$$

- Virtual work expression is sum of the element contributions

$$\delta W = -\begin{Bmatrix}\delta u_{X2}\\ \delta\theta_{Y1}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}} [\frac{1}{L}\begin{bmatrix}EA & 0 & 0\\ 0 & 4EI & 2EI\\ 0 & 2EI & 4EI\end{bmatrix} + \frac{EAu_{X2}}{30}\begin{bmatrix}0 & 0 & 0\\ 0 & 4 & -1\\ 0 & -1 & 4\end{bmatrix})\begin{Bmatrix}u_{X2}\\ \theta_{Y1}\\ \theta_{Y2}\end{Bmatrix} + \begin{Bmatrix}p\\ 0\\ 0\end{Bmatrix}].$$

- Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\left(\frac{1}{L}\begin{bmatrix} EA & 0 & 0 \\ 0 & 4EI & 2EI \\ 0 & 2EI & 4EI \end{bmatrix} + \frac{EAu_{X2}}{30}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} u_{X2} \\ \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} + \begin{Bmatrix} p \\ 0 \\ 0 \end{Bmatrix} = 0.$$

- The remaining task is to solve the (non-linear) equations for the values of the loading parameter p making the solution non-unique and the corresponding modes. The first equation gives (solving the axial force(s) N of the beams as functions of the loading parameters is always the first step)

$$\frac{1}{L}EAu_{X2} + p = 0 \quad \Leftrightarrow \quad u_{X2} = -\frac{pL}{EA}.$$

- When the solution is substituted there, the remaining equations simplify to the homogeneous form

$$\left(\frac{EI}{L}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix} - \frac{pL}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\right)\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix} = 0.$$

- A non-trivial solution (zero rotations satisfy the equations always) is possible only if the matrix in parenthesis is singular

$$\det\left(\frac{EI}{L}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix} - \frac{pL}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\right) = \left(4\frac{EI}{L} - 4\frac{pL}{30}\right)^2 - \left(2\frac{EI}{L} + \frac{pL}{30}\right)^2 = 0 \quad \Rightarrow$$

$$\frac{pL^2}{EI} \in \{12, 60\}.$$

- The smallest of the values is the critical one

$$p_{\mathrm{cr}} = 12\frac{EI}{L^2}. \quad \leftarrow$$

- Stability analysis by the Mathematica code gives

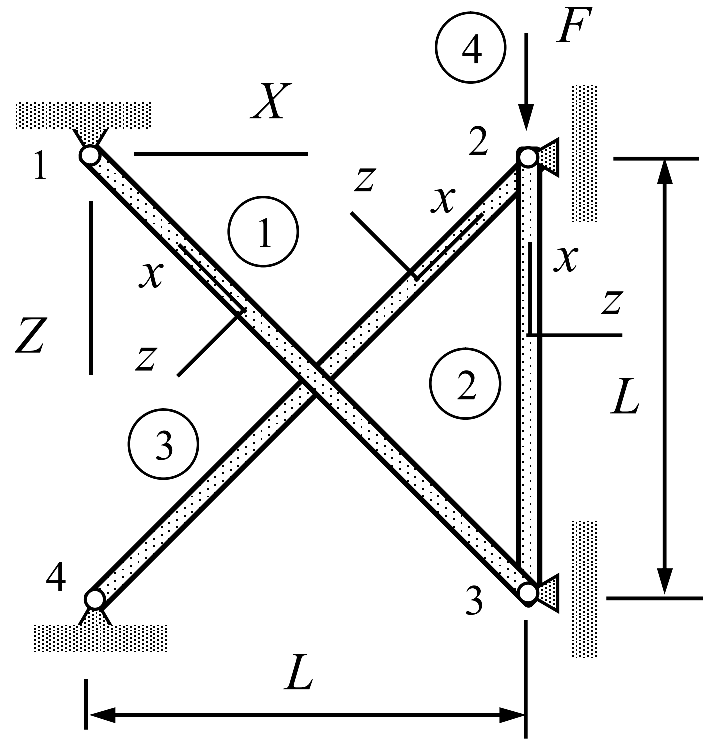
	type	properties	geometry
1	BEAM	{{E, G}, {A, I, I}, {0, 0, 0}}	Line[{1, 2}]
2	FORCE	{-p, 0, 0}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, θY[1], 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, θY[2], 0}

$p[1] \to \frac{12 E I}{L^2} \quad \{uX[2] \to 0, \theta Y[1] \to -1, \theta Y[2] \to 1\}$

$p[2] \to \frac{60 E I}{L^2} \quad \{uX[2] \to 0, \theta Y[1] \to 1, \theta Y[2] \to 1\}$

EXAMPLE 3.2. Consider the truss shown in which elements 1 and 3 are modelled as bars and element 2 as a beam. Determine the critical value of force F for buckling of the beam element. Cross-sectional area of element 1 and 3 are $\sqrt{8}A$. Cross sectional area of element 2 is A and the second moment of area I. Young's modulus of the material is E. Assume that $\theta_{Y3} = -\theta_{Y2}$.



Answer $F_{\mathrm{cr}} = 36\dfrac{EI}{L^2}$

- The non-zero nodal displacements/rotations are θ_{Y2}, $\theta_{Y3} = -\theta_{Y2}$, u_{Z2}, and u_{Z3}. Virtual work expressions of the elements are (here the axial force is given by $N = EA(u_{x2} - u_{x3})/L = EA(u_{Z3} - u_{Z2})/L$)

$$\delta W^1 = -\begin{Bmatrix} -\delta u_{Z3} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{E\sqrt{8}A}{\sqrt{8}L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -u_{Z3} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} (\frac{1}{L} \begin{bmatrix} EA & -EA & 0 \\ -EA & EA & 0 \\ 0 & 0 & 4EI + NL^2/3 \end{bmatrix} \begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^3 = -\begin{Bmatrix} 0 \\ -\delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{E\sqrt{8}A}{\sqrt{8}L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -u_{Z2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^4 = \delta u_{Z2} F = \begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix}.$$

- Virtual work expression is the sum of element contributions

$$\delta W = -\begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{1}{L}\begin{bmatrix} 2EA & -EA & 0 \\ -EA & 2EA & 0 \\ 0 & 0 & 4EI + NL^2/3 \end{bmatrix} \begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix}\right).$$

- Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\frac{1}{L}\begin{bmatrix} 2EA & -EA & 0 \\ -EA & 2EA & 0 \\ 0 & 0 & 4EI + NL^2/3 \end{bmatrix} \begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix} = 0 \quad \text{where} \quad N = \frac{EA}{L}(u_{Z3} - u_{Z2}).$$

- The remaining task is to solve the (non-linear) equations for the values of the loading parameter F making the solution non-unique (the corresponding modes might be of some interest also). The first two equations give (solving the axial force(s) N of the beams as functions of the loading parameters is always the first step)

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_{Z3} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{Z3} \\ u_{Z2} \end{Bmatrix} = \frac{FL}{3EA}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}.$$

- When the solution is substituted there, the axial force expression and the remaining third equation give

$$N = \frac{EA}{L}(u_{Z3} - u_{Z2}) = -\frac{F}{3} \quad \Rightarrow \quad (4EI - \frac{1}{3}\frac{FL^2}{3})\theta_{Y2} = 0 \ .$$

- A non-trivial solution $\theta_{Y2} \neq 0$ is possible only if

$$4EI - \frac{FL^2}{9} = 0 \quad \Leftrightarrow \quad F_{\text{cr}} = 36\frac{EI}{L^2} \ .$$

- Stability analysis by the Mathematica code gives

	type	properties	geometry
1	BAR	$\{\{E\}, \{2\sqrt{2}\,A\}, \{0\}\}$	Line[{1, 3}]
2	BEAM	{{E, G}, {A, I, I}, {0, 0, 0}}	Line[{2, 3}]
3	BAR	$\{\{E\}, \{2\sqrt{2}\,A\}, \{0\}\}$	Line[{4, 2}]
4	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, θY[2], 0}
3	{L, 0, L}	{0, 0, uZ[3]}	{0, -θY[2], 0}
4	{0, 0, L}	{0, 0, 0}	{0, 0, 0}

$F[1] \to \frac{36\,E\,I}{L^2} \quad \{uZ[2] \to 0,\ uZ[3] \to 0,\ \theta Y[2] \to 1\}$

3.4 ELEMENT CONTRIBUTIONS

Virtual work expressions for the beam and plate elements combine virtual work densities of the model and approximation depending on the element shape and type. To derive the expression:

- Start with the virtual work densities $\delta w_{\Omega}^{\text{int}}$, $\delta w_{\Omega}^{\text{sta}}$, and $\delta w_{\Omega}^{\text{ext}}$ of the formulae collection.

- Represent the unknown functions by interpolation of the nodal displacement and rotations (see formulae collection). Substitute the approximations into the density expressions.

- Integrate the virtual work density over the domain occupied by the element to get δW.

ELEMENT APPROXIMATION

In MEC-E8001 element approximation is a polynomial interpolant of the nodal displacement and rotations in terms of shape functions. In stability analysis, shape functions depend on *x, y,* and *z*.

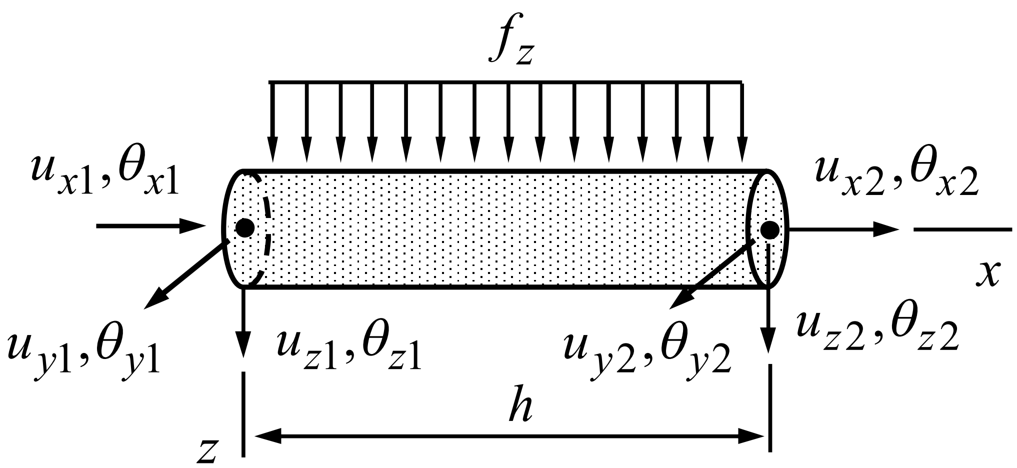
Approximation $\mathbf{u} = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ *always of the same form!*

Shape functions $\mathbf{N} = \{N_1(x,y,z) \;\; N_2(x,y,z) \;\; \ldots \;\; N_n(x,y,z)\}^{\mathrm{T}}$

Parameters $\mathbf{a} = \{\mathrm{a}_1 \;\; \mathrm{a}_2 \;\; \ldots \;\; \mathrm{a}_n\}^{\mathrm{T}}$

Nodal parameters $\mathrm{a} \in \{u_x, u_y, u_z, \theta_x, \theta_y, \theta_z\}$ may be just displacement or rotation components or a mixture of them (as with the beam model).

BEAM MODEL



Coupling term: $\delta w_{\Omega}^{\text{sta}} = -\dfrac{d\delta v}{dx} N \dfrac{dv}{dx} - \dfrac{d\delta w}{dx} N \dfrac{dw}{dx}$, where $N = EA\dfrac{du}{dx}$.

The additional coupling term is part of the virtual work density of internal forces $\delta w_{\Omega} = (\delta w_{\Omega}^{\text{int}} + \delta w_{\Omega}^{\text{sta}}) + \delta w_{\Omega}^{\text{ext}}$ and assumes that $S_y = S_z = I_{yz} = 0$. The coupling of the bar and bending modes is the most significant non-linear term.

- The coupling terms of the bending and bar modes follow from the generic non-linear expression and the kinematic assumption of the beam model in $xz-$plane bending $u_x = u - zdw/dx - ydv/dx$, $u_y = v(x)$, and $u_z = w(x)$. First, the Green-Lagrange axial strainn and stress-strain relationship simplify to (only the most significant terms)

$$E_{xx} = \frac{du}{dx} - z\frac{d^2w}{dx^2} - y\frac{d^2v}{dx^2} + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2 \text{ and } S_{xx} = CE_{xx},$$

- Assuming that $S_y = S_z = I_{yz} = 0$, integration over the cross-section gives the virtual work densities of the bar mode, bending modes, and the additional coupling term (again, only the most significant terms for stability analysis)

$$\delta w_\Omega^{\text{sta}} = -\int_A \delta E_{xx} S_{xx} dA = -\frac{d\delta v}{dx} N \frac{dv}{dx} - \frac{d\delta w}{dx} N \frac{dw}{dx}, \text{ where } N = EA\frac{du}{dx}.$$

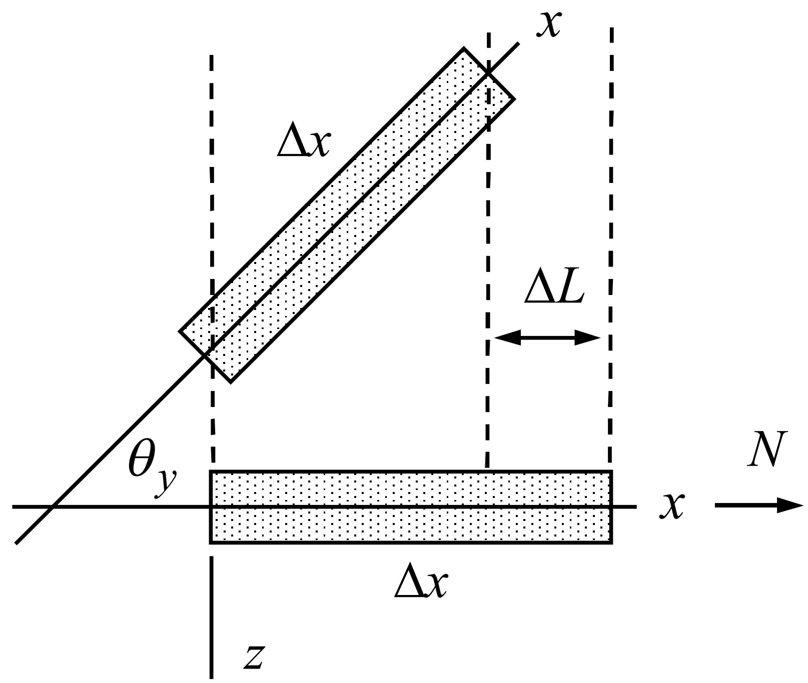
- Derivation of the coupling term, based on the virtual work of the external axial force, is also possible. The axial displacement of the free end of a cantilever *due to the bending only* can be obtained by considering an inextensible material element of length Δx. The length change in the direction of the force is given by (Taylor series $\cos(x) = 1 - x^2/2 + \ldots$)

$$\Delta L = \Delta x - \Delta x \cos\theta_y \quad \Rightarrow$$

$$\frac{dL}{dx} = 1 - \cos\theta_y \approx \frac{1}{2}\theta_y^2 = \frac{1}{2}(-\frac{dw}{dx})^2 \quad \Rightarrow$$

$$u(L) = -\int_0^L \frac{1}{2}(\frac{dw}{dx})^2 dx \quad \Rightarrow$$

$$\delta u(L) = -\int_0^L \frac{d\delta w}{dx}\frac{dw}{dx} dx.$$



- Virtual work of the external force due to the bending effect is therefore given by

$$\delta W^{\text{sta}} = N\delta u(L) = N\int_0^L \frac{d\delta w}{dx}\frac{dw}{dx}dx.$$

- In the simultaneous bending in both directions, the length change of an inextensible material element Δx in the axial direction is given by

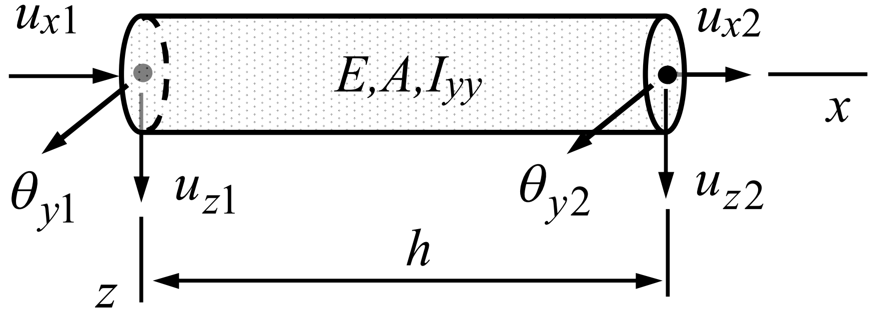
$$\Delta L = \Delta x - \Delta x\cos\theta_y\cos\theta_z \approx \Delta x - \Delta x(1-\frac{1}{2}\theta_y^2)(1-\frac{1}{2}\theta_z^2) \approx \Delta x\frac{1}{2}(\theta_y^2+\theta_z^2) \Rightarrow$$

$$\Delta L \approx \Delta x\frac{1}{2}(\frac{dw}{dx}\frac{dw}{dx}+\frac{dv}{dx}\frac{dv}{dx}) \Rightarrow \delta u(L) = -\int_0^L (\frac{d\delta w}{dx}\frac{dw}{dx}+\frac{d\delta v}{dx}\frac{dv}{dx})dx$$

Hence, the coupling term is the sum of coupling terms of the planar problems!

BENDING-BAR COUPLING (*xz*-plane)

Assuming that $v = 0$, $\phi = 0$, a cubic approximation to $w(x)$ in terms of nodal displacements/rotations u_{z1}, u_{z2}, θ_{y1}, and θ_{y2}, and a linear approximation to $u(x)$ in terms of the nodal displacements u_{x1}, u_{x2},



$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h}\begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix}\begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \text{ where } N = EA\frac{u_{x2} - u_{x1}}{h}.$$

- Virtual work density of the bending-bar mode coupling term in the $xz-$plane is given by

$$\delta w_{\Omega}^{\text{sta}} = -N \frac{d\delta w}{dx} \frac{dw}{dx} \quad \text{where } N = EA \frac{du}{dx}$$

and the cross-sectional area A and Young's modulus E may depend on x. Element approximations (simplest possible) are $du / dx = (u_{x2} - u_{x1}) / h$ and

$$w = \frac{1}{h^3} \begin{Bmatrix} (h-x)^2(h+2x) \\ -h(h-x)^2 x \\ (3h-2x)x^2 \\ (h-x)x^2 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} \Rightarrow \frac{dw}{dx} = \frac{1}{h^3} \begin{Bmatrix} -6(h-x)x \\ -h(h-3x)(h-x) \\ 6(h-x)x \\ h(2h-3x)x \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}.$$

- Integration over the domain occupied by the element gives

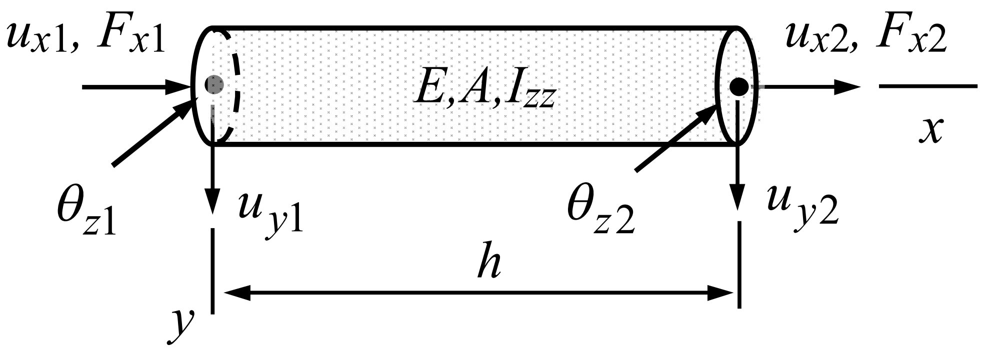
$$\delta W^{\text{sta}} = \int_0^h \delta w_\Omega^{\text{sta}} dx = -N \int_0^h \frac{d\delta w}{dx}\frac{dw}{dx} dx \quad \left(N = EA\frac{du}{dx} \text{ is constant here}\right) \quad \Rightarrow$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \text{ where } N = EA\frac{u_{x2}-u_{x1}}{h}. \;\Leftarrow$$

- The non-linear additional term couples the bending and the bar modes of the beam. In the one-sided coupling, bending mode is affected by the bar mode but not vice versa.

BENDING-BAR COUPLING (*xy*-plane)

Assuming that $u=0$, $w=0$, $\phi=0$, and a cubic approximation to $v(x)$ in terms of nodal displacements/rotations u_{y1}, u_{y2}, θ_{z1}, and θ_{z2}, and linear approximation to $u(x)$ in terms of nodal displacements u_{x1}, u_{x2},



$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h}\begin{bmatrix} 36 & 3h & -36 & 3h \\ 3h & 4h^2 & -3h & -h^2 \\ -36 & -3h & 36 & -3h \\ 3h & -h^2 & -3h & 4h^2 \end{bmatrix}\begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix}, \text{ where } N = EA\frac{u_{x2}-u_{x1}}{h}.$$

PLATE MODEL

Virtual work density combines the thin-slab and plate bending modes. Assuming that the material coordinate system is placed at the geometric mid-plane, bending mode is affected by the thin slab mode but not vice versa. The additional coupling term for stability analysis

Coupling: $\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} \partial\delta w / \partial x \\ \partial\delta w / \partial y \end{Bmatrix}^{\text{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix}$, where $\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$

depends on the in-plane stress resultants N_{xx}, N_{yy}, and $N_{xy} = N_{yx}$ of the thin-slab mode. The additional coupling term is part of the virtual work density of internal forces $\delta w_{\Omega} = \delta w_{\Omega}^{\text{int}} + \delta w_{\Omega}^{\text{sta}} + \delta w_{\Omega}^{\text{ext}}$. As stability term affects only the bending mode, dependence of the stress resultants on the loading parameter can be obtained from a thin-slab problem.

- The coupling term of the plate bending and thin-slab loading modes follows from the generic non-linear virtual work density of the internal forces and the kinematic assumptions of the Kirchhoff plate model $u_x = u - z\partial w / \partial x$, $u_y = v - z\partial w / \partial y$, and $u_z = w(x, y)$. If only the terms used already in the beam case are accounted for, Green-Lagrange strain and the corresponding second Piola-Kirchhoff stress components

$$\begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix} \approx \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} - z \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix} + \begin{Bmatrix} (\partial w / \partial x)^2 / 2 \\ (\partial w / \partial y)^2 / 2 \\ (\partial w / \partial x)(\partial w / \partial y) \end{Bmatrix},$$

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{xy} \end{Bmatrix} = [E]_\sigma \begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix}.$$

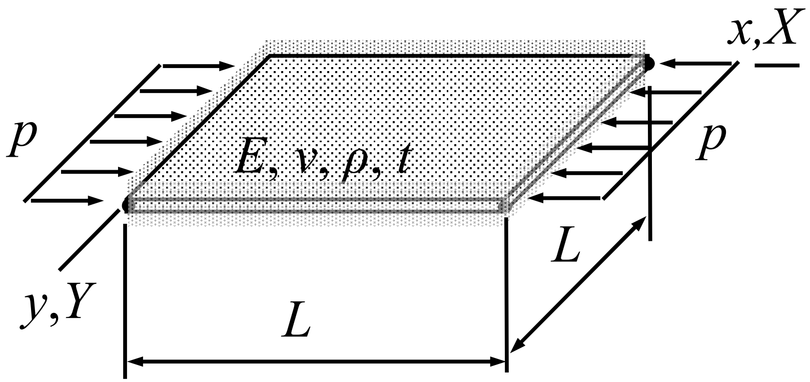
- Assuming that the material coordinate system is placed at the geometric mid-plane, integration of the virtual work density gives the virtual work density of the thin-slab mode, virtual work density of plate bending mode, and the coupling term (considering only the most significant terms)

$$\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} \partial\delta w / \partial x \\ \partial\delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix},$$

where the in-plane stress resultants are given by

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}.$$

EXAMPLE 3.3. Determine the critical value of the in-plane loading p_{cr} making the plate of the figure to buckle. Use the plate model and the continuous polynomial approximation $w(x,y) = a_0(xy/L^2)(1-x/L)(1-y/L)$. Assume that the edge conditions are such that solution to the in-plane stress resultants is given by $N_{xx} = -p$ and $N_{yy} = N_{xy} = 0$ (solution to the thin-slab problem).



Answer $p_{\text{cr}} = 44\dfrac{Et^3}{12L^2(1-\nu^2)}$ (exact $p_{\text{cr}} = 4\pi^2\dfrac{Et^3}{12L^2(1-\nu^2)} \approx 39.5\dfrac{Et^3}{12L^2(1-\nu^2)}$).

- Assuming that the material coordinate system is chosen so that the linear plate bending and thin slab modes decouple, the plate model virtual work densities of the bending mode and the coupling term are given by ($N_{xx} = -p$ and $N_{yy} = N_{xy} = 0$)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix} \text{ where } D = \frac{t^3}{12} \frac{E}{1-\nu^2},$$

$$\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} \partial \delta w / \partial x \\ \partial \delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix} = \frac{\partial \delta w}{\partial x} p \frac{\partial w}{\partial x}.$$

- When the approximation is substituted there, virtual work expressions of the plate bending mode and that of the coupling between the thin-slab and bending modes simplify to

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{22}{45} \frac{D}{L^2} a_0,$$

$$\delta W^{\text{sta}} = \int_0^L \int_0^L \delta w_\Omega^{\text{sta}} dxdy = \delta a_0 \frac{1}{90} p a_0.$$

- Virtual work expression is the sum of the two parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta a_0 (\frac{22}{45} \frac{D}{L^2} - \frac{1}{90} p) a_0.$$

- Principle of virtual work $\delta W = 0 \, \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta a_0 (\frac{22}{45} \frac{D}{L^2} - \frac{1}{90} p) a_0 = 0 \quad \forall \delta a_0 \quad \Rightarrow \quad (\frac{22}{45} \frac{D}{L^2} - \frac{1}{90} p) a_0 = 0.$$

For a nontrivial solution $a_0 \neq 0$, the loading parameter needs to take the value

$$p_{\mathrm{cr}} = 44\frac{D}{L^2}\ .$$

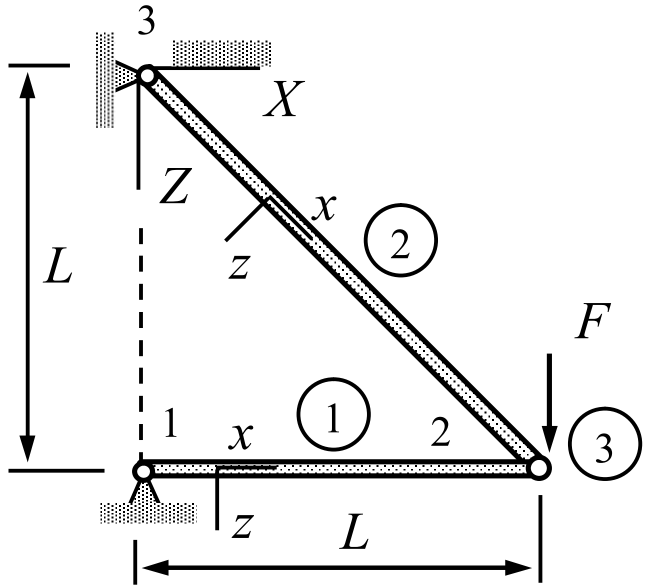
STABILITY ANALYSIS OF TRUSS SIMPLIFIED

If the beams are connected by joints not capable for transmitting moments, one may use the fact that the bar model predicts the axial forces correctly. Then, the first step is a linear displacement analysis for finding the displacements of the nodes and thereby the axial forces $N(p)$ as functions of the loading parameter. After that, the buckling loads of each beam under compression follows from the buckling criterion (N is negative in compression)

$$-N(p) = \pi^2 \frac{EI}{L^2}$$

for a simply supported beam. The first beam to buckle (or the smallest p given by the conditions above) defines the critical load p_{cr}.

EXAMPLE 3.4. A beam truss is loaded by a vertical point force having magnitude F and acting in the positive or negative direction of the Z-axis. Determine the critical load magnitude F_{cr} for buckling of beam 1 or 2 of the truss. Cross-sectional area of element 1 is A and that for element 2 $\sqrt{8}A$, Young's modulus E is constant, and the second moment of area is I for both beams. The beams are connected by frictionless joints.



Answer $F_{\mathrm{cr}} = \dfrac{\pi^2}{\sqrt{8}}\dfrac{EI}{L^2}$ when $F < 0$.

- The relationships between the nodal displacement components in the material and structural systems are $u_{x1} = 0$ and $u_{x2} = u_{X2}$. Element contribution δW^1 to the virtual work expression of the structure is

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}) = -\frac{EA}{L} u_{X2} \delta u_{X2}.$$

- For element 2, $u_{x3} = 0$ and $u_{x2} = (u_{X2} + u_{Z2}) / \sqrt{2}$. Element contribution takes the form

$$\delta W^2 = -\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ \delta u_{X2} + \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} (\frac{E\sqrt{8}A}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ u_{X2} + u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}) \quad \Leftrightarrow$$

$$\delta W^2 = -\frac{EA}{L}(\delta u_{X2} + \delta u_{Z2})(u_{X2} + u_{Z2}).$$

- Virtual work expression of the point force follows from the definition of work. The direction may be up or down and hence F may also be negative (which means up)

$$\delta W^3 = \delta u_{Z2} F\,.$$

- Virtual work expression of a structure is obtained as the sum of the element contributions

$$\delta W = -\frac{EA}{L}\delta u_{X2} u_{X2} - \frac{EA}{L}(\delta u_{X2} + \delta u_{Z2})(u_{X2} + u_{Z2}) + \delta u_{Z2} F \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix}\right).$$

- Using the principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus

$$\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{LF}{EA}\begin{Bmatrix} -1 \\ 2 \end{Bmatrix}.$$

- For buckling of beam 1, the axial force should be compression (negative) and therefore the external force should be acting downwards.

$$N = \frac{EA}{L}(u_{x2} - u_{x1}) = \frac{EA}{L}u_{X2} = -F \quad \Rightarrow \quad F_{\text{cr}} = \pi^2 \frac{EI}{L^2} \quad \text{when} \quad F > 0.$$

- For buckling of beam 2, the axial force should be compression (negative) and therefore the external force should be acting upwards. When $F < 0$

$$N = \frac{E\sqrt{8}A}{\sqrt{2}L}(u_{x2} - u_{x3}) = \sqrt{2}\frac{EA}{L}(u_{X2} + u_{Z2}) = -\sqrt{2}F \quad \Rightarrow \quad F_{\text{cr}} = \frac{\pi^2}{\sqrt{8}}\frac{EI}{L^2}. \quad \leftarrow$$