EXERCISE SET 3,
MS-A0503, FIRST COURSE IN PROBABILITY AND STATISTICS

Explorative exercises

I will expect that you study the explorative problems BEFORE the first lecture of the week. In a manner of speaking, the lectures will contain the "solutions" to the explorative problems, meaning that by solving the problems you have taken important steps towards developing the theory on your own, before I present the theory on the lectures. For the explorative problems, it is VERY STRONGLY RECOMMENDED that you work on them in groups.

Problem 1.

(1) A machine consists of three components, whose life lengths are exponential random variables with parameters 1, 0.5, and 2 respectively. The machine needs all its components to function. Let X be the life length of the machine (which is obviously a random variable). What is the distribution of X?

(2) A machine consists of a collection of k components, whose life lengths are exponential random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively. The machine needs all its components to function. Let X be the life length of the machine (which is obviously a random variable). What is the distribution of X?

Problem 2. The *expected value* of a random variable is usually defined in two different ways for discrete and continuous random variables:

(1) If X is discrete with feasible values $a_1, a_2, \ldots$ and probability mass function p, then
$$E(X) = \sum_i a_i p(a_i).$$

(2) If X is continuous with probability density function f, then
$$E(X) = \int_{-\infty}^{\infty} x f(x).$$

Can you find a formula in terms of the CDF that is valid for both discrete and continuous variables? Hint: Partial integration.

Problem 3. What is the expected value of an exponential random variable with rate λ?

Problem 4. If X is a random variable with expected value μ, then its *variance* is $\mathrm{var}(X) = E((X-\mu)^2)$, and its *standard deviation* is $\sigma = \sqrt{\mathrm{var} X}$.

(1) Argue why this is a reasonable measure of how "spread out" the variable X is.

(2) Show that the variance can also be computed as $\mathrm{var}(X) = E(X^2) - \mu^2$.

(3) Choose some of your favourite random variables, among those we have discussed in class. Compute or look up (in the course book or on Wikipedia) the expected value and standard deviation. What is the probability that $|X - \mu| > \sigma$?

Homework problems

The homework problems are reported during the second exercise session of the week. You are allowed and encouraged to work in groups, but every student should be prepared to present the solutions individually. During the last exercise session of the week, the teacher will ask you to mark what problems you have solved, and you get points according to how many problems you marked as solved. If you mark a problem as solved, however, you should also be prepared to present your solution in front of the class.

Homework 1. A satellite system consists of 4 components and can function adequately if at least 2 of the 4 components are in working condition. If each component is, independently, in working condition with probability 0.6, what is the probability that the system functions adequately?

Homework 2. The probability of error in the transmission of a binary digit over a communication channel is $1/10^3$. Write an expression for the exact probability of more than 3 errors when transmitting a block of 10^3 bits. What is its approximate value? Assume independence.

Homework 3. An IQ test produces scores that are normally distributed with mean value 100 and standard deviation 14.2. The top 1 percent of all scores are in what range? Hint: You probably need to look up some values of the normal distribution. I suggest you do so in Mellin's statistical tables (on the course homepage), as this is the resource you will have available on the exam.

Additional problems

Ross: Chapter 5, Problems 2, 4, 25, 32, 33

Week 3, Problem 1

Let X be the number of functioning components.

$X \sim Bin(4, 0.6)$.

$$\mathbb{P}[\text{system works}] = \mathbb{P}[X \geq 2]$$

$$= \mathbb{P}[X=2] + \mathbb{P}[X=3] + \mathbb{P}[X=4]$$

$$= \binom{4}{2} \cdot 0.6^2 \cdot 0.4^2 + \binom{4}{3} 0.6^3 \cdot 0.4 + \binom{4}{4} 0.6^4$$

$$= 0.36 \left[6 \cdot 0.16 + 4 \cdot 0.24 + 1 \cdot 0.36 \right]$$

$$= 0.36 \cdot 2.28 = 0.8208.$$

$$\boxed{\text{Answer: The system is working with probability } 0.8208.}$$

Week 3, Problem 2

$X = \#\text{ errors} \sim Bin(1000, 10^{-3})$

$$\mathbb{P}[X>3] = 1 - \mathbb{P}[X=0] - \mathbb{P}[X=1] - \mathbb{P}[X=2] - \mathbb{P}[X=3]$$

$$= 1 - 0.999^{1000} - 1000 \cdot 0.999^{999} \cdot 0.001 - \frac{1000 \cdot 999}{2} 0.999^{998} \cdot 0.001^2$$

$$- \frac{1000 \cdot 999 \cdot 998}{6} 0.999^{997} \cdot 0.001^3$$

$$= 1 - 0.999^{1000}\left(1 + \frac{1000 \cdot 0.001}{0.999}\left(1 + \frac{999}{2} \cdot \frac{0.001}{0.999}\left(1 + \frac{998}{3} \frac{0.001}{0.999}\right)\right)\right)$$

Let $N = 1000$. Then

$$\mathbb{P}[X>3] = 1 - \left(\frac{N-1}{N}\right)^N \left(1 + N \cdot \frac{1}{N-1}\left(1 + \frac{N-1}{2} \frac{1}{N-1}\left(1 + \frac{N-2}{3} \cdot \frac{1}{N-1}\right)\right)\right)$$

$$\approx 1 - \left(\frac{N-1}{N}\right)^N \left(1 + 1\left(1 + \frac{1}{2}\left(1 + \frac{1}{3}\right)\right)\right)$$

$$1 - \left(1 - \frac{1}{N}\right)^N \cdot \frac{8}{3} \approx 1 - \frac{1}{e} \cdot \frac{8}{3} \boxed{\approx 0.19}$$