

ELEC-E3530

ELEC-E3530

Integrated Analog Systems L4

SC Ladder Filters

SC Ladder Filters

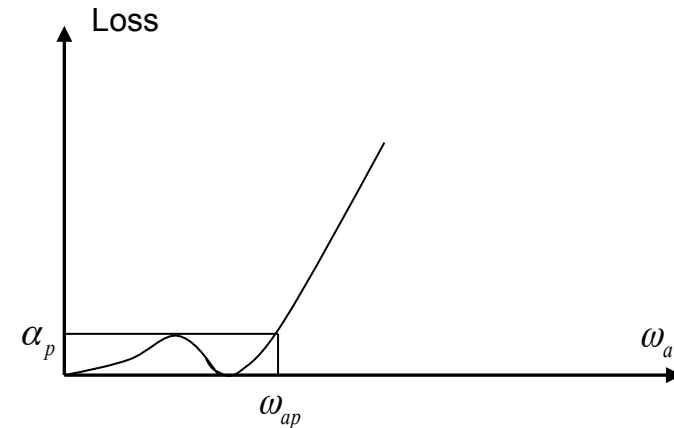
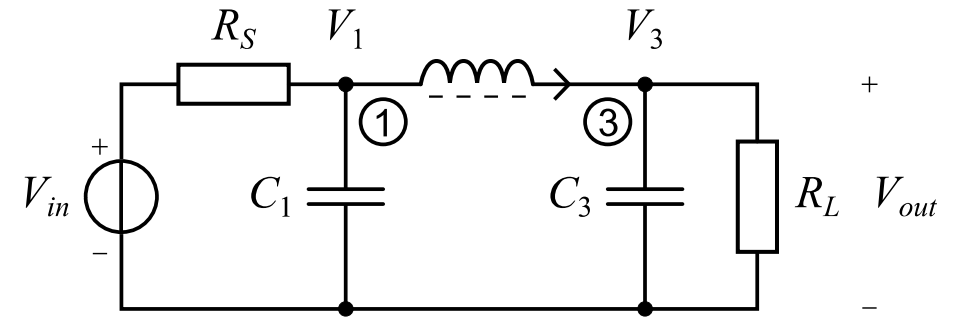
SC-ladder filters

- The LC-ladder filters' sensitivity for component spread is zero at the maximum of the transfer function
- Close to the maximum the sensitivity is relative to the reflection coefficient i.e. sensitivity is small
- Sensitivity for the source and load resistors ($R_s=R_L$) spread is constant i.e. frequency independent constant error

⇒ filters with a high Q value can be implemented

The LC-filter's sensitivity characteristics will be inherited by the SC equivalent IF

- 1) the transfer functions are the same
- 2) the filters coefficients will be determined by one-to-one mapping



Losses in the LC-two ports

$$\varepsilon_L(\omega) = 1 - \frac{\Delta L}{L} + j \frac{1}{Q(\omega)}$$

$$Q(\omega) = \frac{\omega L}{R_L}$$

The attenuation on the pass-band:

$$|\Delta a(\omega)| < \frac{|\varepsilon_{L1} c r(\omega)|}{1 - |r(\omega)|^2} \omega i(\omega)$$

$$r(\omega) = \frac{z(j\omega) - R}{z(j\omega) + R}$$

$i(\omega)$ = group delay time
 $r(\omega)$ = reflection coefficient

Active SC integrator:

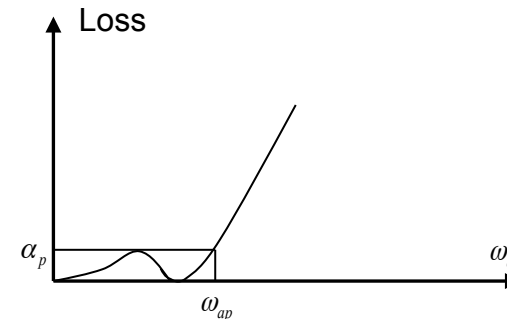
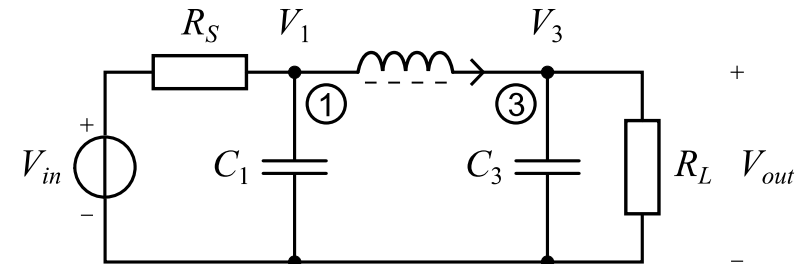
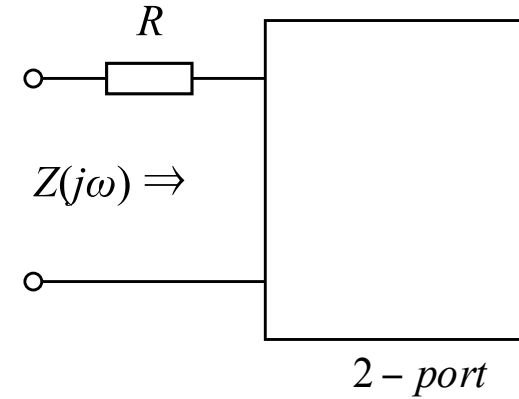
$$H(e^{j\omega T}) = \frac{1}{j\omega R x_i} (1 + m(\omega)) e^{jp(\omega)}$$

$$\approx \frac{1 + m(\omega) + jp(\omega)}{j\omega R x_i}$$

$m(\omega)$ = gain error
 $p(\omega)$ = phase error

$$\Rightarrow m(\omega) \Rightarrow -\frac{\Delta L}{L}$$

$$p(\omega) \Rightarrow -\frac{1}{Q_L(\omega)}$$



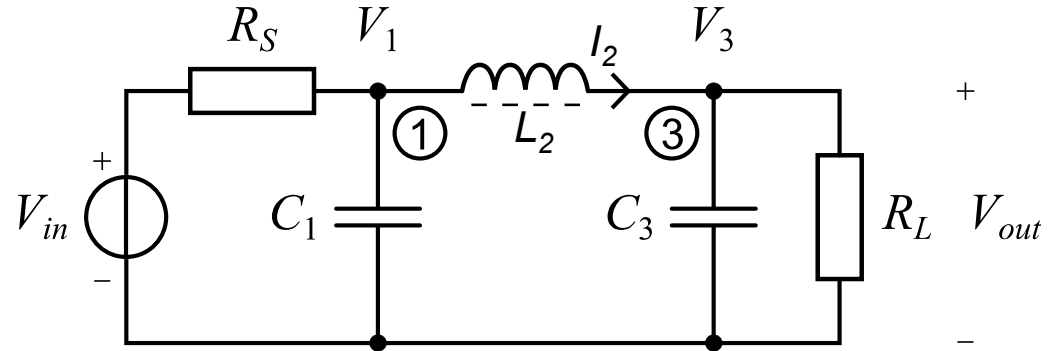
Ladder filter design

For a capacitor:

$$V_c(s) = \frac{1}{sC} I_c(s)$$

For an inductor:

$$I_L(s) = \frac{1}{sL} V_L(s)$$



- Thus the capacitor voltage is obtained by integrating the capacitor current.
- The inductor current is obtained by integrating the inductor voltage

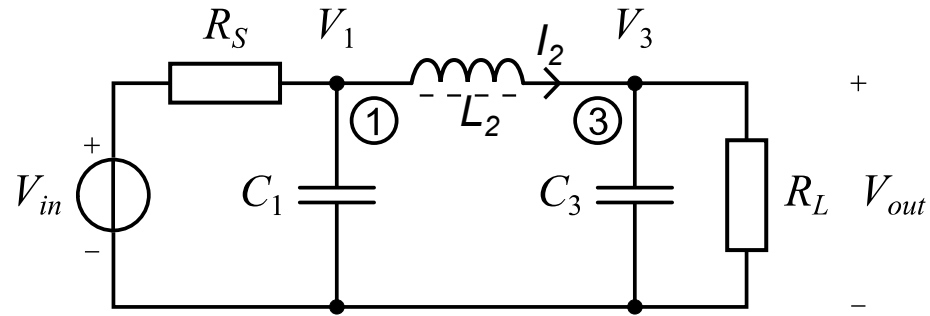
⇒ select the state variables to be

- * the capacitor voltage
- * the inductor current

We select the state variables to be:

$$V_1, I_2, V_3$$

Signal-flow graph



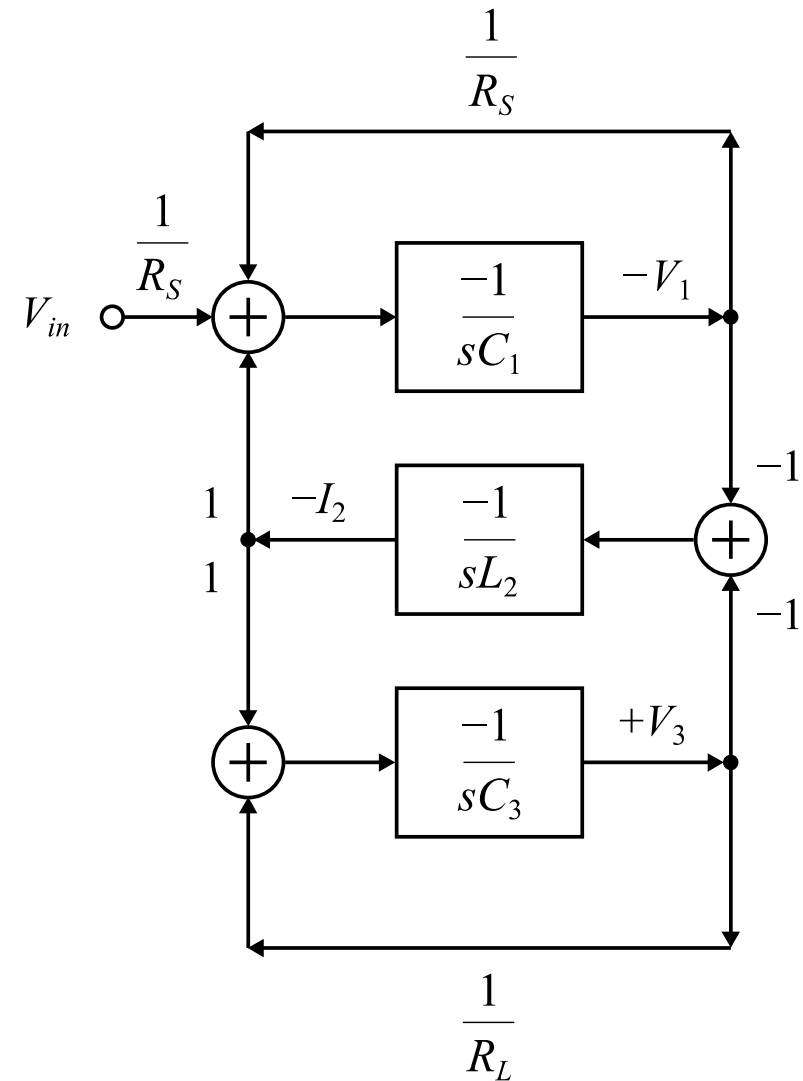
The state equations are:

$$-V_1 = \frac{-1}{sC_1} \left(\frac{V_{in} - V_1}{R_S} - I_2 \right)$$

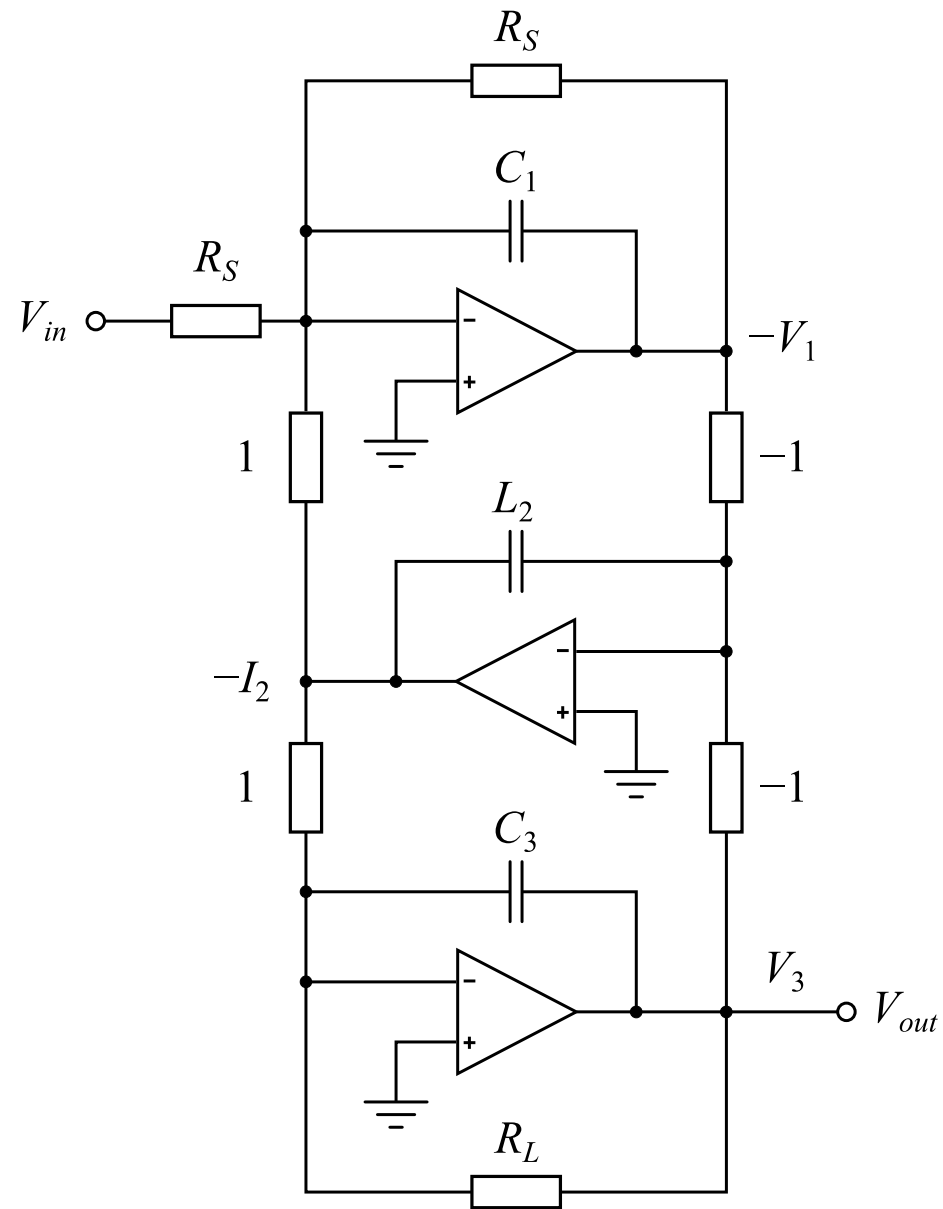
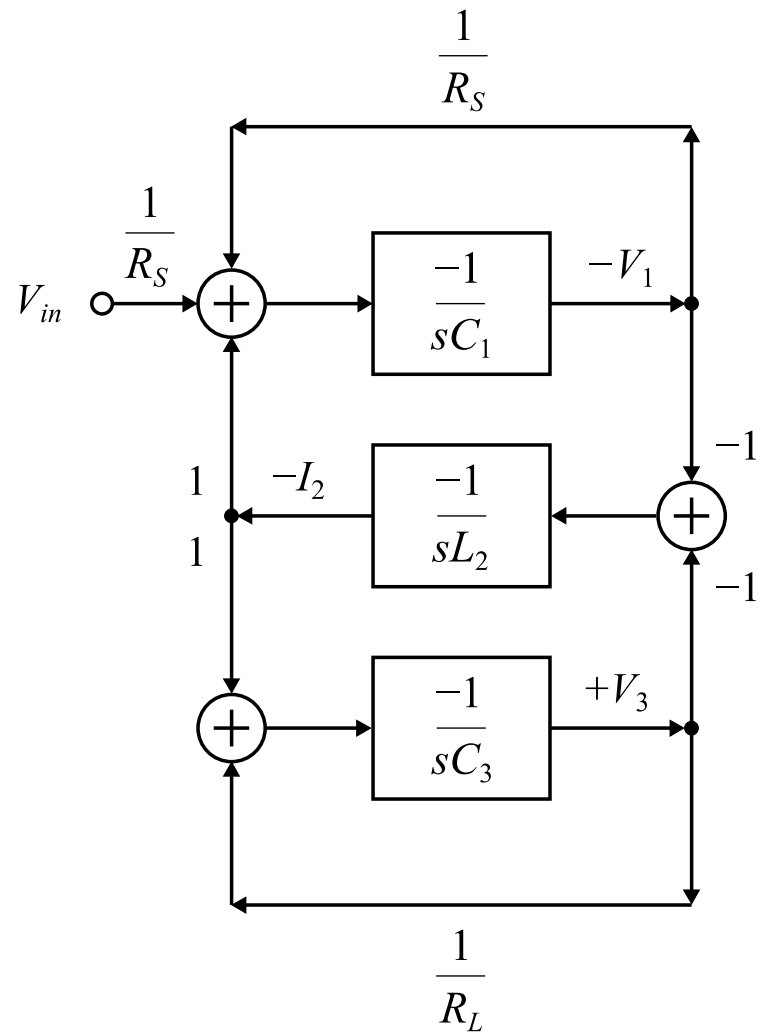
$$-I_2 = \frac{-1}{sL_2} (V_1 - V_3)$$

$$V_3 = \frac{-1}{sC_3} \left(-I_2 + \frac{V_3}{R_L} \right)$$

The sign convention is selected by the requirement that the integrators are realized with the **negative feedback amplifiers** i.e. $H(s) = -1/s$!!



RC-active realization



Scaling the middle integrator

$$-I_2 = \frac{-1}{sL_2}(V_1 - V_3)$$

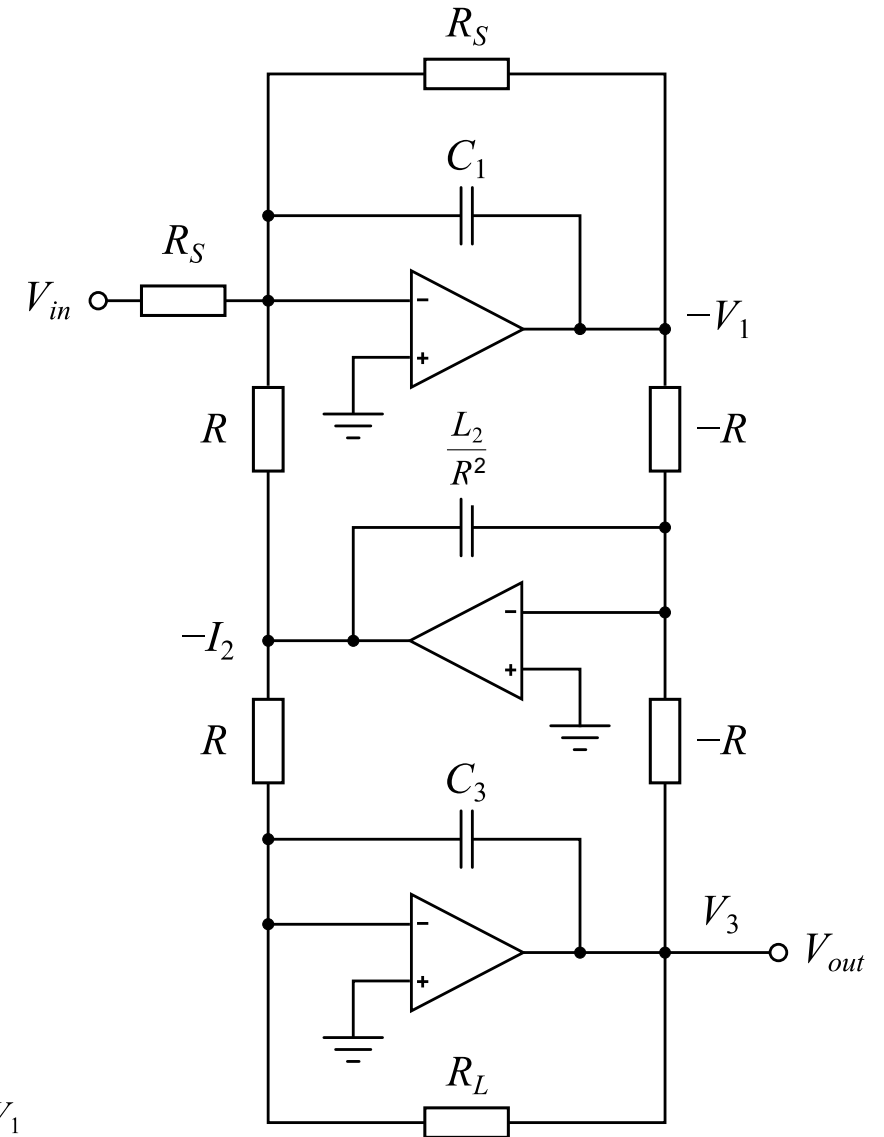
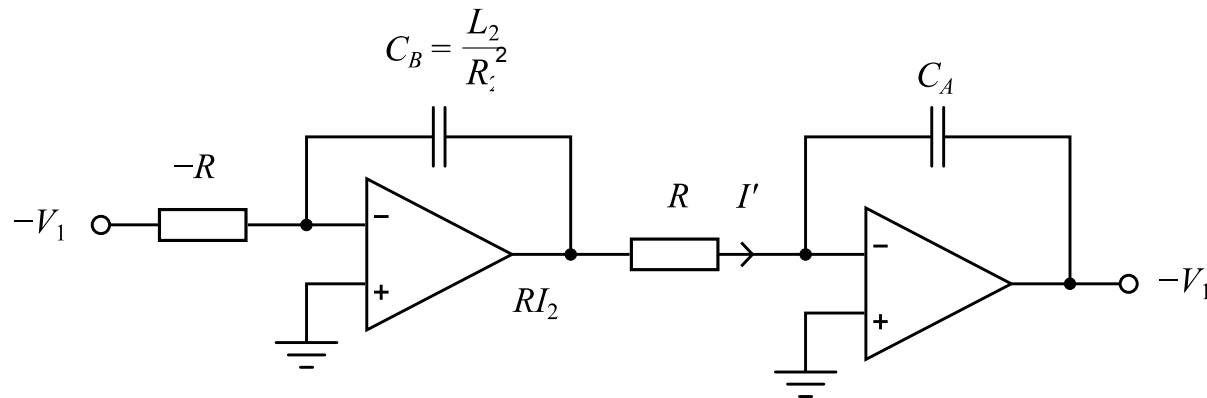
Scaling the input with R

$$-I_2 = \frac{-R}{sL_2} \left(\frac{V_1}{R} - \frac{V_3}{R} \right)$$

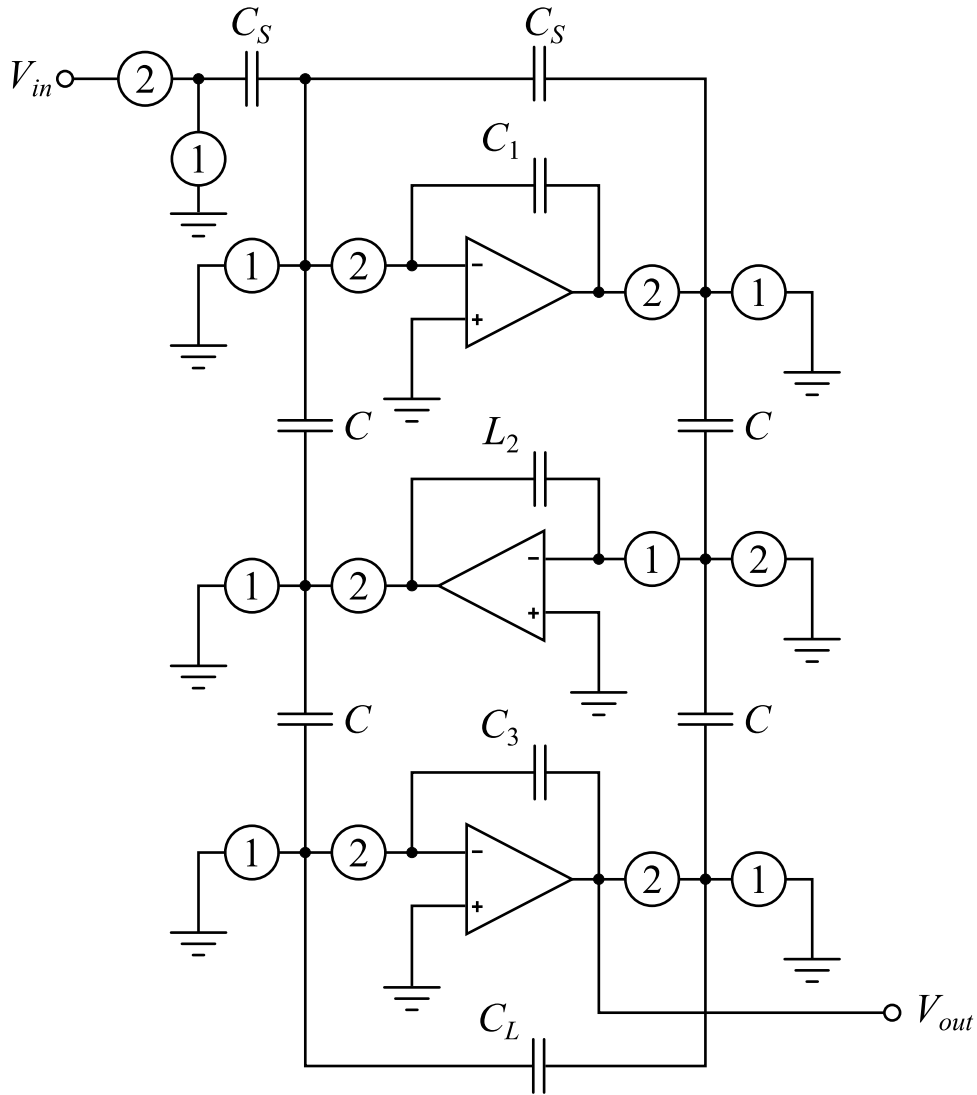
Scaling the output with R (transforming $-I_2$ into voltage)

$$(-V_2) = -I_2 R = \frac{-R^2}{sL_2} \left(\frac{V_1}{R} - \frac{V_3}{R} \right)$$

Also all other impedances connected to the output branch have to be scaled with R



SC realization of LC-ladder filter



The capacitor values for SC realization:

$$R_{SC} = \frac{T}{C_{SC}} \Rightarrow C_{SC} = \frac{T}{R_{SC}}$$

$$\Rightarrow C_s = \frac{T}{R_s}$$

$$C = \frac{T}{1} = T$$

$$C_L = \frac{T}{R_L}$$

Comments on one-to-one mapping:

SC realization requires two capacitors to realize R_s . It also has four switched capacitors with value $C=T$.
 $\Rightarrow$ Sensitivity behaviour is not as good as in the passive LC-prototype.

IF the losses at the pass-band are $< 0,1$ dB , the sensitivity characteristics are still very good !!

Losses in SC integrator

The transfer function for the SC integrator:

$$H(z) = -\frac{C_1}{C_2} \frac{1}{1 - z^{-1}}$$

$$z = e^{j\omega T}$$

$$H(e^{j\omega T}) = -\frac{C_1}{C_2} \frac{1}{1 - e^{-j\omega T}} \approx -\frac{C_1}{C_2} \frac{T}{j\omega - \omega^2 \frac{T}{2}}$$

This is a lossy integrator.

equating gain coefficients and the lossy terms

$$\frac{C_1}{C_2} = \frac{1}{R_1 C_1}$$

$$\Rightarrow R_1 = \frac{T}{C_1}$$

$$\frac{1}{RC_2} = \omega^2 \frac{T}{2}$$

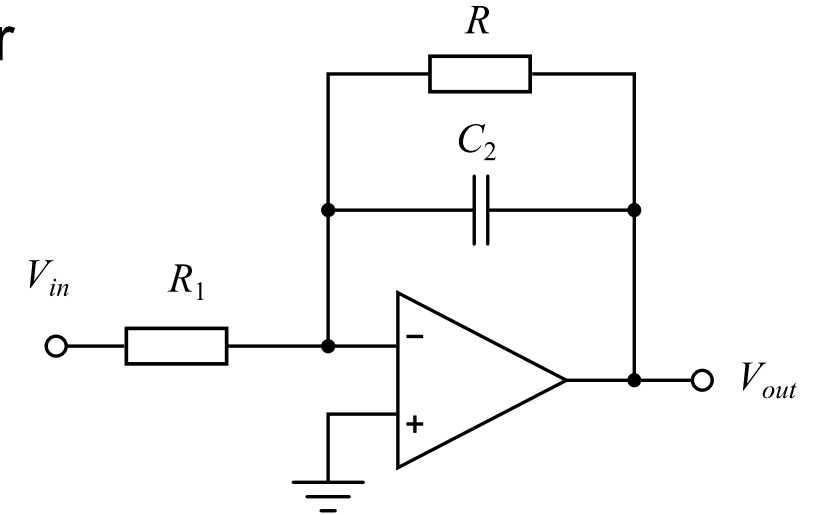
$$R = \frac{2}{\omega^2 T C_2}$$

the Q - factor is

$$\Rightarrow Q = \omega C_2 R = \frac{2}{\omega T}$$

Note: typically $\omega T \ll 1 \Rightarrow Q \gg 1$

The approximation caused by the Z-transformation is equivalent to the losses in the L and C components in the passive filter.



The transfer function of lossy RC integrator:

$$H(\omega) = -\frac{R}{R_1} \frac{1}{1 + j\omega C_2 R} = -\frac{1}{R_1 C_2} \frac{1}{j\omega - \frac{1}{RC_2}}$$

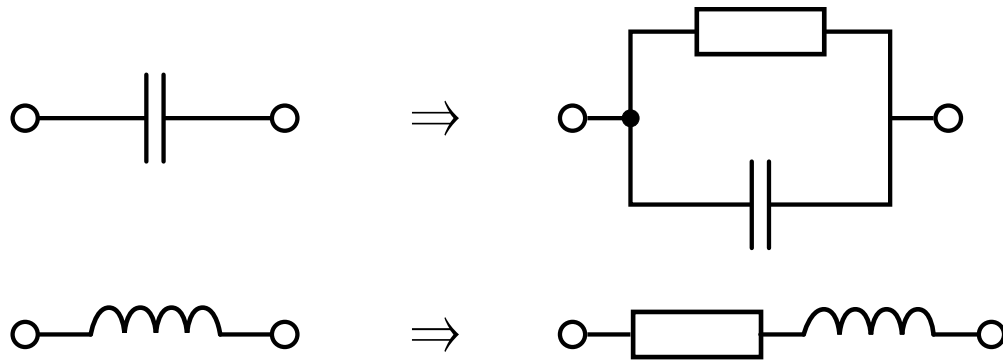
Losses of SC-ladder filter

The transfer function of the middle integrator

$$H(z) = \frac{C_1}{C_2} \frac{z^{-1}}{1 - z^{-1}}$$
$$H(e^{j\omega T}) = \frac{C_1}{C_2} \frac{1}{e^{j\omega T} - 1}$$
$$\approx \frac{C_1}{C_2} \frac{1}{j\omega T - \omega^2 \frac{T^2}{2}}$$

This is a lossy integrator $\Rightarrow Q_L = -\frac{2}{\omega T}$

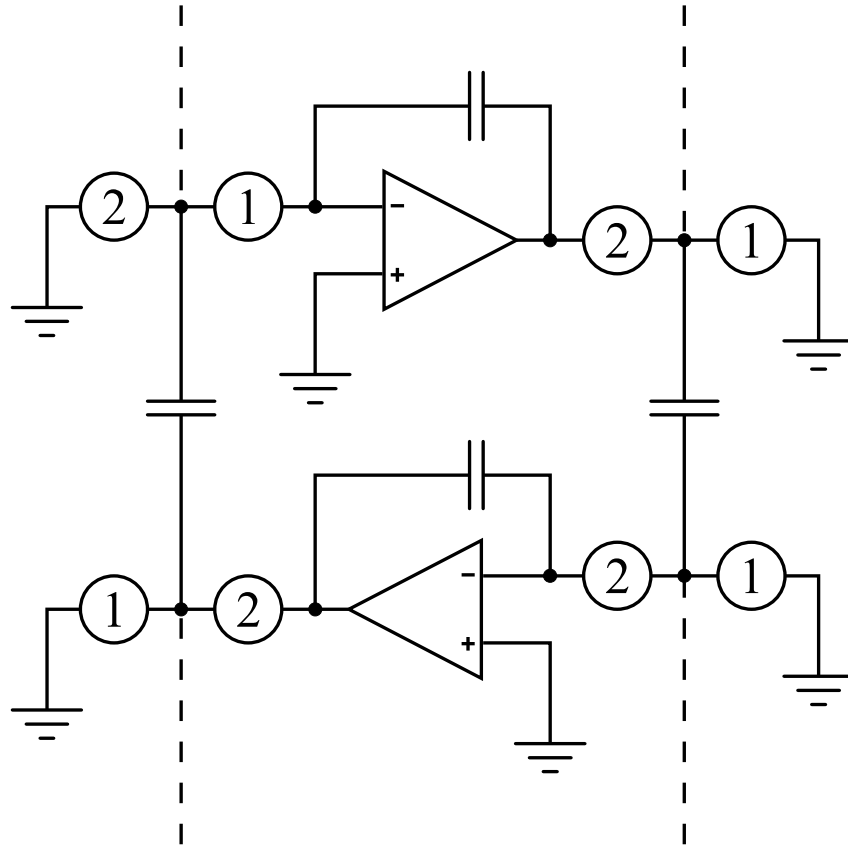
Thus the approximation caused by the z-transformation can be described by the losses of the passive L and C elements in the prototype.



However, the Q-values of the L and C elements have opposite signs
 $\Rightarrow$ in the loop connection they compensate each other's effect

Lossless discrete integrators (LDI)

The SC-ladder implementation has inverting and non-inverting integrators alternating



Transfer functions of inverting and non-inverting integrators:

$$H(z)_{inv} = -\frac{\left(\frac{C_1}{C_2}\right)}{1 - z^{-1}}$$

$$H(z)_{noninv} = \frac{\left(\frac{C_1}{C_2}\right)z^{-1}}{1 - z^{-1}}$$

Transfer function of the loop connection:

$$H(z) = H(z)_{inv} \cdot H(z)_{noninv} = -\frac{\left(\frac{C_1}{C_2}\right)\left(\frac{C_1}{C_2}\right)z^{-1}}{1 - z^{-1}} \frac{1}{1 - z^{-1}}$$

$$= \frac{-K^2 z^{-1}}{(1 - z^{-1})^2} = \frac{-K^2}{\left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)^2}$$

Inserting $z=e^{j\omega T}$ we obtain

$$H(e^{j\omega T}) = \frac{K^2}{4\sin^2\left(\frac{\omega T}{2}\right)}$$

This is real thus the loop connection does not have **losses**!
Comparing with s-domain integrators

$$H_1(s)H_2(s) = -\frac{1}{j\omega RC} \frac{1}{j\omega RC} = \frac{1}{\omega^2 (RC)^2} \quad \text{this is real!}$$

Thus only frequency distortion occurs! ($\omega \rightarrow \sin \frac{\omega T}{2}$)
Let us reform $H(z)$ as follows:

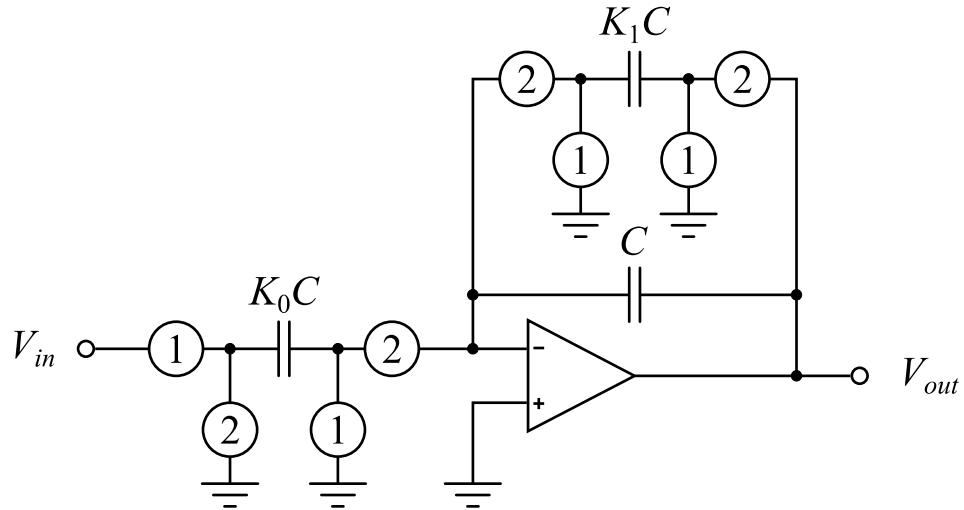
$$H(z) = \frac{-K^2}{\left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)^2} = \underbrace{\frac{-K}{\left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)}}_{\text{LDI-integ.}} \underbrace{\frac{K}{\left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)}}_{\text{LDI-integ.}}$$

These are called Lossless-discrete-integrators.

Notice: The realization of the termination impedances R_s and R_L is not exact!
(i.e. LDI-transformation)

Lossy integrators realizing terminating impedances R_s and R_L

Lossy SC integrator:



Transfer function

$$H(z) = \pm \frac{K_0 z^{-\frac{1}{2}}}{1 + K_1 - z^{-1}}$$

Correct transfer function with LDI-transformation

$$H(z) = \pm \frac{K_0 z^{-\frac{1}{2}}}{1 + K_1 z^{-\frac{1}{2}} - z^{-1}}$$

Term $K_1 z^{-1/2}$ not realizable

Let us insert $z = e^{j\omega T}$

$$H(e^{j\omega T}) = \pm \frac{K_0}{j2 \sin \frac{\omega T}{2} + K_1 \left[\cos \frac{\omega T}{2} + j \sin \frac{\omega T}{2} \right]}$$

K_1 produces a real part and an imaginary part to the denominator $\Rightarrow$ not accurate i.e. not lossless

Equating K_1 and K_0 at the polefrequency ω_1

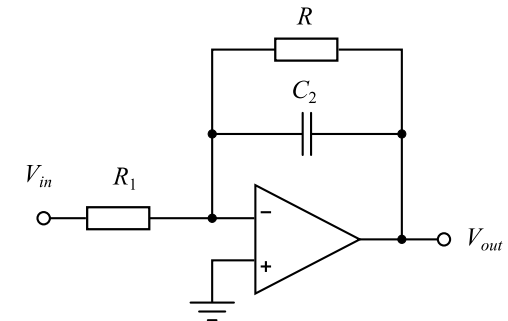
$$K_1 = \frac{2 \sin \frac{\omega_1 T}{2}}{\cos \frac{\omega_1 T}{2} - \sin \frac{\omega_1 T}{2}}$$

$$K_0 = G K_1 \cos \frac{\omega_1 T}{2}$$

for which $G = \frac{R}{R_1}$, $\omega_1 = \frac{2\pi}{\tau_1}$ and $\tau_1 = R_1 C_2$ in lossy RC-integrator

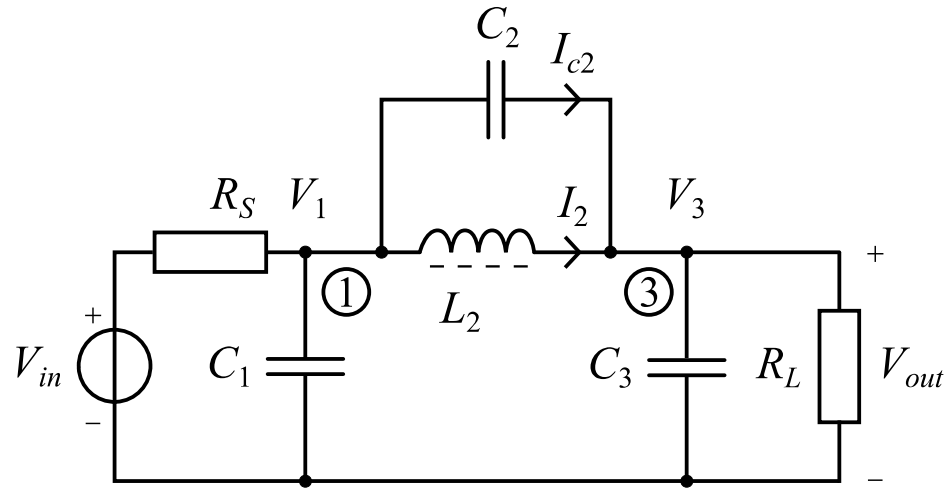
The transfer function of lossy RC integrator:

$$H(\omega) = -\frac{R}{R_1} \frac{1}{1 + j\omega C_2 R} = -\frac{1}{R_1 C_2} \frac{1}{j\omega - \frac{1}{RC_2}}$$



Transfer zeros (elliptical filters)

3rd order elliptical low-pass filter



State-variables

- capacitor voltages
- inductor currents

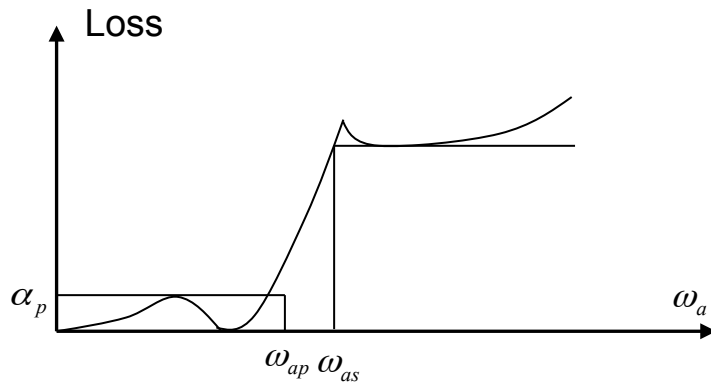
⇒ 4 state equations but only 3 needed

The voltage of C_2 is already given as the difference of the two state-variables V_1 and V_3 .

The capacitor C_2 has a current given as:

$$I_{c2} = sC_2(V_1 - V_3)$$

The state equations are obtained from the current equation at nodes 1 and 3 and from the current equation of L_2 .



State equations

Select state variables to be V_1 , I_2 , V_3 .

Solve current equation at node 1

$$(V_{in} - V_1) \frac{1}{R_S} = sC_1 V_1 + (V_1 - V_3) sC_2 + I_2$$

Form an integrator

$$s(C_1 + C_2)V_1 = \frac{1}{R_S}V_{in} - \frac{1}{R_S}V_1 + sC_2V_3 - I_2$$

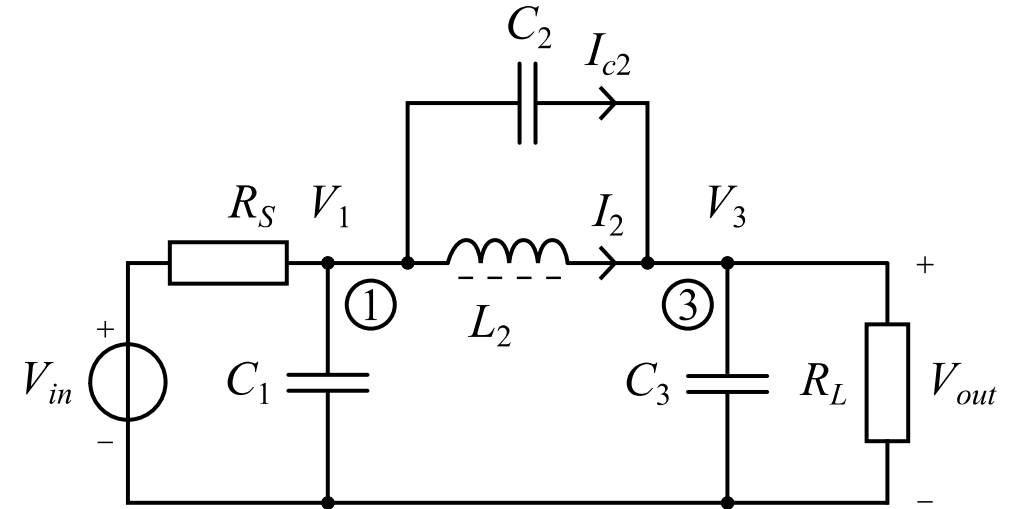
$$-V_1 = \frac{-1}{s(C_1 + C_2)} \left[\frac{1}{R_S}V_{in} - \frac{1}{R_S}V_1 - I_2 + sC_2V_3 \right]$$

Solve V_3 from the current equation at node 3

$$(V_1 - V_3)sC_2 + I_2 = sC_3V_3 + \frac{1}{R_L}V_3$$

$$s(C_2 + C_3)V_3 = sC_2V_1 + I_2 - \frac{1}{R_L}V_3$$

$$V_3 = \frac{-1}{s(C_2 + C_3)} \left[-sC_2V_1 - I_2 + \frac{1}{R_L}V_3 \right]$$



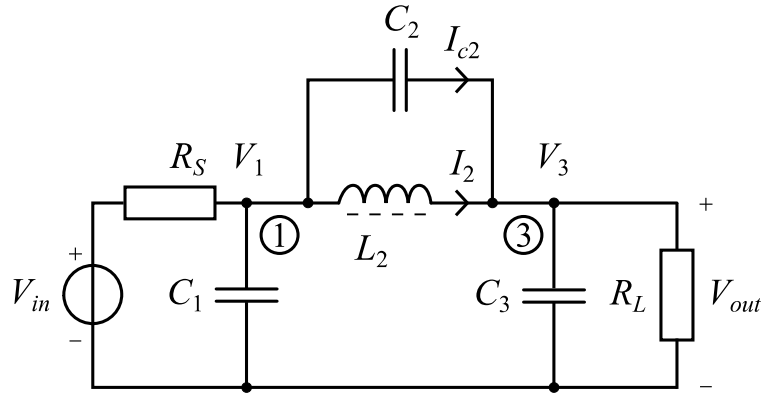
Inductor L_2 current

$$V_1 - V_3 = sL_2 I_2$$

$$-I_2 = \frac{-1}{sL_2} (V_1 - V_3)$$

Note: sign-convention according to negative feedback realization.

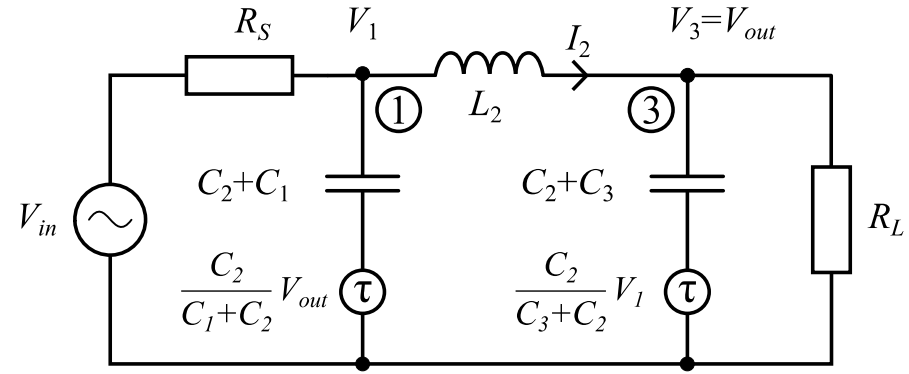
Circuit transformation method to find state-equations



Currents in nodes ① and ③:

$$\textcircled{1} \quad (V_{in} - V_1) \frac{1}{R_S} = (V_1 - V_3) s C_2 + (V_1 - V_3) \frac{1}{s L_2} + s C_1 V_1$$

$$\textcircled{3} \quad (V_1 - V_3) s C_2 + (V_1 - V_3) \frac{1}{s L_2} = V_3 \cdot s C_3 + V_3 \cdot \frac{1}{R_L}$$



again in a changed coupling

$$\begin{aligned} \textcircled{1} \quad (V_{in} - V_1) \frac{1}{R_S} &= (V_1 - V_3) \frac{1}{s L_2} + \left(V_1 - \frac{C_2}{C_1 + C_2} \cdot V_3 \right) s (C_1 + C_2) \\ &= (V_1 - V_3) \frac{1}{s L_2} + V_1 s (C_1 + C_2) - V_3 s C_2 \\ &= (V_1 - V_3) \frac{1}{s L_2} + V_1 s C_1 + (V_1 - V_3) s C_2 \end{aligned}$$

$$\begin{aligned} \textcircled{3} \quad \frac{1}{s L_2} (V_1 - V_3) &= \left(V_3 - \frac{C_2}{C_2 + C_3} V_1 \right) s (C_2 + C_3) + V_3 \frac{1}{R_L} \\ &= V_3 s (C_2 + C_3) - V_1 s C_2 + V_3 \frac{1}{R_L} \\ &= (V_3 - V_1) s C_2 + V_3 s C_3 + V_3 \frac{1}{R_L} \end{aligned}$$

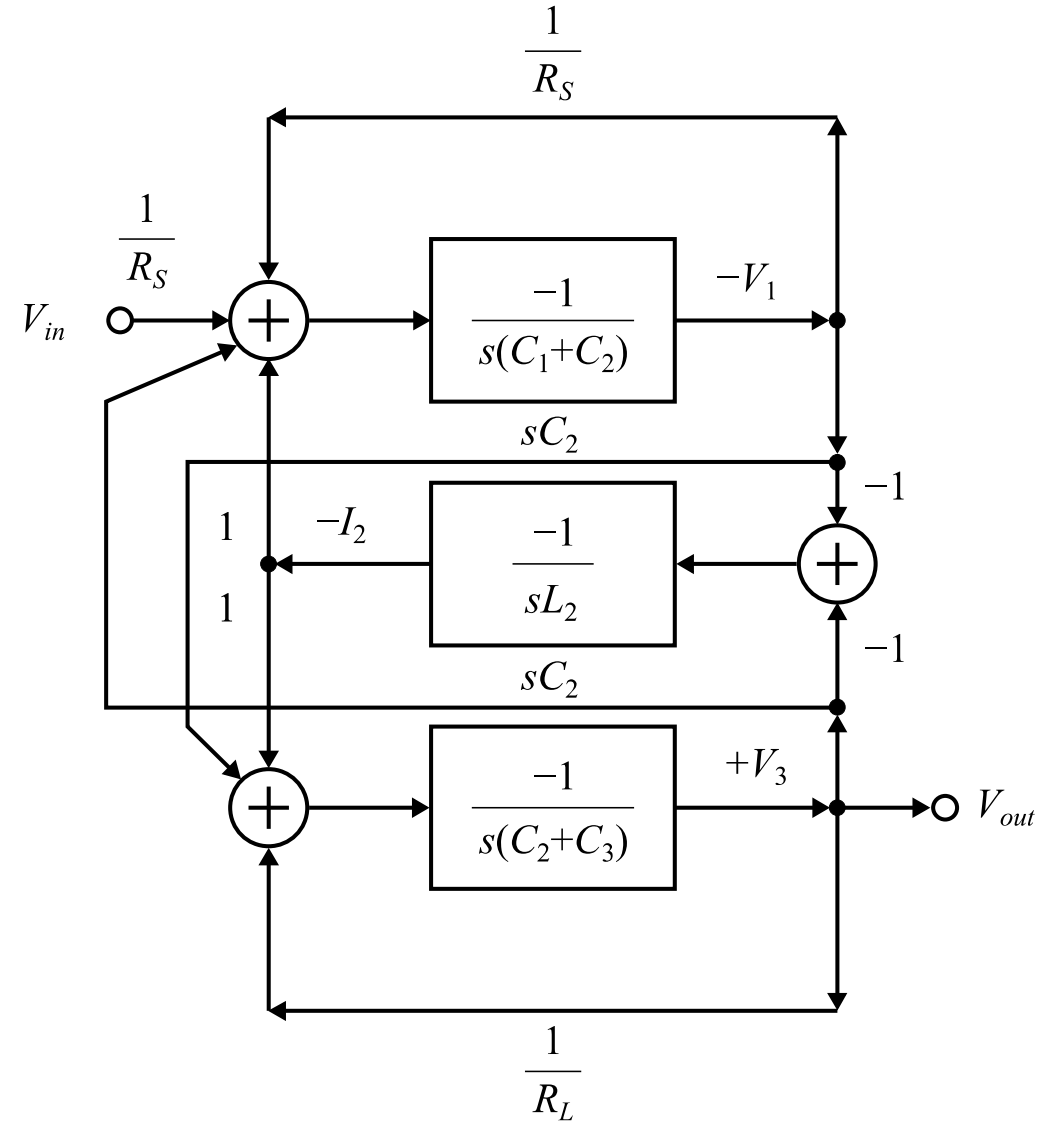
Signal-flow graph

State-equations

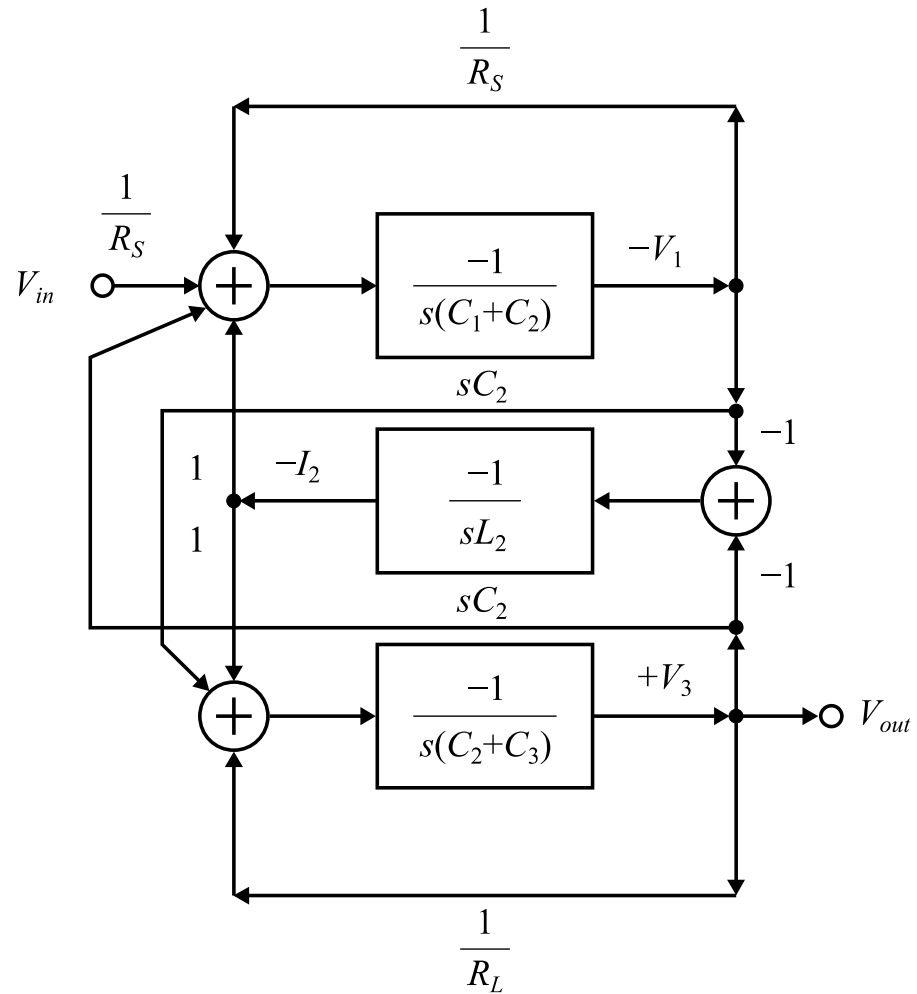
$$-V_1 = \frac{-1}{s(C_1 + C_2)} \left(\frac{V_{in} - V_1}{R_S} + sC_2V_3 - I_2 \right)$$

$$-I_2 = \frac{-1}{sL_2} (V_1 - V_3)$$

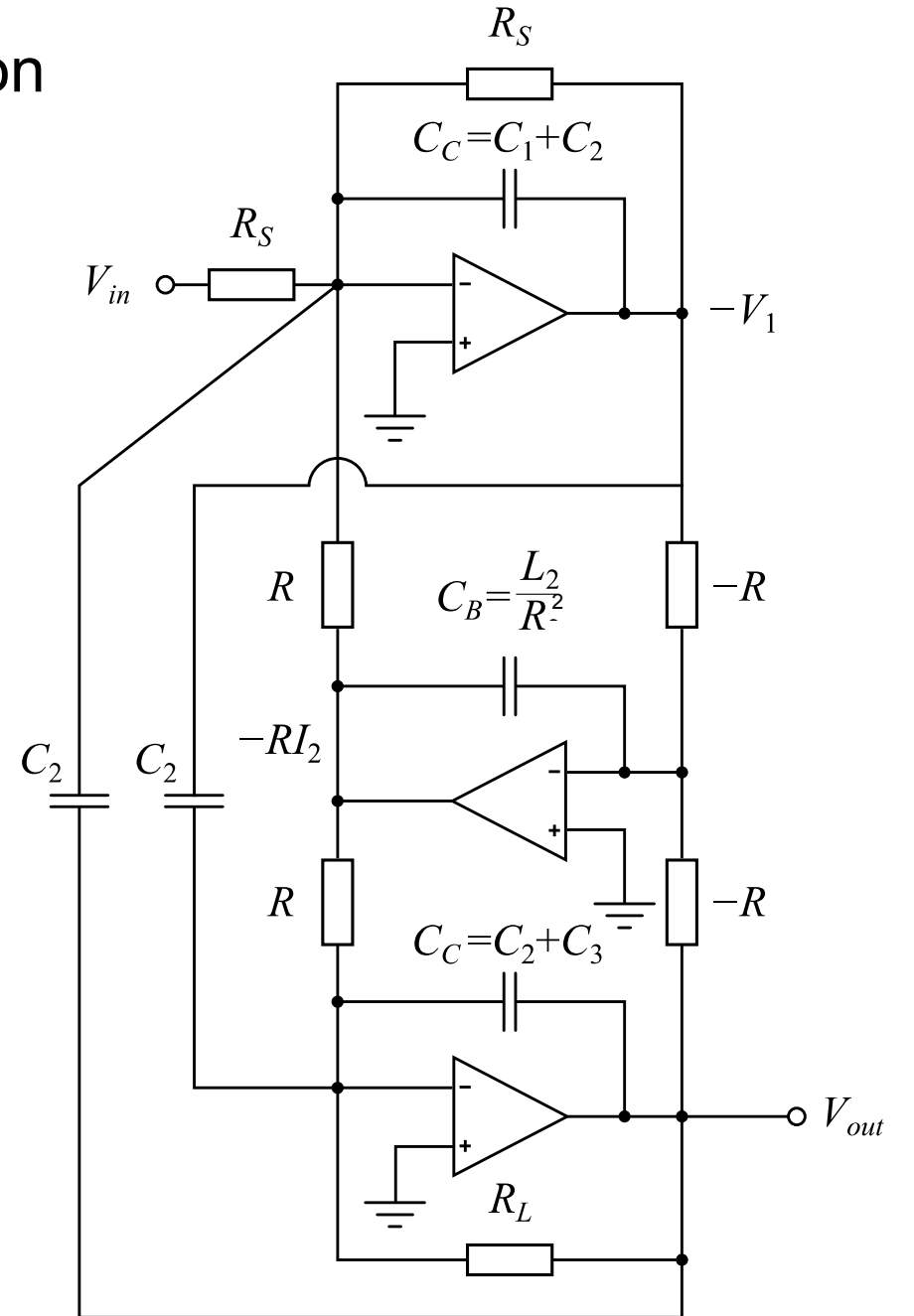
$$V_3 = \frac{-1}{s(C_2 + C_3)} \left(-sC_2V_1 - I_2 + \frac{V_3}{R_L} \right)$$



RC-active realization



Impedance scaling of middle integrator:



Elliptical SC-filter

The switched-capacitors are given by:

$$R_{sc} = \frac{T}{C_{sc}} \Rightarrow C_{sc} = \frac{T}{R_{sc}}$$

Thus:

$$C_s = \frac{T}{R_s}$$

$$C = T$$

$$C_L = \frac{T}{R_L}$$

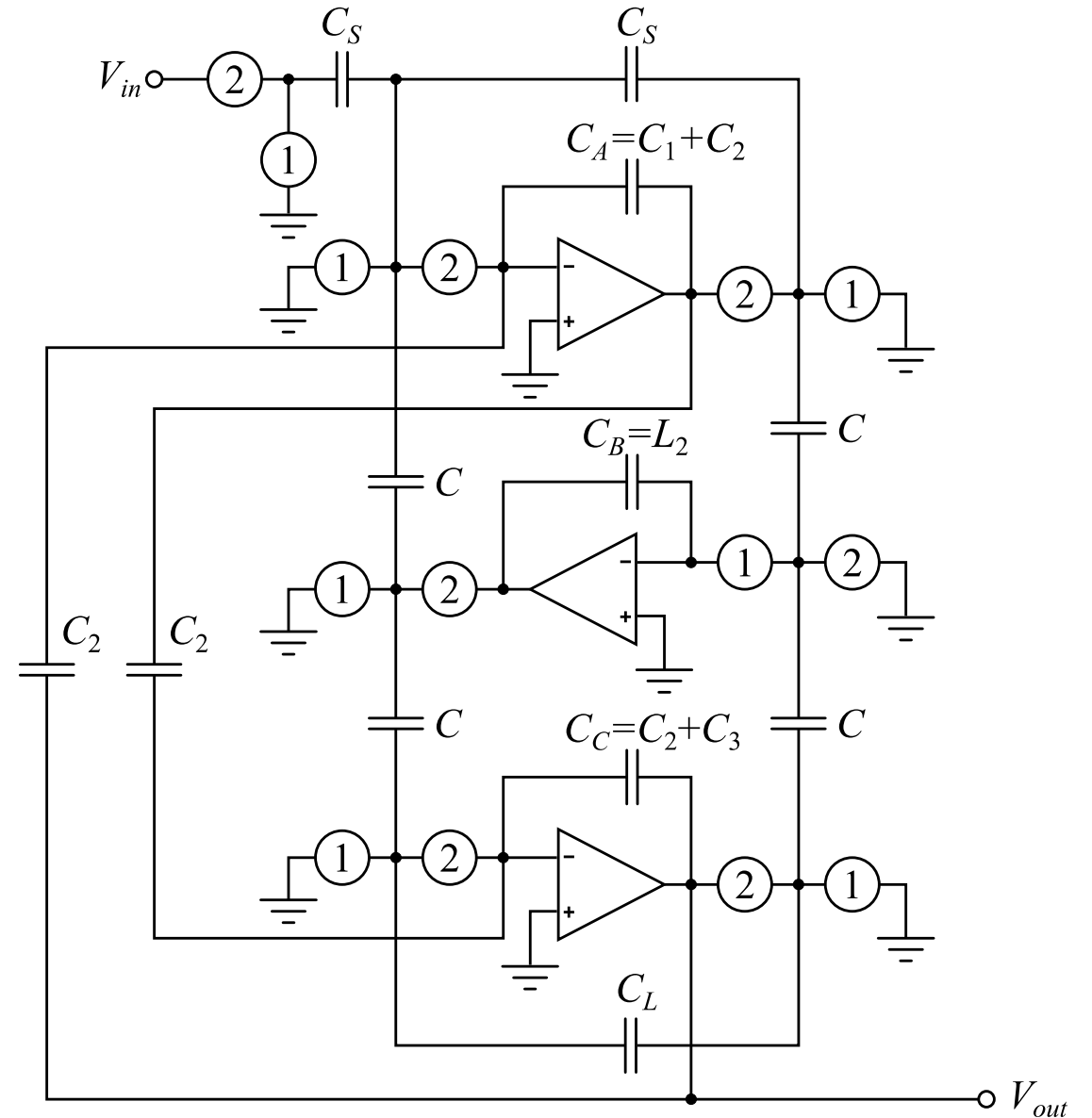
and the integrator capacitors are

$$C_A = C_1 + C_2$$

$$C_B = L_2$$

$$C_C = C_2 + C_3$$

C_2 is included in four different capacitors, while in the passive LC-ladder filter it appears only once
→ effects sensitivity!



Exact design of SC-ladder filter

The errors caused by the LDI-transform can be avoided by using the bilinear transform:

$$s = \frac{2}{T} \frac{z - 1}{z + 1}$$

A 3rd-order LCR-filter implementation by using the bilinear transformation

For the bilinear transformation the state equations have to be changed as follows:

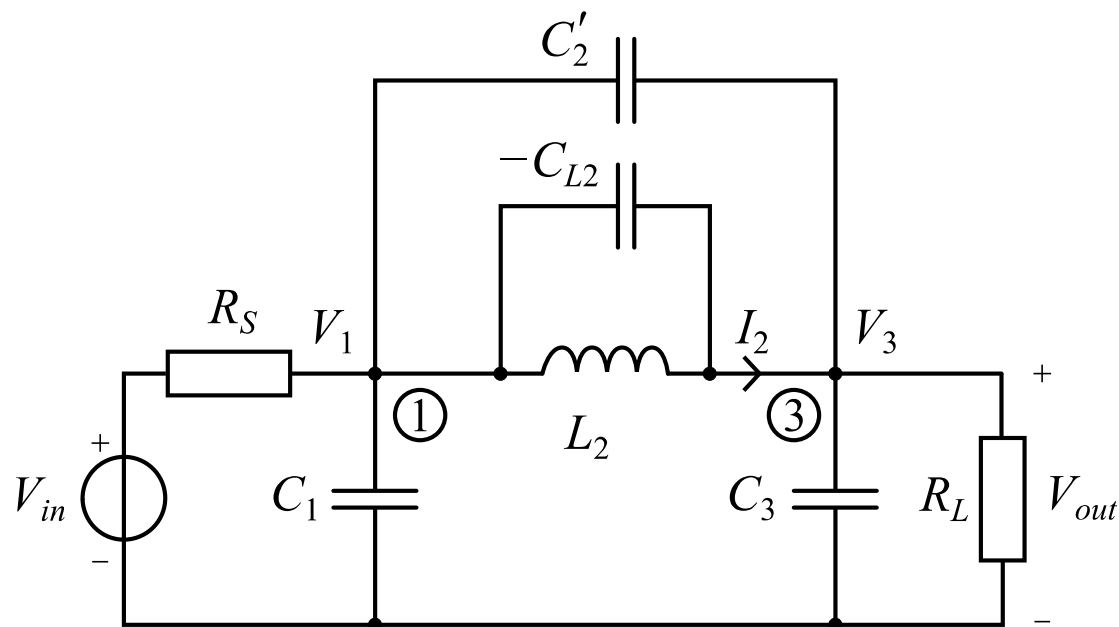
$$\left. \begin{aligned} C'_2 &= C_2 + C_{L2} \\ -C_{L2} &= \frac{-T^2}{4L_2} \end{aligned} \right\} \Rightarrow C_2 = C'_2 - C_{L2}$$

The new state equations:

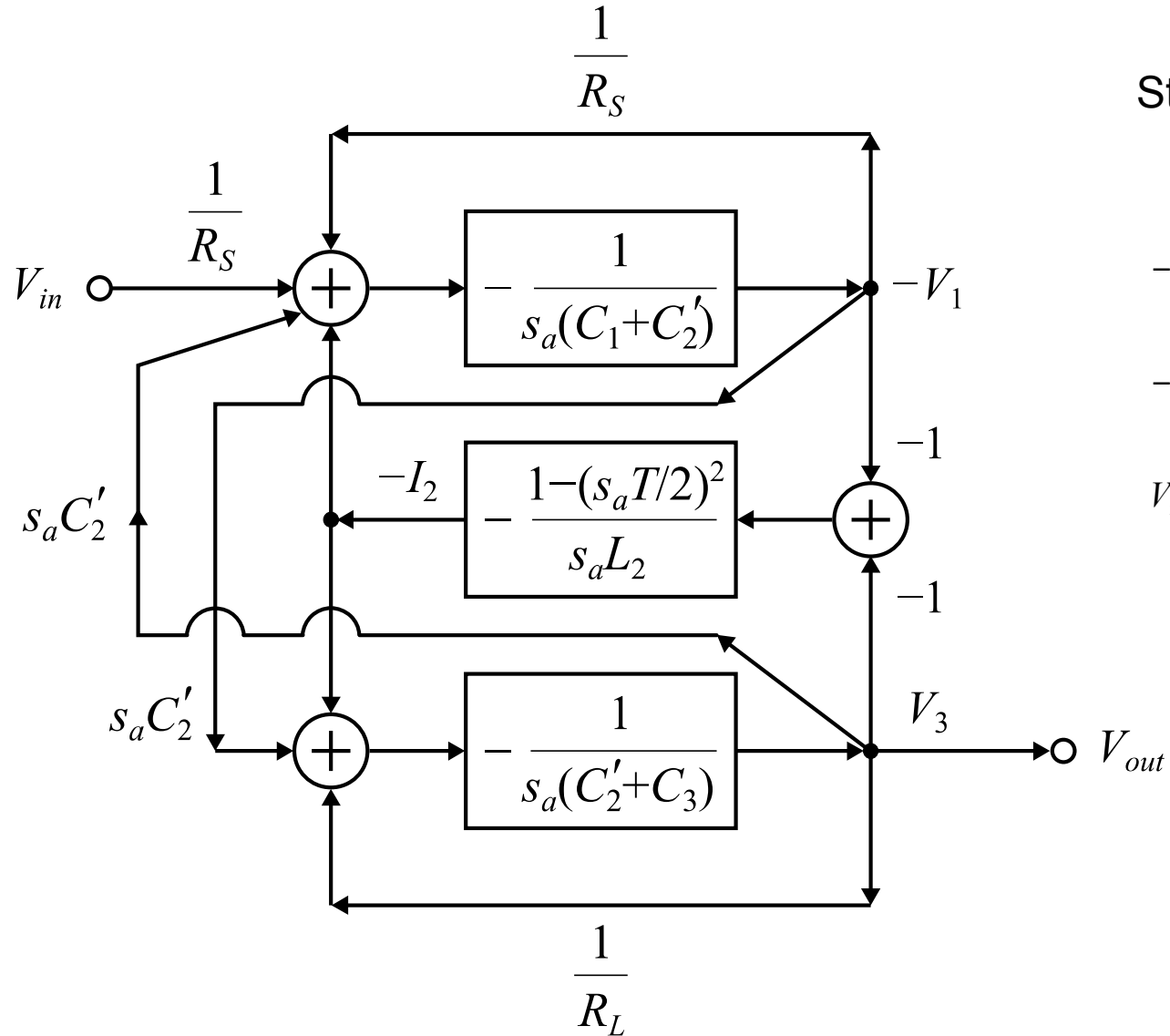
$$-V_1 = \frac{-1}{s_a(C'_1 + C'_2)} \left[\frac{V_{in} - V_1}{R_s} + sC'_2 V_3 - I_2 \right]$$

$$-I_2 = - \left[-s_a C_{L2} + \frac{1}{s_a L_2} \right] [V_1 - V_3] \triangleq - \frac{1 - (s_a T/2)^2}{s_a L_2} [V_1 - V_3]$$

$$V_3 = \frac{-1}{s_a(C'_2 + C_3)} \left[-s_a C'_2 V_1 - I_2 + \frac{V_3}{R_L} \right]$$



Signal-flow graph



State equations:

$$-V_1 = \frac{-1}{s_a(C_1 + C_2')} \left[\frac{V_{in} - V_1}{R_s} + sC_2'V_3 - I_2 \right]$$

$$-I_2 = - \left[-s_a|C_{L2}| + \frac{1}{s_a L_2} \right] [V_1 - V_3] \triangleq - \frac{1 - (s_a T/2)^2}{s_a L_2} [V_1 - V_3]$$

$$V_3 = \frac{-1}{s_a(C_2' + C_3)} \left[-s_a C_2' V_1 - I_2 + \frac{V_3}{R_L} \right]$$

Implementation of the middle segment:

$$Q(s_a) = \frac{-I_2}{s_a} = -\frac{1 - (s_a T/2)^2}{s_a^2 L_2} [V_1 - V_3]$$

Bilinear transformation $\Rightarrow$

$$Q(z) = -\frac{1 - \left[\frac{z-1}{z+1}\right]^2}{\frac{4L_2}{T^2} \left[\frac{z-1}{z+1}\right]^2} [V_1(z) - V_3(z)] = -\frac{z4C_{L2}}{(z-1)^2} [V_1(z) - V_3(z)]$$

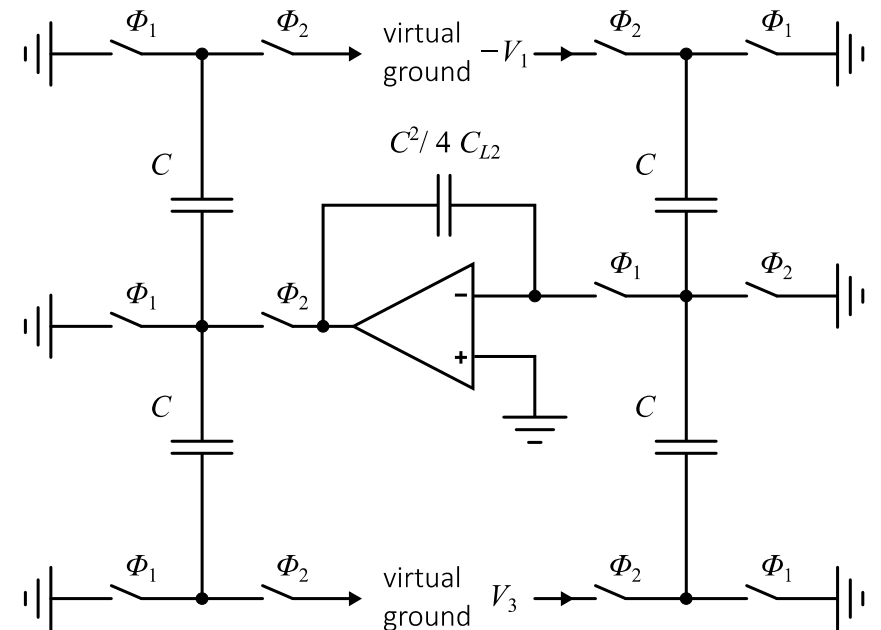
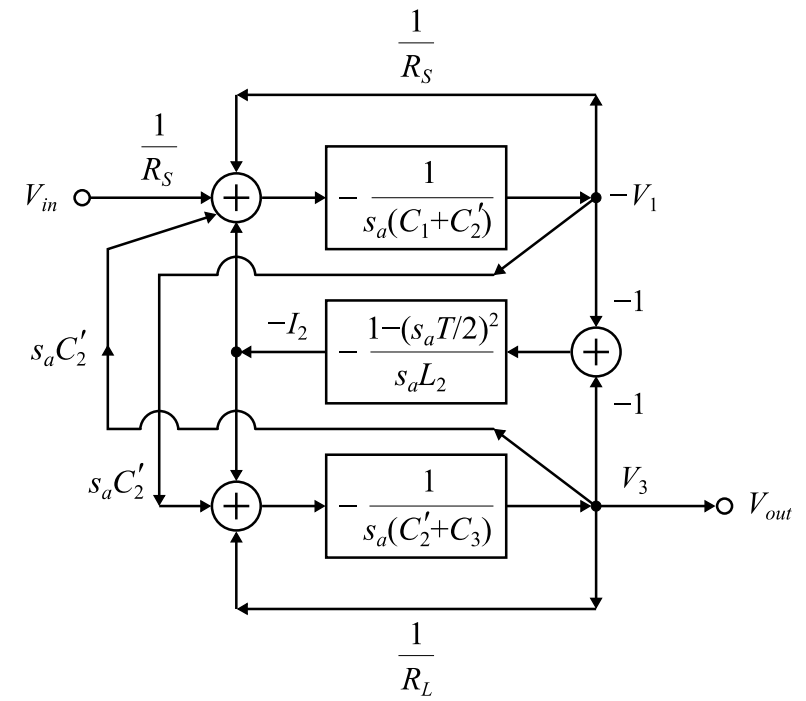
$$\Rightarrow (1 - z^{-1})Q(z) = \frac{z^{-1}4C_{L2}}{1 - z^{-1}} (V_3 - V_1)$$

Notice!

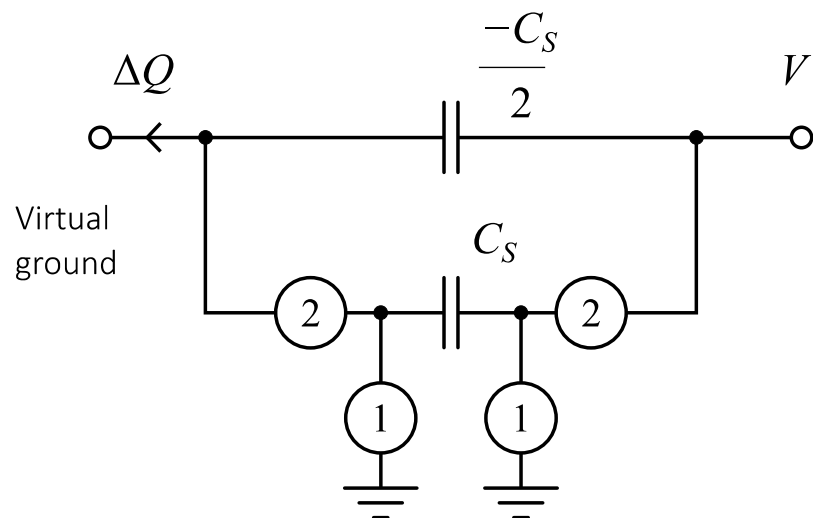
$$(1 - z^{-1})Q(z) \triangleq q(t_n) - q(t_{n-1}) = \Delta q$$

$$\Rightarrow (1 - z^{-1})Q(z) = C \left[\frac{-4C_{L2}/C^2}{1 - z^{-1}} \underbrace{(Cz^{-1}[V_3 - V_1])}_{\hat{Q}} \right]$$

$\uparrow$
 $Q = VC \quad V = \frac{Q}{C}$



Feedback loops $1/R_S$ and $1/R_L$

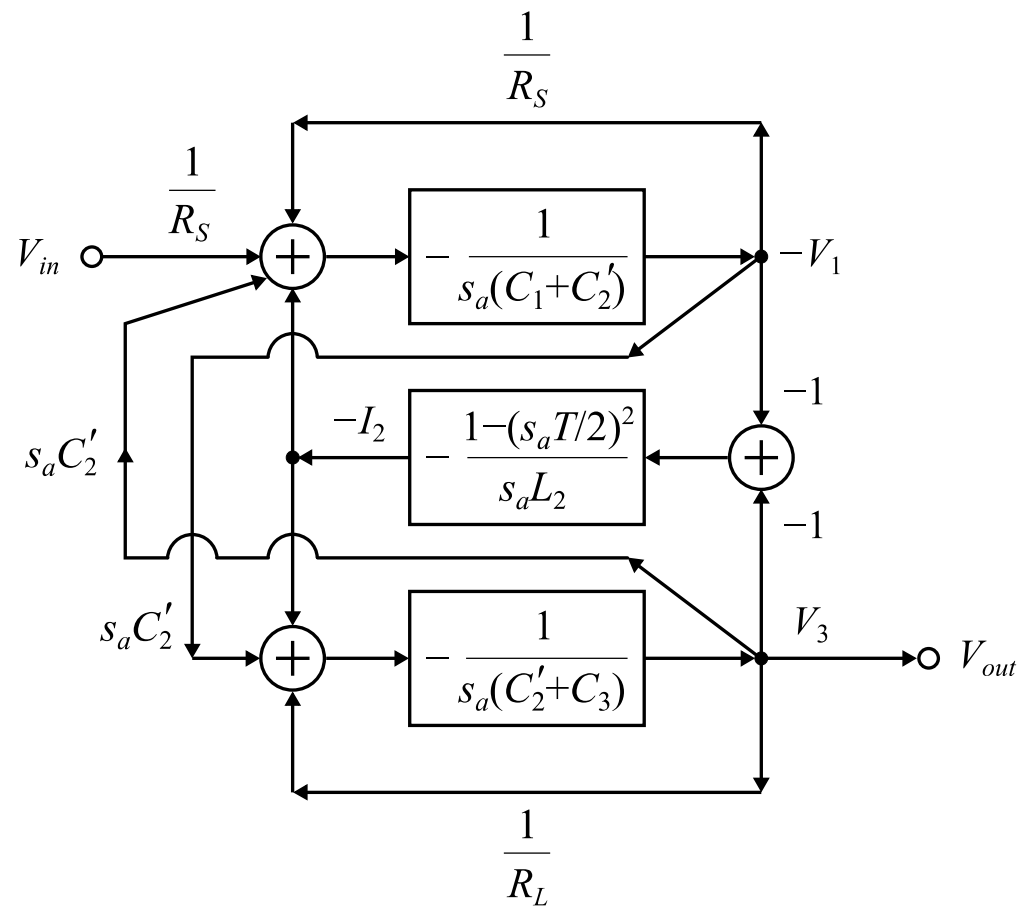


$$\frac{\Delta Q}{V} = -\frac{C_S}{2}(1 - z^{-1}) + C_S = \frac{C_S}{2}(1 + z^{-1})$$

$$\Delta q = q(t_n) - q(t_{n-1})$$

$$\Delta Q = (1 - z^{-1})Q(z) = \frac{C_S}{2}(1 - z^{-1})V(z)$$

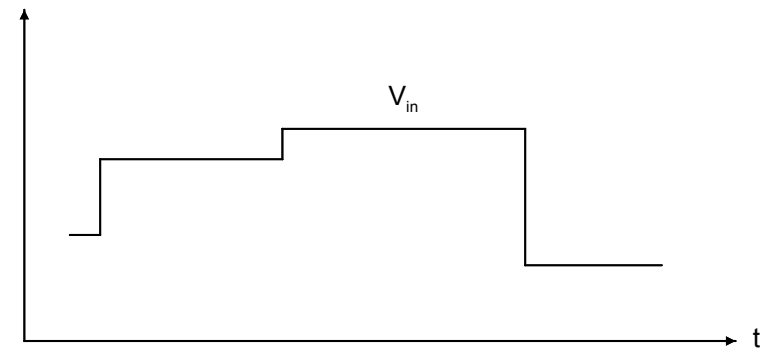
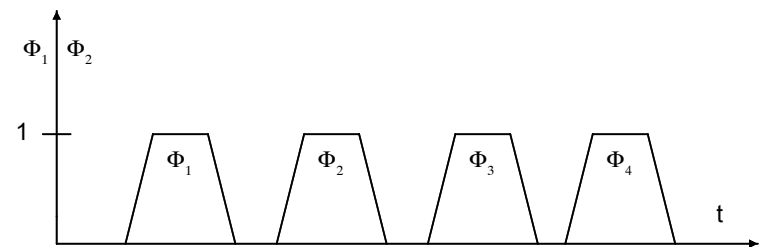
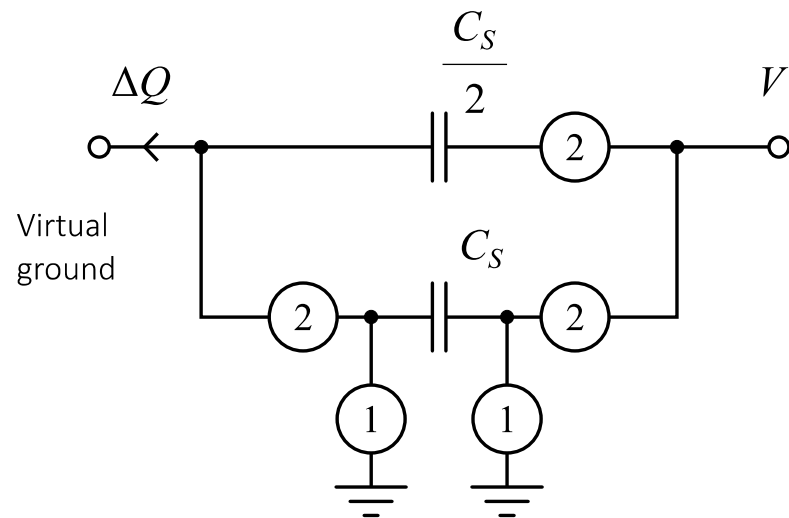
$$C_S = \frac{T}{R_S} \quad Q(z) = \frac{T}{2R_S} \frac{1 + z^{-1}}{1 - z^{-1}} V(z)$$



Reverse transformation (bilinear)

$$Q(s) = \frac{V(s)}{sR_s}$$

Optional implementation of feedback resistor with bilinear transformation:



The movement of the charge in the input branch:

$$I_{IN} = \frac{dQ_{IN}}{dt} = \frac{V_{IN}}{R_S} \Rightarrow Q_{IN}(s_a) = \frac{V_{IN}}{s_a R_S}$$

Bilinear transformation:

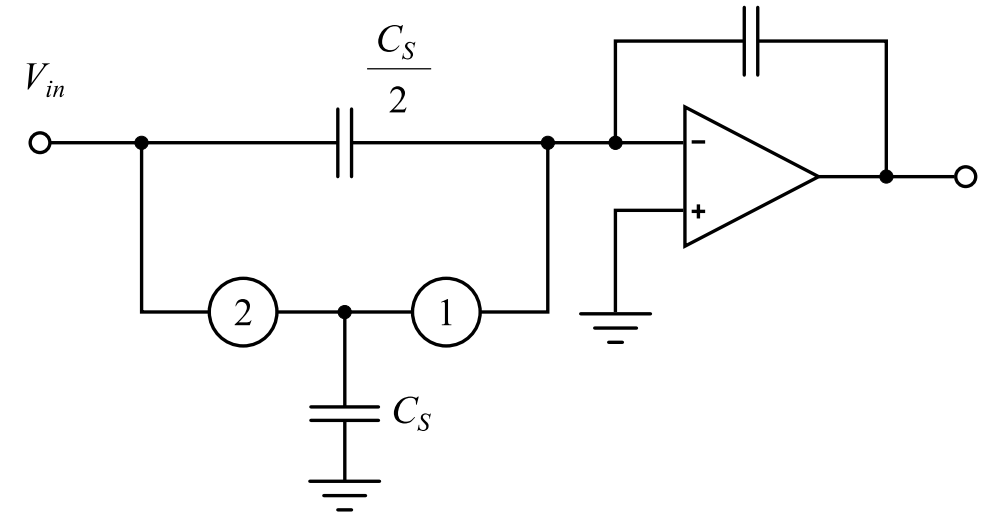
$$Q_{IN}(z) = \frac{T}{2} \frac{z+1}{z-1} \frac{V_{IN}(z)}{R_S}$$

$$(1 - z^{-1})Q_{IN}(z) = \frac{T}{2R_S} (1 + z^{-1})V_{IN}(z)$$

In time-domain:

$$\Delta q_{IN} = q_{IN}(t_n) - q_{IN}(t_{n-1}) = \frac{C_S}{2} [v_{IN}(t_n) + v_{IN}(t_{n-1})]$$

$$= \frac{C_S}{2} [v_{IN}(t_n) - v_{IN}(t_{n-1})] + C_S v_{IN}(t_{n-1})$$



- transfer function zeros $s_a C_2'$:

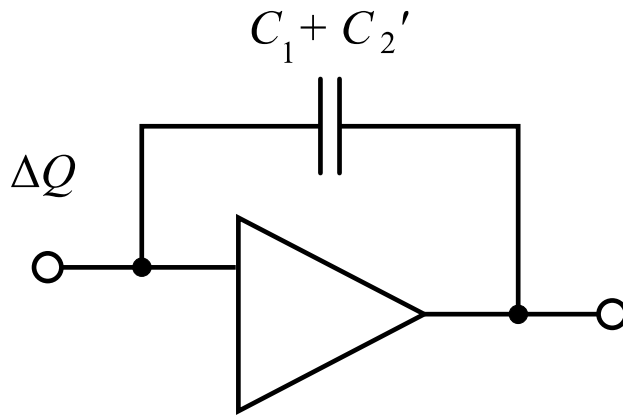
$$\frac{Q}{V} = C_2'$$

Implementation with non-switched capacitors C_2'

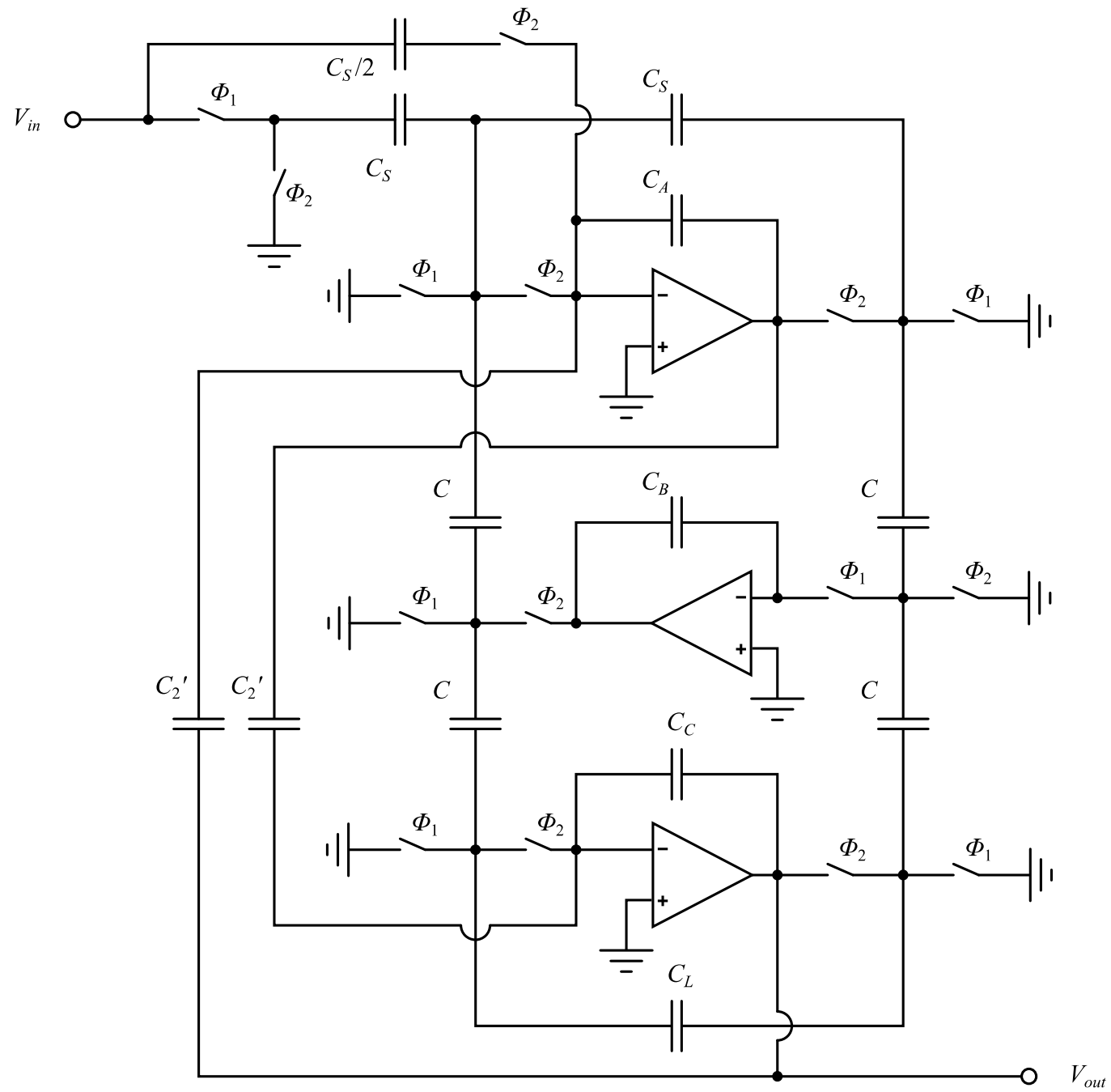
- current to voltage conversion

$$-\frac{1}{C_1 + C_2'}$$

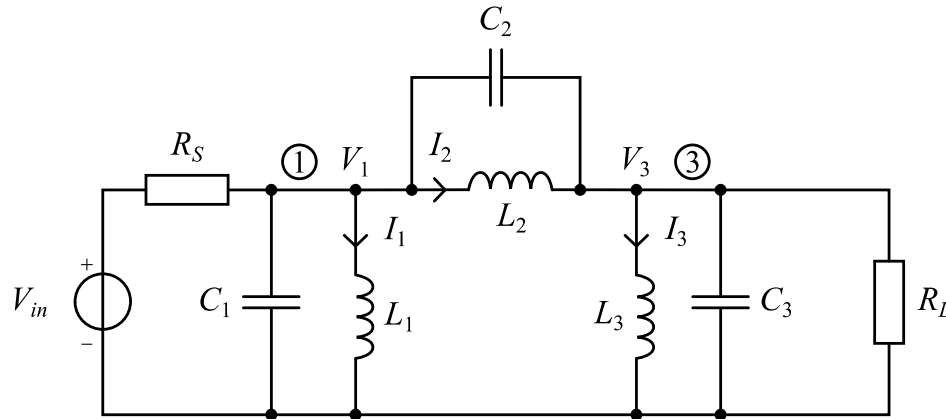
Implementation with integrator (amplifier + feedback capacitor)



$$V_1 = \frac{\Delta Q}{C_1 + C_2'}$$



Band-pass filters



State equations:

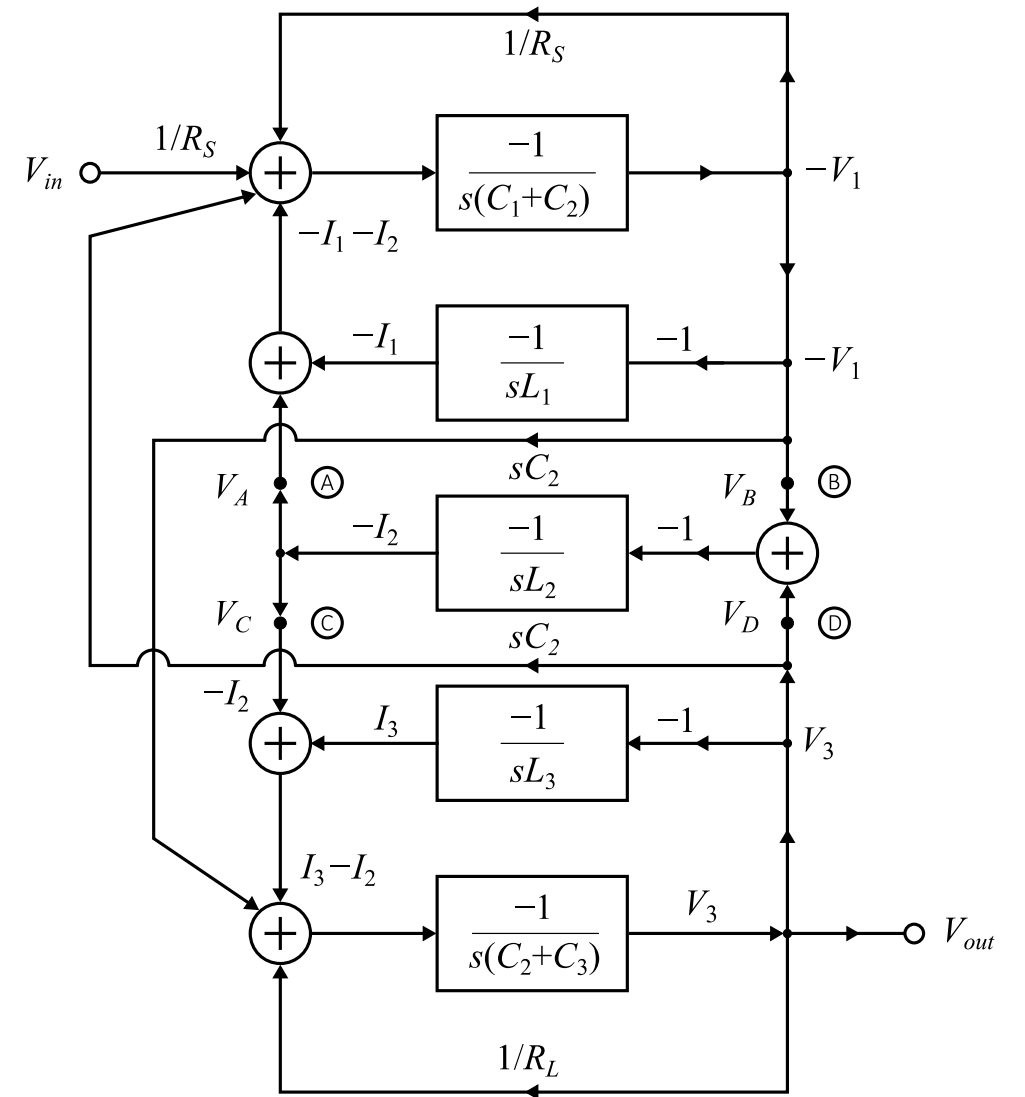
$$-V_1 = \frac{-1}{s(C_1 + C_2)} \left[\frac{V_{in} - V_1}{R_S} + sC_2V_3 - I_1 - I_2 \right]$$

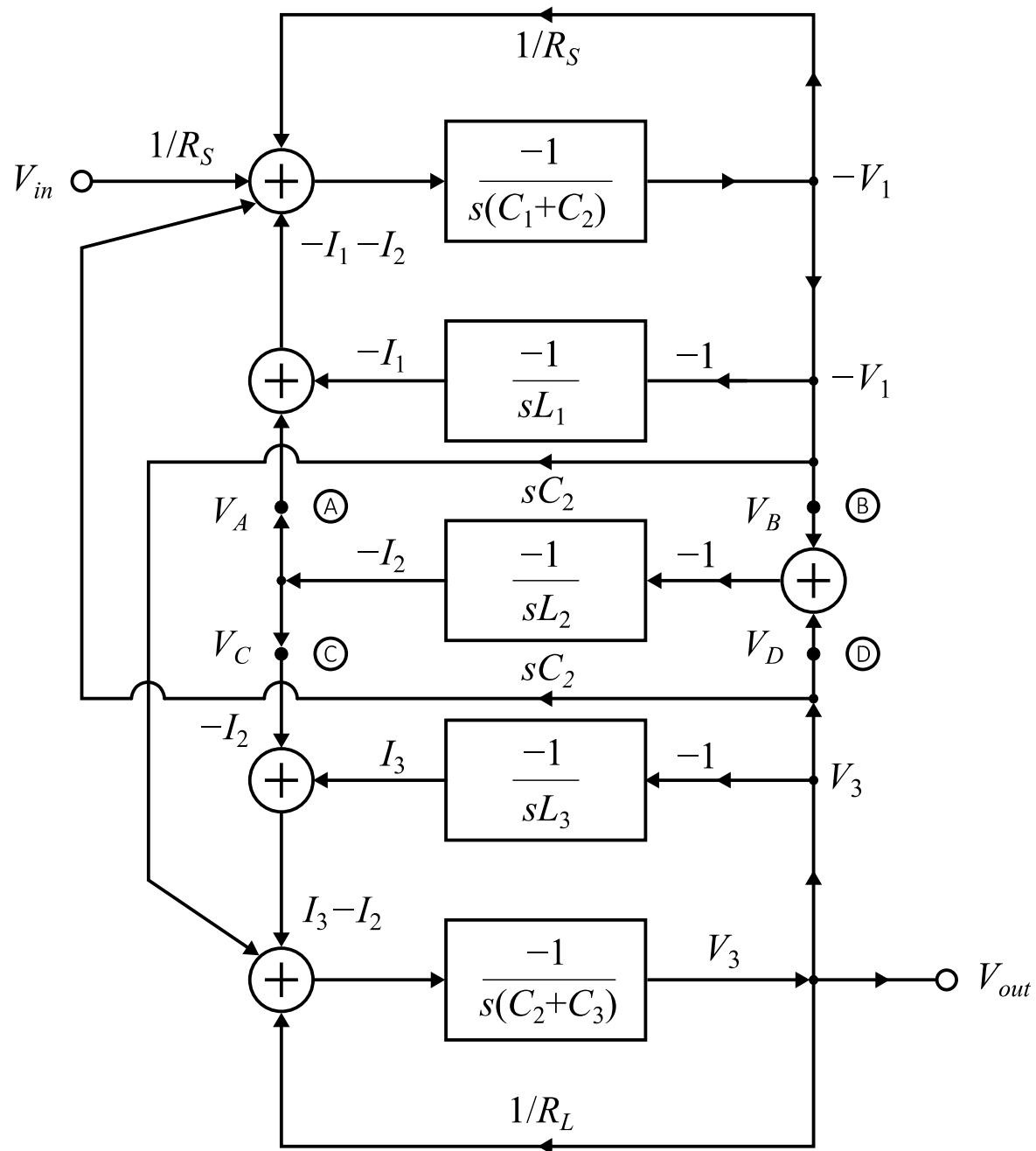
$$-I_1 = \frac{-V_1}{sL_1}$$

$$-I_2 = \frac{-V_1 + V_3}{sL_2}$$

$$-I_3 = \frac{-V_3}{sL_3}$$

$$V_3 = \frac{-1}{s(C_2 + C_3)} \left[-sC_2V_1 + I_3 - I_2 + \frac{V_3}{R_L} \right]$$





Offset-voltages cause a problem of DC-instability!

$$H_{AB} = \frac{V_B}{V_A} = \frac{-sL_1}{s^2L_1(C_1 + C_2) + \frac{sL_1}{R_S} + 1}$$

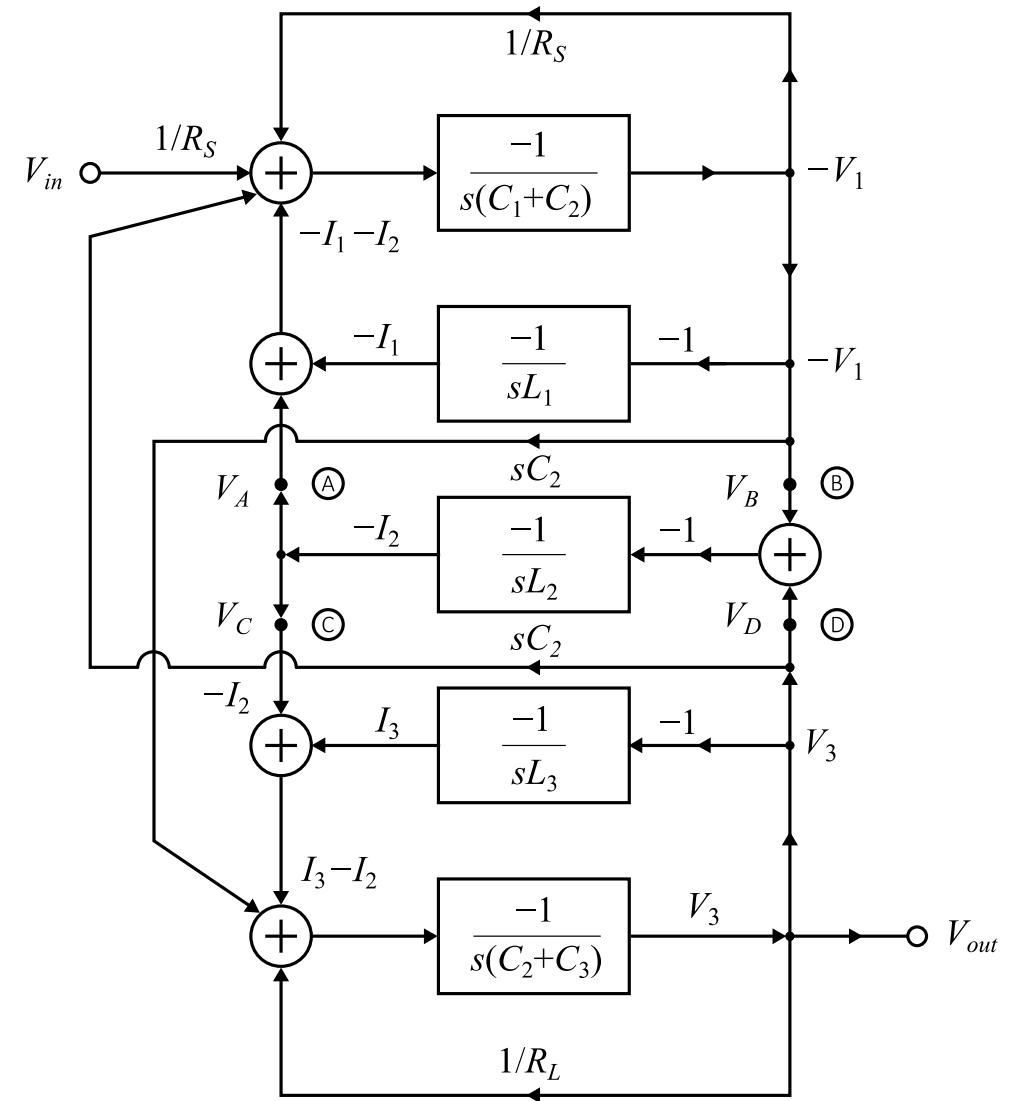
$$H_{CD} = \frac{V_C}{V_D} = \frac{-sL_3}{s^2L_3(C_2 + C_3) + \frac{sL_3}{R_L} + 1}$$

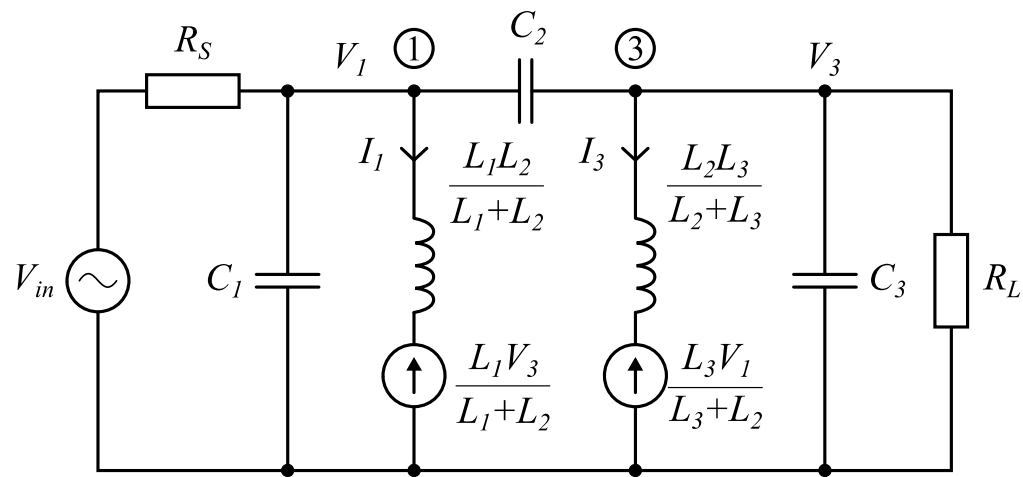
nyt $s \rightarrow 0 \Rightarrow H_{AB} \rightarrow 0$ ja $H_{CD} \rightarrow 0$

$\Rightarrow -\frac{1}{sL_2}$ - integrator has no DC - feedback!!

$\Rightarrow$ saturates!!

Also $\frac{1}{sL_1}$ - and $\frac{1}{sL_3}$ - integrators saturate!





Current equation of node ①

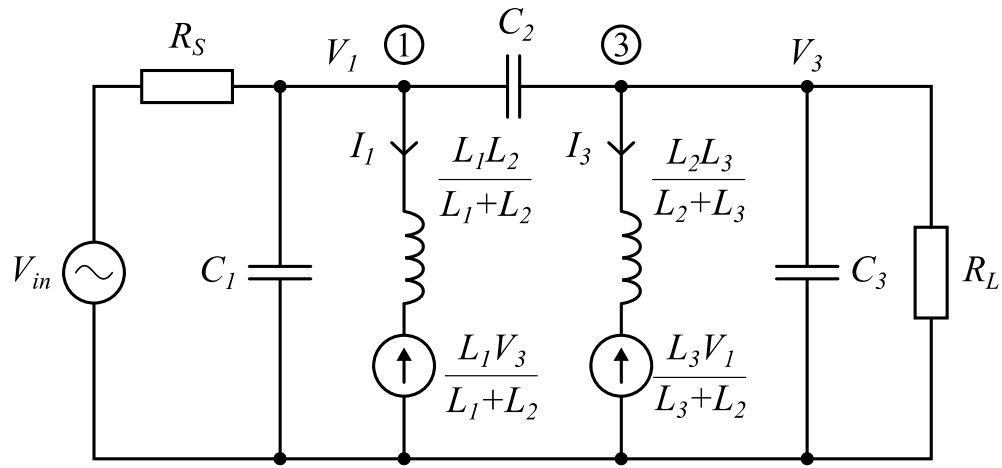
$$0 = \frac{V_{in} - V_1}{R_S} - sC_1 V_1 - sC_2 (V_1 - V_3) - \frac{1}{\frac{sL_1 L_2}{L_1 + L_2}} \left(V_1 - \frac{L_1 V_3}{L_1 + L_2} \right)$$

$$= \dots - \left[\frac{1}{\frac{sL_1 L_2}{L_1 + L_2}} V_1 - \frac{1}{sL_2} V_3 \right]$$

$$= \dots - \left[\left(\frac{1}{sL_2} + \frac{1}{sL_1} \right) V_1 - \frac{1}{sL_2} V_3 \right]$$

$$= \dots - \left[\frac{1}{sL_1} V_1 + \frac{1}{sL_2} (V_1 - V_3) \right]$$

$$0 = \frac{V_{in} - V_1}{R_S} - sC_1 V_1 - \frac{V_1}{sL_1} - sC_2 (V_1 - V_3) - \frac{1}{sL_2} (V_1 - V_3)$$



new state equations:

$$-V_1 = \frac{-1}{s(C_1 + C_2)} \left[\frac{V_{in} - V_1}{R_S} + sC_2 V_3 - I_{(1)} \right],$$

$$-I_{(1)} = -(I_1 + I_2) = \frac{-1}{sL_{12}} \left[V_1 - \frac{L_1 V_3}{L_1 + L_2} \right],$$

$$V_3 = \frac{-1}{s(C_2 + C_3)} \left[-sC_2 V_1 - I_{(3)} + \frac{V_3}{R_L} \right],$$

$$-I_{(3)} = I_3 - I_2 = \frac{-1}{sL_{32}} \left[-V_3 + \frac{L_3 V_1}{L_3 + L_2} \right],$$

where $L_{12} = L_1 \parallel L_2 = \frac{L_1 L_2}{L_1 + L_2}$.

$$L_{32} = L_3 \parallel L_2$$

