Aalto University
School of Science

CS-E4530 Computational Complexity Theory

Lecture 5: NP and Nondeterminism

Aalto University
School of Science
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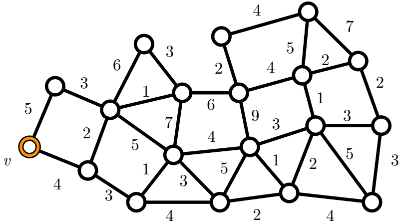
Agenda

- Polynomial-time verifiers
- Examples of polynomial-time verifiers
- The language class NP
- Nondeterministic Turing Machines
- NP-completeness

Travelling Salesman Problem

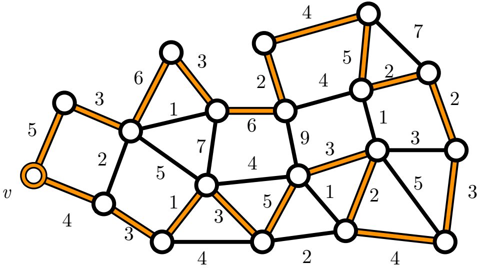
Travelling Salesman Problem (Decision Version)

- **Instance:** Graph $G = (V,E)$ with positive edge weights, integer $W \geq 0$, a vertex $v \in V$.
- **Question:** Is there a tour starting from vertex v that visits all other vertices exactly once and then returns to v with weight at most W?



Travelling Salesman Problem

- **We don't know how to solve TSP in polynomial time**
- **We can *verify* the correctness of a solution:**
  - *Solution:* a tour $T = (v_1, v_2, \ldots, v_n)$
  - *Verification:* check that T is a valid tour, T visits all vertices once, and has weight at most W
  - Verification takes polynomial time



k-colouring

Definition

Let k be a fixed positive integer, and let $G = (V, E)$ be an undirected graph. A *k-colouring* of G is a function

$$c\colon V \to \{1, 2, \dots, k\}$$

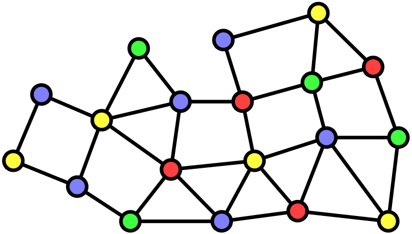
such that for adjacent vertices v and u, we have $c(v) \neq c(u)$.

k-colouring problem (k-COL)

- **Instance:** Graph $G = (V, E)$.
- **Question:** Is there a k-colouring of G?

k-colouring

- **We don't know how to solve k-colouring in polynomial time**
- **We can *verify* the correctness of a solution:**
  - *Solution:* a k-colouring $c\colon V\colon \{1,2,\ldots,k\}$
  - *Verification:* check that for all edges $\{u,v\} \in E$, we have $c(u) \neq c(v)$
  - Verification takes polynomial time



Polynomial-time Verifiers

Definition (Polynomial-time Verifier)

Let $L \subseteq \{0,1\}^*$. A *polynomial-time verifier* for L is a polynomial-time Turing machine M such that for some polynomial function $p \colon \mathbb{N} \to \mathbb{N}$ the following holds:

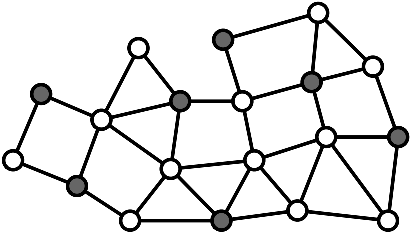
- if $x \in L$, there is a string $u \in \{0,1\}^*$ with $|u| \leq p(|x|)$ so that $M\big((x,u)\big) = 1$, and
- if $x \notin L$, we have $M\big((x,u)\big) = 0$ for all $u \in \{0,1\}^*$.

If $M\big((x,u)\big) = 1$, we call u the *certificate* or *witness* for x.

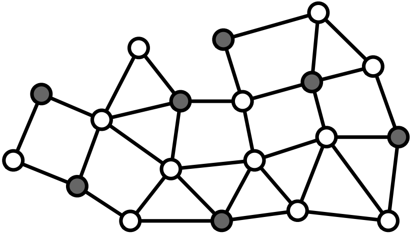
Maximum Independent Set

Maximum independent set (MaxIS)

- **Instance:** Graph $G = (V,E)$ and an integer $k \geq 1$.
- **Question:** Is there a set of vertices I such that $|I| \geq k$ and for all $u,v \in I$, we have that $\{u,v\} \notin E$?



Maximum Independent Set



- *Certificate:* A vertex set $I \subseteq V$ of size k ($O(k \log n)$ bits)
- *Verifier:*
  - Check that I has correct size
  - Check that for each edge $\{u, v\} \in V$, either $u \notin I$ or $v \notin I$

Subset Sum

Subset sum

- **Instance:** A list of integers $a_1, a_2, \ldots, a_n$ and an integer T.
- **Question:** Is there a subset of the input list that sums up to T?

- *Certificate:* A subset S of the input list
- *Verifier:*
  - Check that S is a valid subset of input
  - Compute the sum of S and check that it is T

Composite Numbers

Composite number

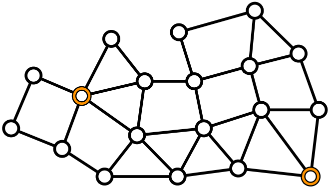
- **Instance:** An integer N.
- **Question:** Are there numbers p and q with $p,q \notin \{1,N\}$ such that $pq = N$? (That is, N is not a prime number.)

- *Certificate:* Numbers p and q ($O(\log|N|)$ bits)
- *Verifier:* Check that $pq = N$

Connectivity

Connectivity

- **Instance:** Graph $G = (V, E)$, two vertices s and t.
- **Question:** Is there a path from s to t in G?

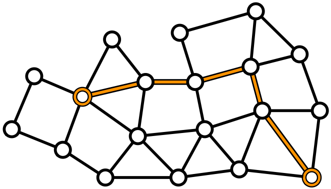


- *Certificate:* A path $P = (v_1, v_2, \ldots, v_k)$ in graph G
- *Verifier:* Check that P is a valid path from s to t

Connectivity

Connectivity

- **Instance:** Graph $G = (V, E)$, two vertices s and t.
- **Question:** Is there a path from s to t in G?



- *Certificate:* A path $P = (v_1, v_2, \ldots, v_k)$ in graph G
- *Verifier:* Check that P is a valid path from s to t

NP

Definition (NP)

The class NP is the class of all languages $L \subseteq \{0,1\}^*$ that have a polynomial-time verifier.

NP and Other Classes

- **Consider the following time complexity classes:**
  - *Polynomial time:* $\mathsf{P} = \bigcup_{d=1}^{\infty} \mathsf{DTIME}(n^d)$
  - *Exponential time:* $\mathsf{EXP} = \bigcup_{d=1}^{\infty} \mathsf{DTIME}(2^{n^d})$

Theorem

$\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{EXP}$

- **Proof (**$\mathsf{P} \subseteq \mathsf{NP}$**):**
  - Use length-0 string ε as certificate
- **Proof (**$\mathsf{NP} \subseteq \mathsf{EXP}$**):**
  - Try all possible certificates of length $p(|x|)$
  - $O(2^{p(n)})$ possibilities + checking

P vs NP

- **Do all polynomial-time verifiable problems also have polynomial-time algorithms?**
  - Formally: does it hold that $\mathrm{P} = \mathrm{NP}$?
  - This is the famous *P vs. NP question*
  - This seems to be a really difficult problem
- **In practice, we tend to assume** $\mathrm{P} \neq \mathrm{NP}$
  - We will use this assumption to prove that certain problems are difficult
  - Gives *conditional lower bounds*

Nondeterministic Turing Machines

- **We give an alternative definition of** NP **in terms of** ***polynomial-time nondeterministic Turing machines***
  - NP stands for *nondeterministic polynomial time*
- **NDTM is an** ***abstract*** **model of computation**
  - Does not correspond to any physical method of computation
  - Purely a *conceptual tool*
  - Can be viewed as an abstraction of computation that tries all possible solutions

Nondeterministic Turing Machines

- **A *nondeterministic Turing machine* M is a Turing machine with following special features:**
  - M has a special *accept state* q_{accept}
  - M has two *transition functions* δ_1 and δ_2
  - M does not have an output tape
- **An *execution* of nondeterministic Turing machine M:**
  - Start from the starting state as usual
  - Apply *either* δ_1 or δ_2 at each step
  - Halt when reaching q_{accept} or q_{h}
- **For each input, a NDTM has *multiple possible executions***

Nondeterministic Turing Machines

Definition

A NDTM M decides language L in time $T(n)$ if:

- For any $x \in L$, there is at least one execution on input x that reaches state q_{accept}
- For any $x \notin L$, all executions halt without entering q_{accept}
- All executions on input $x \in \{0,1\}^*$ run for at most $T(|x|)$ steps

Nondeterministic Time Complexity

Definition (Class NTIME)

Let $T\colon \mathbb{N} \to \mathbb{N}$ be a function. The class NTIME$(T(n))$ is the set of languages L for which there exists a nondeterministic Turing machine M and a constant $c > 0$ such that M decides L and runs in time $c \cdot T(n)$.

NP: Alternative Definition

Theorem

$\mathsf{NP} = \bigcup_{d=1}^{\infty} \mathsf{NTIME}(n^d)$.

- **Proof ($\bigcup_{d=1}^{\infty} \mathsf{NTIME}(n^d) \subseteq \mathsf{NP}$):**
  - Let $L \in \bigcup_{d=1}^{\infty} \mathsf{NTIME}(n^d)$
  - *We have:* $p(n)$-time NDTM M for L, where p is polynomial
  - *We want:* Polynomial-time verifier M' for L

  - For any $x \in L$, M has an accepting execution
  - *Certificate:* a string $u \in \{0,1\}^{p(|x|)}$
  - *Verifier:* Simulate M, use u to choose which transition function to use ($0 \to \delta_1$, $1 \to \delta_2$), check that the execution ends in q_{accept}

NP: Alternative Definition

Theorem

NP $= \bigcup_{d=1}^{\infty}$ NTIME(n^d).

- **Proof (NP $\subseteq \bigcup_{d=1}^{\infty}$ NTIME(n^d)):**
  - Let $L \in$ NP
  - *We have:* $p(n)$-time verifier M using certificates of length at most $q(n)$ for L, where p, q are polynomial
  - *We want:* Polynomial-time NDTM M' for L

  - Use nondeterminism to generate a certificate u of length at most $q(|x|)$ for input x
  - Concretely: δ_1 writes 0, δ_2 writes 1
  - Deterministically simulate verifier M with (x, u), move to q_{accept} if M accepts

NP-hardness and NP-completeness

Definition

We say that a language L is NP*-hard* if for any language $L' \in$ NP, there is a polynomial-time reduction from L' to L.

Definition

We say that L is NP*-complete* if L is NP-hard and $L \in$ NP.

NP-hardness and NP-completeness

Theorem

- *If L is* NP*-hard and $L \in$* P*, then* P $=$ NP*.*
- *If L is* NP*-complete, then $L \in$* P *if and only if* P $=$ NP*.*

- **Proof (first statement):**
  - Recall: $L' \leq_p L$ and $L \in$ P implies $L' \in$ P
  - If L is NP-hard and $L \in P$, then for any language $L' \in$ NP we have $L' \leq_p L$ and thus $L' \in$ P
  - Thus it follows from the assumption that NP $\subseteq$ P

NP-complete Languages

- NP**-complete problems are the hardest problems in** NP
  - If we believe P $\neq$ NP, then NP-complete languages are not in P
  - Important technique for proving *conditional* lower bounds, as many interesting problems are in NP
  - On the other hand, if one NP-complete problem has a polynomial-time algorithm, then P $=$ NP
- **Typical application:**
  - We have a computational problem L we are interested in
  - Prove that L is NP-complete and conclude there is probably no polynomial-time algorithm

An NP-complete Language

Definition (TMSAT)

- **Instance:** A tuple $(\alpha, x, 1^n, 1^t)$, where $\alpha, x \in \{0,1\}^*$
- **Question:** Is there a string $u \in \{0,1\}^*$ with $|u| \leq n$ such that the Turing machine M_α outputs 1 on input (x, u) within t steps? $(*)$

- TMSAT $= \{(\alpha, x, 1^n, 1^t) :$ Condition $(*)$ holds for $(\alpha, x, 1^n, 1^t)\}$

An NP-complete Language

Theorem

TMSAT *is* NP*-complete.*

- **Proof:**

  (i) TMSAT $\in$ NP, i.e. TMSAT has a polynomial-time verifier.

  - Note that

$$\left|\llcorner(\alpha, x, 1^n, 1^t)\lrcorner\right| \geq |1^n| = n$$
$$\left|\llcorner(\alpha, x, 1^n, 1^t)\lrcorner\right| \geq \left|1^t\right| = t$$

  - That is, n and t are polynomial in $|\llcorner(\alpha, x, 1^n, 1^t)\lrcorner|$
  - *Certificate:* a string $u \in \{0,1\}^*$ with $|u| \leq n$
  - *Verification algorithm:* simulate Turing machine M_α on input (x, u) for t steps, check if it halts and outputs 1

An NP-complete Language

Theorem

TMSAT *is* NP*-complete.*

- **Proof (cont'd):**

  (ii) TMSAT is NP-hard:

  - Let $L \in \text{NP}$
  - By definition, there is a verifier M for L that runs in time $q(n)$ with certificates of size at most $p(n)$, where p, q are polynomial
  - *Reduction:* map $x \mapsto (\llcorner M \lrcorner, x, 1^{p(|x|)}, 1^{q(|x|)})$
  - Correctness follows immediately from definitions

NP-complete Languages

- TMSAT **is not very interesting example**
  - Definition tied directly to the definition of NP
  - Does not really tell us anything new about NP
- **Next objective:** ***find other* NP*-complete languages***
  - Many *natural* problems are NP-complete
  - In fact, we have already seen many examples

NP-completeness via Reductions

Theorem

Let $L_1, L_2 \in \{0,1\}^*$ *be languages. If* L_1 *is* NP*-hard and* $L_1 \leq_p L_2$*, then* L_2 *is* NP*-hard.*

- **Proof:** Follows from the transitivity of $\leq_p$.

Corollary

Let $L_1, L_2 \in \{0,1\}^*$ *be languages. If* L_1 *is* NP*-complete,* $L_1 \leq_p L_2$*, and* $L_2 \in$ NP*, then* L_2 *is* NP*-complete.*

NP-completeness via Reductions

- **Next lectures:**
  - Prove that a problem called *CNF-SAT* is NP-complete
  - Prove other NP-completeness results by building a *tree of reductions* step-by-step, starting from CNF-SAT

- **For example:**
  - We will prove that *3-colouring* is NP-complete via an intermediate problem called *3-SAT*
  - The reduction presented at previous lecture then implies that *4-colouring* is NP-complete

Lecture 5: Summary

- Polynomial-time verifiers
- The class NP
- Nondeterministic Turing machines
- NP-completeness
- Existence of an NP-complete language