1. *Write RATE equations for the population of the upper and lower energy levels and the photon density in a situation, in which two-level atoms are excited from the lower level to the upper level by a mechanism having a constant transition probability. The energy of the system would increase continuously, if it is not assumed that part of the photons exit the system. Assume constant probability for the emission of a photon in a time unit. Calculate the photon density in steady state.*

First we write the equations for the changes in level populations and photon density, i.e., so called RATE equations:

$$\begin{cases} \dfrac{dN_1}{dt} = A_{21}N_2 + B_{21}N_2\varphi - B_{12}N_1\varphi - CN_1 = A_{21}N_2 + B\left(N_2 - N_1\right)\varphi - CN_1 = -\dfrac{dN_2}{dt} \\ \dfrac{d\varphi}{dt} = A_{21}N_2 + B\left(N_2 - N_1\right)\varphi - D\varphi \end{cases}.$$

At steady state the time derivatives are zero so we can solve photon density: $A_{21}N_2 + B\left(N_2 - N_1\right)\varphi - D\varphi = 0 \quad \Rightarrow \quad \varphi = \dfrac{A_{21}N_2}{B\left(N_1 - N_2\right) + D}$.

By inserting the solution to the first equation: $A_{21}N_2 + B\left(N_2 - N_1\right)\dfrac{A_{21}N_2}{B\left(N_1 - N_2\right) + D} - CN_1 = 0 \;\; \Rightarrow$

$$\frac{A_{21}N_2\left(B\left(N_1 - N_2\right) + D\right)}{B\left(N_1 - N_2\right) + D} + \frac{B\left(N_2 - N_1\right)A_{21}N_2}{B\left(N_1 - N_2\right) + D} - \frac{CN_1\left(B\left(N_1 - N_2\right) + D\right)}{B\left(N_1 - N_2\right) + D} = 0 \quad \Rightarrow$$

$$A_{21}N_2D - CN_1D - CN_1BN_1 + CN_1BN_2 = 0 \quad \Rightarrow \quad N_2 = \frac{CDN_1 + CBN_1^2}{A_{21}D + CBN_1} = \frac{CD + CBN_1}{A_{21}D + CBN_1}N_1 .$$

By inserting this back to the photon density solution $\Rightarrow \quad \varphi = \dfrac{A_{21}N_2}{B\left(N_1 - N_2\right) + D} =$

$$\frac{\dfrac{A_{21}\left(D + BN_1\right)N_1}{A_{21}D/C + BN_1}}{BN_1\left(1 - \dfrac{D + BN_1}{A_{21}D/C + BN_1}\right) + D} = \frac{A_{21}\left(D + BN_1\right)N_1}{BN_1\left(A_{21}D/C + BN_1 - D - BN_1\right) + D\left(A_{21}D/C + BN_1\right)} =$$

$$\frac{CA_{21}DN_1 + CA_{21}BN_1N_1}{DA_{21}D + BN_1A_{21}D} = \frac{CN_1\left(D + BN_1\right)}{D\left(D + BN_1\right)} = \frac{C}{D}N_1 .$$

2. *a) Calculate the photon lifetime in a Fabry-Perot cavity if absorption and other losses are negligible. Use reflectivities of R_1 and R_2 for the mirror facets. b) What is the average time a photon created in the middle of cavity spends in the cavity before emission? c) Calculate the same as in b, but with $R_1 = R_2 = R$.*

a) When the electromagnetic wave has progressed through the cavity two times and returned to its starting location, the wave intensity is given by $I = I_0 R_1 R_2$. In other words the lost intensity is $I_0 - I_0 R_1 R_2$. This has taken a time of $t = 2L/v = 2Ln_r/c$, so that

$$\frac{dI}{dt} = -\frac{I_o}{\tau_{ph}} \quad \Rightarrow \quad -\frac{I_o}{\tau_{ph}} = \frac{I_0 R_1 R_2 - I_0}{2Ln_r/c} \quad \Rightarrow \quad \tau_{ph} = \frac{2Ln_r}{c(1-R_1R_2)}.$$

b) We calculate the average distance the photon travels before emission

$$\langle L_{ph} \rangle = \frac{1}{2}L + \frac{1}{2}\left[R_1 L + R_1 R_2 L + R_1^2 R_2 L + R_1^2 R_2^2 L + \quad\right] +$$

$$+\frac{1}{2}\left[R_2 L + R_2 R_1 L + R_2^2 R_1 L + R_2^2 R_1^2 L + \quad\right] =$$

$$= \frac{1}{2}L + L\left[R_1R_2 + (R_1R_2)^2 + \quad\right] + \frac{1}{2}L(R_1+R_2)\left[1 + R_1R_2 + (R_1R_2)^2 + \quad\right] =$$

$$= -\frac{1}{2}L + L\left[1 + R_1R_2 + (R_1R_2)^2 + \quad\right] + \frac{1}{2}L(R_1+R_2)\left[1 + R_1R_2 + (R_1R_2)^2 + \quad\right] =$$

$$= -\frac{1}{2}L + \frac{1}{2}L\frac{2}{1-R_1R_2} + \frac{1}{2}L(R_1+R_2)\frac{1}{1-R_1R_2} = \frac{L}{2}\left\{-1 + \frac{2+R_1+R_2}{1-R_1R_2}\right\} =$$

$$= \frac{L}{2}\left\{\frac{1+R_1+R_2+R_1R_2}{1-R_1R_2}\right\}.$$

Because $L_{ph} = v \cdot t_{ph} = \frac{c}{n_r} \cdot t_{ph} \quad .\Rightarrow. \quad t_{ph} = \frac{n_r}{c}L_{ph} = \frac{n_r L}{2c}\left\{\frac{1+R_1+R_2+R_1R_2}{1-R_1R_2}\right\}.$

c) If $R_1 = R_2 = R$, one has $\langle L_{ph} \rangle = \frac{L}{2}\left\{\frac{1+2R+R^2}{1-R^2}\right\} = \frac{L}{2}\frac{(1+R)^2}{(1+R)(1-R)} = \frac{L}{2}\frac{(1+R)}{(1-R)}$

$$L_{ph} = v \cdot t_{ph} = \frac{c}{n_r} \cdot t_{ph} \quad \Rightarrow \quad t_{ph} = \frac{n_r}{c}L_{ph} = \frac{n_r L}{2c}\left(\frac{1+R}{1-R}\right).$$

3. *Calculate the separation of the longitudinal modes of a GaAs laser, when the length of the laser is 500 μm, wavelength is 860 nm and the refractive index of the active material is given by* $n = 4.18 - \lambda(\mu\mathrm{m}) \cdot 0.63$.

The separation between longitudinal modes $\delta\lambda = \frac{\lambda^2}{2ln_r}\left(1 - \frac{\lambda}{n_r}\frac{dn_r}{d\lambda}\right)^{-1}$, which is obtained via differentiation from the standing wave condition $m\lambda = 2n_r L$ ($m = 1,2,3,...$). Refractive index at the lasing wavelength is $n_r = 4.18 - \lambda(\mu\mathrm{m}) \cdot 0.63 = 3.638$ and the derivative $\frac{dn_r}{d\lambda(\mu\mathrm{m})} = -0.63$.

From here one gets $\delta\lambda = 1.77$ Å.

(Approximative formula where the refractive index is assumed to be constant gives $\delta\lambda_{appr} = \frac{\lambda^2}{2ln_r} = 2.03$ Å. Error of roughly 15 %.)

4. *The threshold currents for diode lasers having lengths of 250 μm and 500 μm are 1 kA/cm² and 0.75 kA/cm², respectively. The reflectivity of the mirror facets is 0.65, the cavity losses are 5.5 cm⁻¹ and the recombination time constant is 0.5 ns. The thickness of the active layer is 0.2 μm. Calculate the injection density corresponding to the transparency condition of the active material.*

The threshold current density for the shorter cavity laser is

$$J_{th,l} = J_{th,0} + C\left[\gamma + \frac{1}{l}\ln\frac{1}{R}\right] \quad , \quad l = 250\ \mu\mathrm{m}$$

and for the longer $J_{th,2l} = J_{th,0} + C\left[\gamma + \frac{1}{2l}\ln\frac{1}{R}\right]$.

Let us solve the constant C first:

$$J_{th,l} - J_{th,2l} = C\left(\frac{1}{l} - \frac{1}{2l}\right)\ln\frac{1}{R} = \frac{C}{2l}\ln\frac{1}{R} \quad \Rightarrow \quad C = \frac{2l\left(J_{th,l} - J_{th,2l}\right)}{\ln(1/R)} = 2902\ \frac{\mathrm{A}}{\mathrm{m}}.$$

Therefore $J_{th,0} = J_{th,l} - C\left[\gamma + \frac{1}{l}\ln\frac{1}{R}\right] = 340\ \frac{\mathrm{A}}{\mathrm{cm}^2}$.

From the transparency current density we get the injected carrier density at transparency:

$$J_{th,0} = \frac{qd}{\tau_r}n_{nom} \quad \Rightarrow \quad n_{nom} = \frac{\tau_r J_{th,0}}{qd} = 5.3 \cdot 10^{16}\ \frac{1}{\mathrm{cm}^3}.$$

5. *Threshold current density of a semiconductor laser is J_{th} = 1.35 kA cm^{-2}. A similar structure is used in a superluminescent LED by artificially raising the threshold current. What is the new threshold current density, if the current flows only through half of the area of the diode? Assume the relation of $J = 1\,\mathrm{kA/cm^2} + g\cdot 5\,\mathrm{A/cm}$ between current density and gain. Refractive index of the active material is 3.5, length of the laser component is 300 µm and the absorption coefficient of the active material at the operation wavelength is α = 3000 cm^{-1}.*

From the equation of J as a function of g we can calculate the threshold gain g_{th}: $J = 1\,\mathrm{kA/cm^2} + g\cdot 5\,\mathrm{A/cm} \quad\Rightarrow\quad g_{th} = \dfrac{J_{th} - 1\,\mathrm{kA/cm^2}}{5\,\mathrm{A/cm}} = 70\dfrac{1}{\mathrm{cm}}$.

Reflection coefficient of the laser mirrors is $R = \left(\dfrac{n_1 - n_2}{n_2 + n_1}\right)^2 = 0.309$.

Because at the threshold the gain must compensate for losses γ and the outgoing light intensity, it follows that $g_{th} = \gamma + \dfrac{1}{l}\ln\left(\dfrac{1}{R}\right)$. Now we can calculate the loss coefficient $\gamma = g_{th} - \dfrac{1}{l}\ln\left(\dfrac{1}{R}\right) = 30.9\dfrac{1}{\mathrm{cm}}$.

When current goes through only half the length of the cavity, $L = l/2$, and the new threshold condition is

$$I = I_0 R^2 e^{-2l\gamma} e^{2Lg_{th}} e^{-2L\alpha} = I_0 \quad\Rightarrow\quad 2Lg_{th} - 2l\gamma - 2L\alpha = 2\ln\left(\frac{1}{R}\right).$$

Notice that the area where the current does not flow is absorbing.

The new threshold gain is

$$g_{th} = \alpha + \frac{1}{L}\left[l\gamma + \ln\left(\frac{1}{R}\right)\right] = \alpha + \frac{l}{L}\left[\gamma + \frac{1}{l}\ln\left(\frac{1}{R}\right)\right] = \alpha + 2\left[\gamma + \frac{1}{l}\ln\left(\frac{1}{R}\right)\right] = 3140\frac{1}{\mathrm{cm}}$$

and the new threshold current is

$J_{th} = 1\dfrac{\mathrm{kA}}{\mathrm{cm}^2} + g_{th}\cdot 5\dfrac{\mathrm{A}}{\mathrm{cm}} = 16.7\dfrac{\mathrm{kA}}{\mathrm{cm}^2}$, which is more than ten times the original J_{th}.