4 NON-LINEAR ANALYSIS

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on the topics of week 5:

- ☐ Large displacement elasticity theory, principle of virtual work

- ☐ Large displacement FEA for solid, thin slab, and bar models

- ☐ Non-linear element contributions of solid, thin slab, and bar models

SOURCES OF NON-LINEARITY

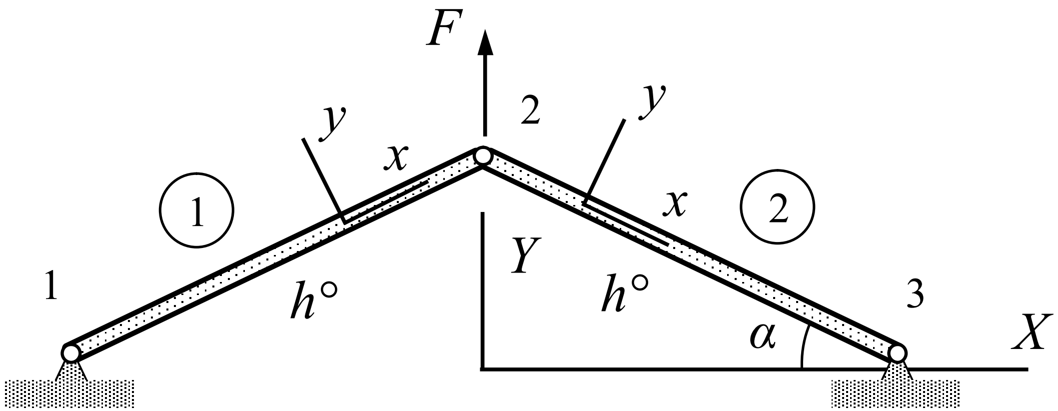
- **Geometry:** Equilibrium equations should be satisfied in deformed geometry depending on displacement. Strain measures of large displacements are always non-linear.

  the source to be considered here!

- **Material:** Constitutive equation $g(\sigma,u)=0$ may be non-linear. Near reference geometry, truncated Taylor series $g^{\circ}+(\partial g/\partial\sigma)^{\circ}\Delta\sigma+(\partial g/\partial u)^{\circ}\Delta u=0$ gives a useful approximation.

- **Contact and follower forces:** Constitutive equation of external forces may be non-linear. Even the simplest contact condition is one sided (inequality) and therefore non-linear.

EXAMPLE. Consider the structure of figure and determine the relationship between the vertical displacement of node 2 (positive upwards) and force F acting on node 2. Assume that the force-length relationship (small strains and large displacements) is given by $N = EA(h/h^{\circ}-1)$ (EA is constant and h° is the length when $N=0$).



Answer $\dfrac{F}{EA}-2(\sin\alpha+\mathrm{a})\dfrac{\sqrt{1+2\mathrm{a}\sin\alpha+\mathrm{a}^2}-1}{\sqrt{1+2\mathrm{a}\sin\alpha+\mathrm{a}^2}}=0$, where $\mathrm{a}=\dfrac{u_{Y2}}{h^{\circ}}$

- Strain definition should not induce stress under rigid body motion of motion of a bar. Strain measure $e = h / h^\circ - 1$, based on the relative length change, satisfies the criterion. At the deformed geometry, when displacement is u_{Y2},

$$h = \left| h^\circ \cos\alpha \vec{I} + (h^\circ \sin\alpha + u_{Y2})\vec{J} \right| = h^\circ \sqrt{1 + 2\mathrm{a}\sin\alpha + \mathrm{a}^2} \quad \Rightarrow$$

$$\delta h = \frac{\partial h}{\partial u_{Y2}} \delta u_{Y2} = \frac{\sin\alpha + \mathrm{a}}{\sqrt{1 + 2\mathrm{a}\sin\alpha + \mathrm{a}^2}} \delta u_{Y2} \Rightarrow$$

$$N = EA(\frac{h}{h^\circ} - 1) = EA(\sqrt{1 + 2\mathrm{a}\sin\alpha + \mathrm{a}^2} - 1), \text{ where } \mathrm{a} = \frac{u_{Y2}}{h^\circ}.$$

- Virtual work expressions of external and internal forces for one bar element, written at the deformed geometry with length h, are $\delta W^{\text{ext}} = F \delta u_{Y2}$ and $\delta W^{\text{int}} = -N \delta h$. As the structure consists of two bars (internal parts of the bars are same by symmetry), virtual work expression of the structure

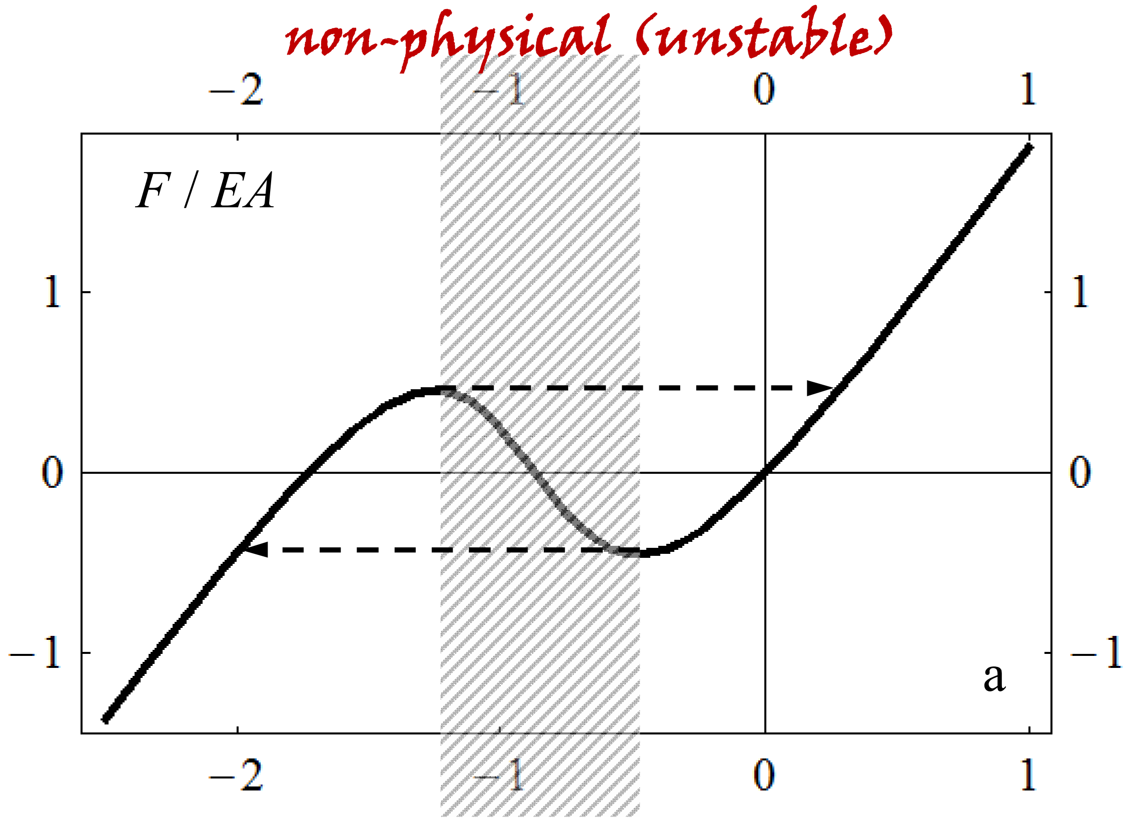
$$\delta W = [F - 2EA(\sin\alpha + \mathrm{a})\frac{\sqrt{1+2\mathrm{a}\sin\alpha + \mathrm{a}^2} - 1}{\sqrt{1+2\mathrm{a}\sin\alpha+\mathrm{a}^2}}]\delta u_{Y2}\,.$$

- Principle of virtual work and the fundamental lemma of variation calculus are valid also in large displacement analysis

$$F - 2EA(\sin\alpha + \mathrm{a})\frac{\sqrt{1+2\mathrm{a}\sin\alpha + \mathrm{a}^2} - 1}{\sqrt{1+2\mathrm{a}\sin\alpha+\mathrm{a}^2}} = 0\,. \quad \leftarrow$$

The remaining –mathematical problem – is to find a solution or solutions to the non-linear algebraic equation.

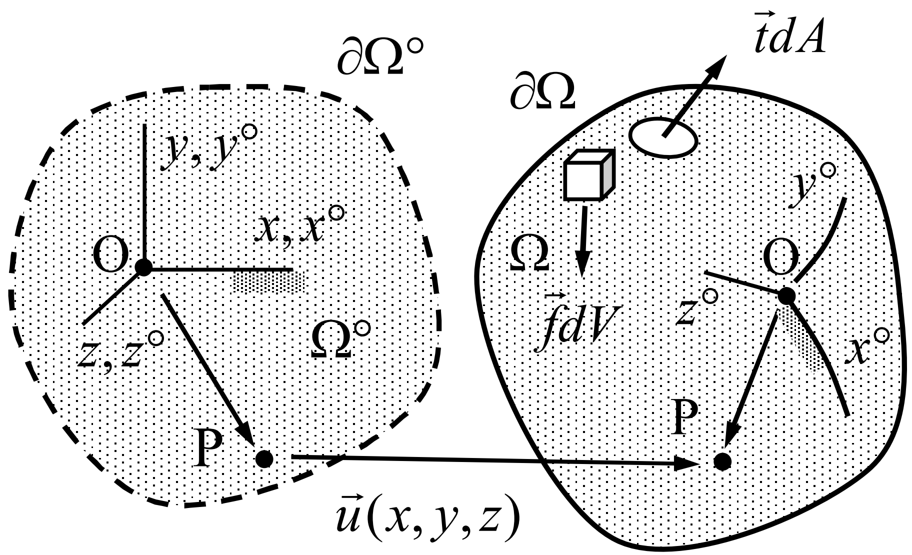
FORCE-DISPLACEMENT RELATIONSHIP $\alpha = \pi / 3$



Numerical method may give mathematically correct but non-physical solutions in non-linear elasticity. In practice, displacement (solution) may not depend continuously on the force (data).

4.1 NON-LINEAR ELASTICITY

Assuming equilibrium at the initial geometry Ω°, the aim is to find a new equilibrium on the deformed geometry Ω, when e.g. external forces acting on the structure are changed.



The local forms of the balance laws are concerned with the deformed domain which depends on the displacement! This brings a severe non-linearity into the boundary value problem for the displacement components.

STRAIN MEASURES

A rigid body motion should not induce strains! A proper strain measure with this respect is always non-linear in displacement components (small strain $|h - h^\circ| \ll h^\circ$)

Linear strain $\varepsilon = \dfrac{h}{h^\circ} - 1$ $\Rightarrow$ $2\vec{\vec{\varepsilon}} = \vec{\vec{F}}_\mathrm{c} + \vec{\vec{F}} - 2\vec{\vec{I}} = \nabla^\circ\vec{u} + (\nabla^\circ\vec{u})_\mathrm{c}$

Green-Lagrange $E = \dfrac{1}{2}[(\dfrac{h}{h^\circ})^2 - 1]$ $\Rightarrow$ $2\vec{\vec{E}} = \vec{\vec{F}}_\mathrm{c} \cdot \vec{\vec{F}} - \vec{\vec{I}} = \nabla^\circ\vec{u} + (\nabla^\circ\vec{u})_\mathrm{c} + \nabla^\circ\vec{u} \cdot (\nabla^\circ\vec{u})_\mathrm{c}$

Superscript $\circ$ refers to the initial geometry and subscript c denotes conjugate tensor. Deformation gradient $\vec{\vec{F}}_\mathrm{c} = \vec{\vec{I}} + \nabla^\circ\vec{u}$ is one of the key quantities of large displacement theory. At the initial geometry, material coordinate system is usually assumed to be Cartesian so that $\nabla^\circ = \vec{i}\partial / \partial x^\circ + \vec{j}\partial / \partial y^\circ + \vec{k}\partial / \partial z^\circ$.

GREEN-LAGRANGE STRAIN MEASURE

The Green-Lagrange strain has the components (in the basis of the initial geometry)

$$\begin{Bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial u_x / \partial x)^2 + (\partial u_y / \partial x)^2 + (\partial u_z / \partial x)^2 \\ (\partial u_x / \partial y)^2 + (\partial u_y / \partial y)^2 + (\partial u_z / \partial y)^2 \\ (\partial u_x / \partial z)^2 + (\partial u_y / \partial z)^2 + (\partial u_z / \partial z)^2 \end{Bmatrix},$$

$$\begin{Bmatrix} 2E_{xy} \\ 2E_{yz} \\ 2E_{zx} \end{Bmatrix} = \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} + \begin{Bmatrix} (\partial u_x / \partial x)(\partial u_x / \partial y) + (\partial u_y / \partial x)(\partial u_y / \partial y) + (\partial u_z / \partial x)(\partial u_z / \partial y) \\ (\partial u_x / \partial y)(\partial u_x / \partial z) + (\partial u_y / \partial y)(\partial u_y / \partial z) + (\partial u_z / \partial y)(\partial u_z / \partial z) \\ (\partial u_x / \partial z)(\partial u_x / \partial x) + (\partial u_y / \partial z)(\partial u_y / \partial x) + (\partial u_z / \partial z)(\partial u_z / \partial x) \end{Bmatrix}.$$

Green-Lagrange $\vec{\vec{E}}$ gives zero strain in a rigid body motion, whereas linear strain $\vec{\vec{\varepsilon}}$ does not. Linear strain $\vec{\vec{\varepsilon}}$ can be taken as an approximation to $\vec{\vec{E}}$ valid when strains and rotations of material elements are small!

- Green-Lagrange measure is based on comparing the *squares of lengths* of a material element at the initial and deformed geometries (mapping $\vec{r} = \vec{\aleph}(x^\circ, y^\circ, z^\circ, t)$ or $\vec{r} = \vec{r}^\circ + \vec{u}(x^\circ, y^\circ, z^\circ, t)$)

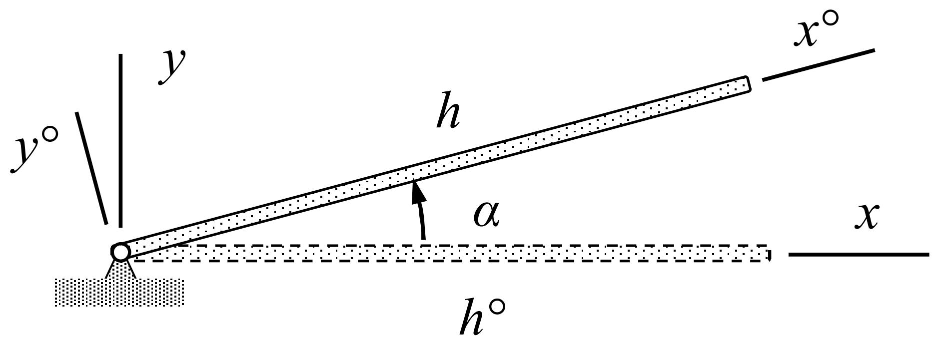
$$(d\vec{r}^\circ \cdot \nabla^\circ \vec{r}) \cdot (d\vec{r}^\circ \cdot \nabla^\circ \vec{r}) - d\vec{r}^\circ \cdot d\vec{r}^\circ = d\vec{r}^\circ \cdot 2\overleftrightarrow{E} \cdot d\vec{r}^\circ \quad \Rightarrow$$

$$\overleftrightarrow{E} = (\nabla^\circ \vec{r}) \cdot (\nabla^\circ \vec{r})_{\mathrm{c}} - \overleftrightarrow{I} = \frac{1}{2}(\overleftrightarrow{F}_{\mathrm{c}} \cdot \overleftrightarrow{F} - \overleftrightarrow{I}) \quad \text{or} \quad \overleftrightarrow{E} = \frac{1}{2}[\nabla^\circ \vec{u} + (\nabla^\circ \vec{u})_{\mathrm{c}} + \nabla^\circ \vec{u} \cdot (\nabla^\circ \vec{u})_{\mathrm{c}}].$$

- The components of the symmetric Green-Lagrange strain follow by substituting the component form of the displacement gradient into the definition. In conjugate $(\nabla \vec{u})_{\mathrm{c}}$, the component matrix is transposed.

EXAMPLE 4.1. Consider a bar whose left end is fixed and right end is free to move. Assuming that a particle of the bar is identified by its coordinates at the initial horizontal geometry and displacement is given by

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} (1+\varepsilon)\cos\alpha - 1 & -\sin\alpha \\ (1+\varepsilon)\sin\alpha & \cos\alpha - 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$



determine the linear strain component ε_{xx} and the Green-Lagrange strain component E_{xx}. The displacement describes rotation with angle α and length increase $\Delta h = \varepsilon h°$.

Answer $E_{xx} = \varepsilon + \frac{1}{2}\varepsilon^2 \approx \varepsilon$ when $|\varepsilon| \ll 1$ and $\varepsilon_{xx} = (1+\varepsilon)\cos\alpha - 1 \approx \varepsilon$ when $|\alpha| \ll 1$

- Partial derivatives of the displacement components are

$$\begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial x \end{Bmatrix} = \begin{Bmatrix} (1+\varepsilon)\cos\alpha - 1 \\ (1+\varepsilon)\sin\alpha \end{Bmatrix} \text{ and } \begin{Bmatrix} \partial u_x / \partial y \\ \partial u_y / \partial y \end{Bmatrix} = \begin{Bmatrix} -\sin\alpha \\ \cos\alpha - 1 \end{Bmatrix}.$$

- Linear and Green-Lagrange axial strain components

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = (1+\varepsilon)\cos\alpha - 1 \text{ and } E_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}(\frac{\partial u_x}{\partial x})^2 + \frac{1}{2}(\frac{\partial u_y}{\partial x})^2 = \varepsilon + \frac{1}{2}\varepsilon^2.$$ ←

The former depends strongly on the rotation angle even when ε is small although pure rotation should not cause any strains. The latter does not depend on the rotation at all. Also, for small length changes, the Green-Lagrange strain is close to the relative change of length $\varepsilon = \Delta h / h^{\circ}$.

KINEMATICS OF LARGE DISPLACEMENTS

Displacement .... $\vec{r} = \vec{r}^\circ + \vec{u}(x^\circ, y^\circ, z^\circ)$

Deformation gradient .... $\overleftrightarrow{F}_{\mathrm{c}} = \overleftrightarrow{I} + \nabla^\circ \vec{u}$

Green-Lagrange .... $2\overleftrightarrow{E} = \overleftrightarrow{F}_{\mathrm{c}} \cdot \overleftrightarrow{F} - \overleftrightarrow{I} = \nabla^\circ \vec{u} + (\nabla^\circ \vec{u})_{\mathrm{c}} + (\nabla^\circ \vec{u}) \cdot (\nabla^\circ \vec{u})_{\mathrm{c}}$

Variation .... $\delta \overleftrightarrow{E} = \overleftrightarrow{F}_{\mathrm{c}} \cdot \delta \overleftrightarrow{\varepsilon} \cdot \overleftrightarrow{F}$ where $2\overleftrightarrow{\varepsilon} = \nabla \vec{u} + (\nabla \vec{u})_{\mathrm{c}}$

Domain element .... $dV = J dV^\circ$

Jacobian .... $J = |\det[F]|$

Nanson .... $\vec{n} dA = J \overleftrightarrow{F}_{\mathrm{c}}^{-1} \cdot \vec{n}^\circ dA^\circ$ or $d\vec{A} = J \overleftrightarrow{F}_{\mathrm{c}}^{-1} \cdot d\vec{A}^\circ$

- Basis vectors of the curvilinear material coordinate system depend on the displacement (basis vectors may not be orthogonal and the lengths may differ from 1). The basis vectors define the edges of the unit cube of the initial geometry. In deformed geometry, the same material element is parallelepiped in shape with edges

$$\begin{Bmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{Bmatrix} = \begin{Bmatrix} \partial\vec{r} / \partial x^\circ \\ \partial\vec{r} / \partial y^\circ \\ \partial\vec{r} / \partial z^\circ \end{Bmatrix} = \begin{bmatrix} \partial x / \partial x^\circ & \partial y / \partial x^\circ & \partial z / \partial x^\circ \\ \partial x / \partial y^\circ & \partial y / \partial y^\circ & \partial z / \partial y^\circ \\ \partial x / \partial z^\circ & \partial y / \partial z^\circ & \partial z / \partial z^\circ \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F]^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$

$$\vec{\vec{F}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \text{ and } \vec{\vec{F}}_{\mathrm{c}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} [F]^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}.$$

- Relationship between the volume elements of domain Ω° of initial geometry and that of the final geometry Ω

$$dV = |(\vec{e}_x \times \vec{e}_y)\cdot\vec{e}_z (dx^\circ dy^\circ dy^\circ)| = J dV^\circ \quad \text{where} \quad J = \left|\det[F]\right| \,. \quad \Leftarrow$$

- The relationship between the variations of the linear and Green-Lagrange strains follows from the strain definitions and the relationship between derivatives in the fixed and material coordinate systems $\nabla^\circ = \overleftrightarrow{F}_\mathrm{c} \cdot \nabla$ (in terms of the basis vectors of the fixed Cartesian coordinate system)

$$\delta\overleftrightarrow{E} = \frac{1}{2}[\nabla^\circ\delta\vec{u} + (\nabla^\circ\delta\vec{u})_\mathrm{c} + \nabla^\circ\delta\vec{u}\cdot(\overleftrightarrow{F} - \overleftrightarrow{I}) + (\overleftrightarrow{F}_\mathrm{c} - \overleftrightarrow{I})\cdot(\nabla^\circ\delta\vec{u})_\mathrm{c}] \quad \Rightarrow$$

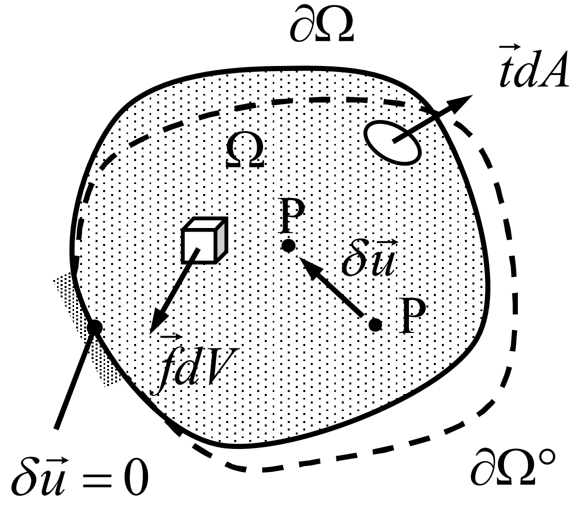
$$\delta\overleftrightarrow{E} = \frac{1}{2}(\nabla^\circ\delta\vec{u}\cdot\overleftrightarrow{F}) + \frac{1}{2}\overleftrightarrow{F}_\mathrm{c}\cdot(\nabla^\circ\delta\vec{u})_\mathrm{c} = \overleftrightarrow{F}_\mathrm{c}\cdot\delta\overleftrightarrow{\varepsilon}\cdot\overleftrightarrow{F}\,. \quad \Leftarrow$$

PRINCIPLE OF VIRTUAL WORK

Principle of virtual work $\delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \;\forall \delta\vec{u}$ is concerned with the deformed domain Ω. To avoid the complications due to a non-constant domain, the description of motion (a is used to express all quantities in the Cartesian $(x, y, z)-$system of the initial geometry:

$$\delta W^{\text{int}} = \int_{\Omega^\circ} -\left(\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ \delta E_{zz} \end{Bmatrix} + \begin{Bmatrix} S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 2\delta E_{xy} \\ 2\delta E_{yz} \\ 2\delta E_{zx} \end{Bmatrix} \right) dV^\circ$$

$$\delta W^{\text{ext}} = \int_{\Omega^\circ} \left(\begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \right) J dV^\circ + \ldots$$



The Green-Lagrange strain measure $\vec{\vec{E}}$ is non-linear. Also, the PK2 stress $\vec{\vec{S}}$ differs from the Cauchy (true) stress $\vec{\vec{\sigma}}$.

- The principle of virtual work $\delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \;\; \forall \delta \vec{u}$ holds at the equilibrium and therefore at the deformed geometry. In non-linear analysis, virtual work density of internal forces is expressed in terms of Green-Lagrange strain measure and PK2 stress with $\delta \vec{\vec{E}} = \vec{\vec{F}}_{\text{c}} \cdot \delta \vec{\vec{\varepsilon}} \cdot \vec{\vec{F}}$ and $dV = J dV^{\circ}$ (tensor identity $\vec{\vec{a}} : (\vec{\vec{b}}_{\text{c}} \cdot \vec{\vec{c}} \cdot \vec{\vec{b}}) = (\vec{\vec{b}} \cdot \vec{\vec{a}} \cdot \vec{\vec{b}}_{\text{c}}) : \vec{\vec{c}}$)

$$\delta W^{\text{int}} = -\int_{\Omega} (\vec{\vec{\sigma}} : \delta \vec{\vec{\varepsilon}}_{\text{c}}) dV = -\int_{\Omega^{\circ}} \vec{\vec{\sigma}} : (\vec{\vec{F}}_{\text{c}}^{-1} \cdot \delta \vec{\vec{E}}_{\text{c}} \cdot \vec{\vec{F}}^{-1}) J dV^{\circ} \quad \Rightarrow$$

$$\delta W^{\text{int}} = -\int_{\Omega^{\circ}} (\vec{\vec{F}}^{-1} \cdot \vec{\vec{\sigma}} \cdot \vec{\vec{F}}_{\text{c}}^{-1} J) : \delta \vec{\vec{E}}_{\text{c}}) dV^{\circ} = -\int_{\Omega^{\circ}} (\vec{\vec{S}} : \delta \vec{\vec{E}}_{\text{c}}) dV^{\circ}. \quad \Leftarrow$$

$$\delta W^{\text{ext}} = \int_{\Omega} (\vec{f} \cdot \delta \vec{u}) dV + ... = \int_{\Omega^{\circ}} (\vec{f} \cdot \delta \vec{u}) J dV^{\circ} + ... \quad \Leftarrow$$

The component forms follow by substituting the representations in the $(\vec{i}, \vec{j}, \vec{k})$ basis.

ELASTIC MATERIAL

Under the assumption of large displacement and small strains the Green-Lagrange strain measure does not differ much from the linear setting with small displacements and small strains. Constitutive equations

$$\begin{Bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \end{Bmatrix} = \frac{1}{C} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix} \text{ and } \begin{Bmatrix} 2E_{xy} \\ 2E_{yz} \\ 2E_{zx} \end{Bmatrix} = \frac{1}{G} \begin{Bmatrix} S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix},$$

with material parameters C (which replaces E), ν, and $G = C/(2+2\nu)$ are same as those of the linear case, are assumed to simplify the setting. Also, the uni-axial and two-axial (plane) stress and strain relationships follows just by using strains instead of engineering strains and C instead of E.

KINETICS OF LARGE DISPLACEMENTS

Piola-Kirchhoff 1............... $J\vec{\vec{\sigma}} = \vec{\vec{P}} \cdot \vec{\vec{F}}_{\mathrm{c}}$

Piola-Kirchhoff 2............... $J\vec{\vec{\sigma}} = \vec{\vec{F}} \cdot \vec{\vec{S}} \cdot \vec{\vec{F}}_{\mathrm{c}} \qquad (\vec{\vec{F}} \cdot \vec{\vec{S}} = \vec{\vec{P}})$

Force element .... $d\vec{F} = \vec{t}dA = \vec{n} \cdot \vec{\vec{\sigma}}dA = \vec{\vec{\sigma}}_{\mathrm{c}} \cdot \vec{n}dA = \vec{\vec{P}} \cdot \vec{n}^{\circ}dA^{\circ}$

Virtual work density .... $\delta w_{V^{\circ}}^{\mathrm{int}} = -\vec{\vec{S}} : \delta\vec{\vec{E}}_{\mathrm{c}} = -\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_{\mathrm{c}}J$

Elastic material.................. $\vec{\vec{S}} = \lambda \mathrm{tr}(\vec{\vec{E}})\vec{\vec{I}} + 2\mu\vec{\vec{E}}$

Analysis uses the PK2 stress concept. Cauchy (true) stress follows from the relationship between the quantities. In practice, the simple constitutive equation applies to isotropic material subjected to small strains (displacements may be large).

4.2 NON-LINEAR FEA

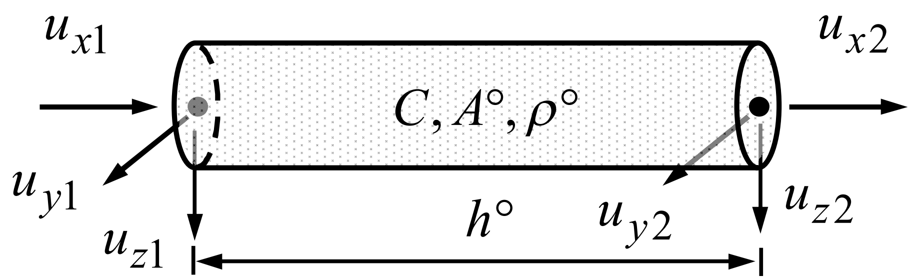
- Model the structure as a collection of beam, plate, etc. elements. Derive the element contributions δW^e and express the nodal displacement and rotation components of the material coordinate system in terms of those in the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get $\delta W = -\delta \mathbf{a}^{\mathrm{T}} \mathbf{F}(\mathbf{a})$

- Use the principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the system equations $\mathbf{F}(\mathbf{a}) = 0$. Find a physically meaningful solution by any of the standard numerical methods for non-linear algebraic equation systems.

BAR ELEMENT

Virtual work expression can be expressed in a concise form in terms of initial and deformed lengths of a bar element

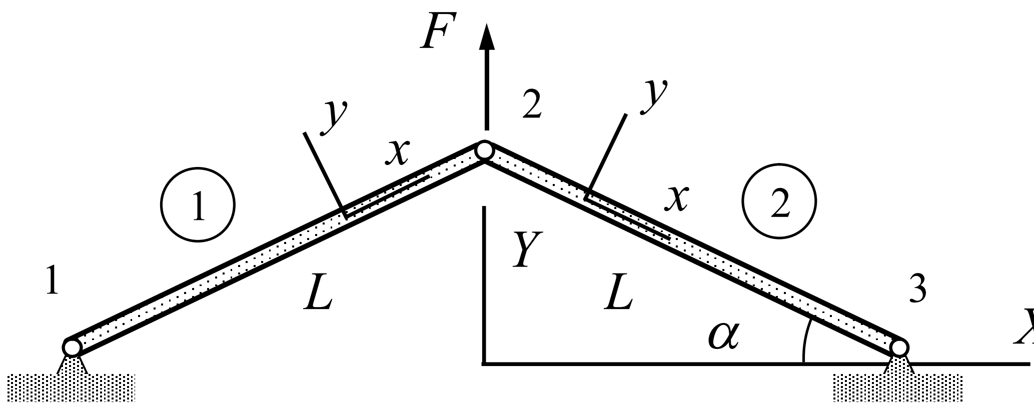


$$\delta W^{\text{int}} = -\delta h \frac{h}{h^\circ} C A^\circ \frac{1}{2}[(\frac{h}{h^\circ})^2 - 1],$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} g_x \delta u_{x1} + g_y \delta u_{y1} + g_z \delta u_{z1} \\ g_x \delta u_{x2} + g_y \delta u_{y2} + g_z \delta u_{z2} \end{Bmatrix}^{\text{T}} \frac{\rho^\circ A^\circ h^\circ}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

Length squared $h^2 = (h^\circ + u_{x2} - u_{x1})^2 + (u_{y2} - u_{y1})^2 + (u_{z2} - u_{z1})^2$ of the deformed element depends also on the nodal displaments in the $y-$ and $z-$directions. Transformation into the components of the structural system follows the lines of the linear theory.

EXAMPLE 4.2. Consider the bar structure shown subjected to large displacements. Determine the relationship between the vertical displacement of node 2 (positive upwards) and force F acting on node 2. Use the principle of virtual work and assume the constitutive equation $S_{xx} = CE_{xx}$, in which Green-Lagrange strain $E_{xx} = [(h/h^{\circ})^2 - 1]/2$ and C is constant. Cross-sectional area of the initial geometry is A°.



Answer $\dfrac{F}{CA^{\circ}} - 2(\sin\alpha + \mathrm{a})(\mathrm{a}\sin\alpha + \dfrac{1}{2}\mathrm{a}^2) = 0$ where $\mathrm{a} = \dfrac{u_{Y2}}{h^{\circ}}$

- In (geometrically) non-linear analysis, equilibrium equations are satisfied at the deformed geometry, although the mathematics is related with the initial geometry. Virtual work expressions of internal forces of the bar element and the point force are

$$\delta W^{\text{int}} = -\delta h \frac{h}{h^\circ} C A^\circ \frac{1}{2}[(\frac{h}{h^\circ})^2 - 1] \quad \text{and} \quad \delta W^{\text{ext}} = F \delta u_{Y2} .$$

- For element 1, the relationship between the displacement components in the material coordinate system are $u_{x2} = u_{Y2} \sin\alpha$ and $u_{y2} = u_{Y2} \cos\alpha$ giving

$$h^2 = (L + u_{Y2} \sin\alpha)^2 + (u_{Y2} \cos\alpha)^2 = L^2 + 2 u_{Y2} L \sin\alpha + u_{Y2}^2 ,$$

$$E_{xx} = \frac{1}{2}[(\frac{h}{L})^2 - 1] = \mathrm{a} \sin\alpha + \frac{1}{2}\mathrm{a}^2 \quad \Rightarrow \quad \delta E_{xx} = (\sin\alpha + \mathrm{a}) \delta \mathrm{a} \quad \text{where } \mathrm{a} = \frac{u_{Y2}}{L} .$$

- Then, virtual work expression of the structure becomes (the contribution for bar 2 is the same due to the symmetry).

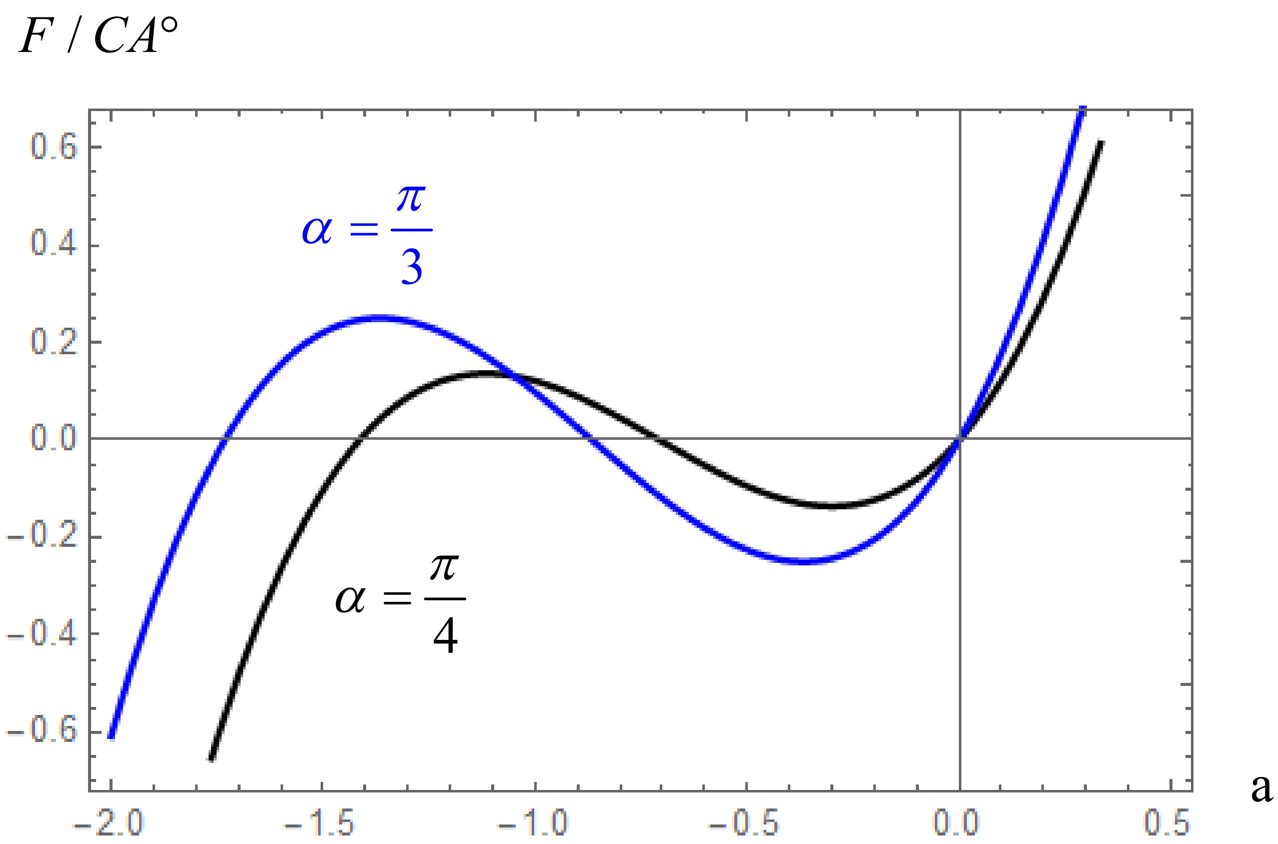
$$\delta W = -\delta u_{Y2}(\sin\alpha + \mathrm{a})2CA^{\circ}(\mathrm{a}\sin\alpha + \frac{1}{2}\mathrm{a}^2) + F\delta u_{Y2}.$$

- Principle of virtual work and the fundamental lemma of variation calculus give

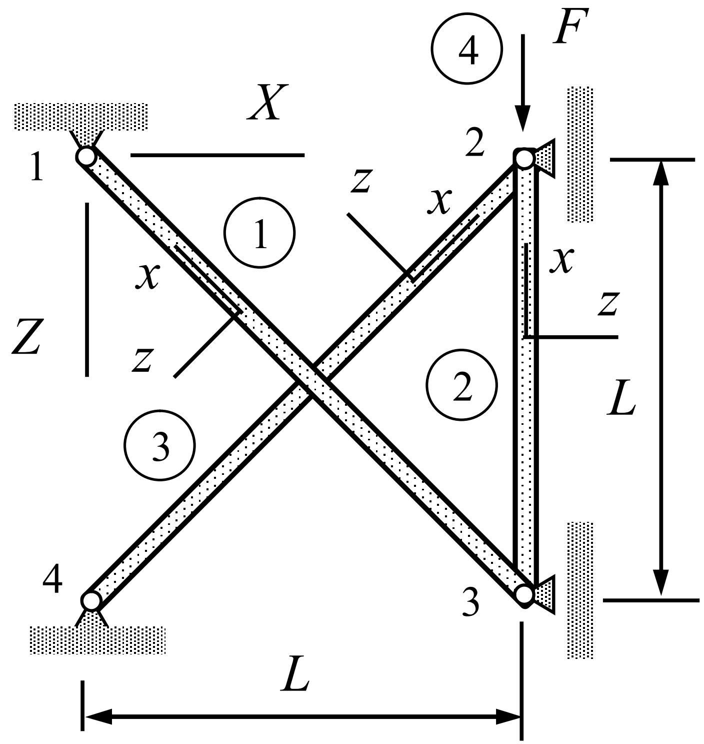
$$\delta W = [F - 2(\sin\alpha + \mathrm{a})CA^{\circ}(\mathrm{a}\sin\alpha + \frac{1}{2}\mathrm{a}^2)]\delta u_{Y2} = 0 \quad \Rightarrow$$

$$F - 2(\sin\alpha + \mathrm{a})CA^{\circ}(\mathrm{a}\sin\alpha + \frac{1}{2}\mathrm{a}^2) = 0. \quad \Leftarrow$$

FORCE-DISPLACEMENT RELATIONSHIP



EXAMPLE 4.3. Determine the nodal displacement u_{Z2} and u_{Z3} of the bar structure shown. Use non-linear bar elements and linear approximations. Cross-sectional areas and length of the initial geometry are $A = 0.01\,\mathrm{m}^2$ and $L = 1\,\mathrm{m}$. Elasticity parameter $C = 100\,\mathrm{Nm}^{-2}$ and external force $F = 0.05\,\mathrm{N}$.



Answer $u_{Z2} \approx 0.085\mathrm{m}$ and $u_{Z3} \approx 0.061\mathrm{m}$

- The physically correct solution is just one of the mathematically correct solutions to the nodal displacements (in this case the number of solutions is 6). The solver for non-linear analysis returns a real valued solution with the minimal norm. In the example, when $L = 1\,\mathrm{m}$, $A = 0.01\,\mathrm{m}^2$, $C = E = 100\,\mathrm{Nm}^{-2}$, and $F = 1/20\,\mathrm{N}$:

	type	properties	geometry
1	BAR	{{E}, {A, {0, 1, 0}}, {0, 0, 0}}	Line[{3, 1}]
2	BAR	{{E}, {A, {0, 1, 0}}, {0, 0, 0}}	Line[{3, 2}]
3	BAR	{{E}, {A, {0, 1, 0}}, {0, 0, 0}}	Line[{4, 2}]
4	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, 0, 0}
3	{L, 0, L}	{0, 0, uZ[3]}	{0, 0, 0}
4	{0, 0, L}	{0, 0, 0}	{0, 0, 0}

{uZ[2] → 0.0854082, uZ[3] → 0.0609567}

4.3 ELEMENT CONTRIBUTIONS

Virtual work expressions for the elements combine virtual work densities of the model and an approximation depending on the element shape and type. To derive the expression for an element:

- ☐ Start with the large displacement versions of the virtual work densities $\delta w_{\Omega^\circ}^{\text{int}}$ and $\delta w_{\Omega^\circ}^{\text{ext}}$ of the formulae collection.

- ☐ Represent the unknown functions by interpolation of the nodal displacement and rotations (see formulae collection). Substitute the approximations into the density expressions.

- ☐ Integrate the virtual work density over the domain occupied by the element at the initial geometry to get δW.

ELEMENT APPROXIMATION

In MEC-E8001 element approximation is a polynomial interpolant of the nodal displacements and rotations in terms of shape functions. In non-linear analysis, shape functions depend on *x, y,* and *z*.

Approximation $\mathbf{u}=\mathbf{N}^{\mathrm{T}}\mathbf{a}$ *always of the same form!*

Shape functions $\mathbf{N}=\{N_1(x,y,z)\;\; N_2(x,y,z)\;\; \ldots\;\; N_n(x,y,z)\}^{\mathrm{T}}$

Parameters $\mathbf{a}=\{\mathrm{a}_1\;\; \mathrm{a}_2\;\; \ldots\;\; \mathrm{a}_n\}^{\mathrm{T}}$

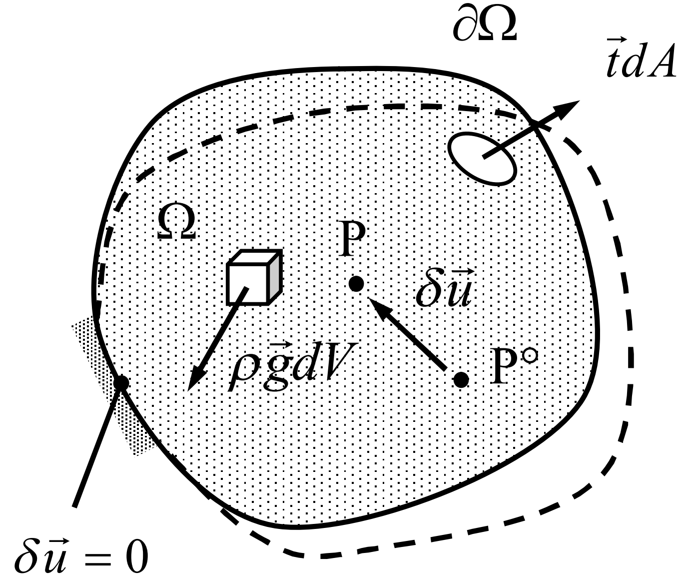
Nodal parameters $\mathrm{a}\in\{u_x,u_y,u_z,\theta_x,\theta_y,\theta_z\}$ may be just displacement or rotation components or a mixture of them (as with the beam model).

SOLID

The model does not contain kinetic or kinematic assumptions in addition to those of non-linear elasticity theory

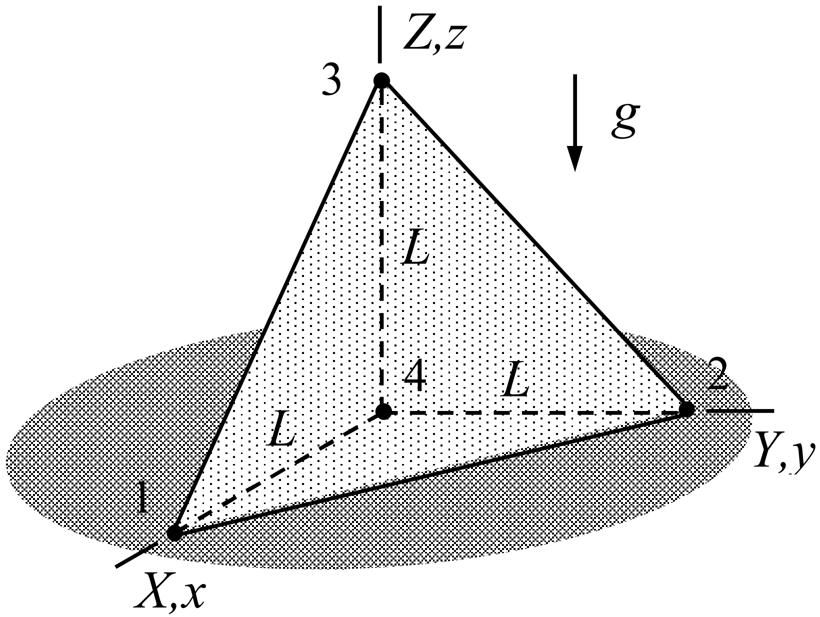
$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ \delta E_{zz} \end{Bmatrix}^{\mathrm{T}} [C] \begin{Bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \end{Bmatrix} - \begin{Bmatrix} 2\delta E_{xy} \\ 2\delta E_{yz} \\ 2\delta E_{zx} \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} 2E_{xy} \\ 2E_{yz} \\ 2E_{zx} \end{Bmatrix},$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \rho^\circ \begin{Bmatrix} g_x \\ g_y \\ g_z \end{Bmatrix}.$$



The solution domain can be represented, e.g., by tetrahedron elements with linear interpolation of the displacement components $u(x,y,z)$, $v(x,y,z)$, and $w(x,y,z)$

EXAMPLE 4.4. A tetrahedron of edge length L, density ρ, and elastic properties C and ν is subjected to its own weight on a horizontal floor. Determine the equilibrium equation for the displacement u_{Z3} of node 3 with one tetrahedron element and linear approximation. Assume that $u_{X3} = u_{Y3} = 0$ and that the bottom surface is fixed and that the geometry and density described is concerned with the initial geometry (gravity omitted).



Answer: $(1+\text{a})\text{a}(1+\frac{1}{2}\text{a})+F=0$ where

$$F=\frac{1}{4}\frac{1-\nu-2\nu^2}{1-\nu}\frac{\rho g L^3}{C} \quad \text{and} \quad \text{a}=\frac{u_{Z3}}{L}\,.$$

- Linear shape functions can be deduced directly from the figure $N_1 = x/L$, $N_2 = y/L$, $N_3 = z/L$, and $N_4 = 1 - x/L - y/L - z/L$. Only the shape function of node 3 is actually needed as the other nodes are fixed. Approximations to the displacement components are

$$u_x = u_y = 0 \text{ and } u_z = \frac{z}{L}u_{Z3}, \text{ giving } \frac{\partial u_z}{\partial x} = \frac{\partial u_z}{\partial y} = 0 \text{ and } \frac{\partial u_z}{\partial z} = \frac{1}{L}u_{Z3}.$$

- When the approximation is substituted there, the non-zero Green-Lagrange strain component takes the form

$$E_{zz} = \frac{1}{L}u_{Z3} + \frac{1}{2L^2}u_{Z3}^2 \quad \Rightarrow \delta E_{zz} = \frac{1}{L}\delta u_{Z3} + \frac{1}{L^2}u_{Z3}\delta u_{Z3}.$$

- Virtual work densities of the internal and external forces simplify to (we assume that the material is described by the constitutive equation of linear elasticity theory in which the Young's modulus E is replaced by elasticity parameter C)

$$\delta w_{\Omega^\circ}^{\text{int}} = -\delta E_{zz} S_{zz} = \frac{-C(1-\nu)}{(1+\nu)(1-2\nu)} \delta u_{Z3} (\frac{1}{L} + \frac{1}{L^2} u_{Z3})(\frac{1}{L} u_{Z3} + \frac{1}{2L^2} u_{Z3}^2),$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = -\delta u_z \rho g = -\frac{z}{L} \rho g \delta u_{Z3}.$$

- Virtual work expressions are obtained as integrals of densities over the volume occupied by the body at the initial geometry. With $\mathrm{a} = u_{Z3} / L$

$$\delta W^{\text{int}} = \int_{V^\circ} \delta w_{V^\circ}^{\text{int}} dV = \delta w_{V^\circ}^{\text{int}} \frac{L^3}{6} = -\frac{L^2}{6} \frac{1-\nu}{(1+\nu)(1-2\nu)} C \delta u_{Z3} (1+\mathrm{a})(\mathrm{a} + \frac{1}{2}\mathrm{a}^2),$$

$$\delta W^{\text{ext}} = \int_{V^\circ} \delta w_{V^\circ}^{\text{ext}} dV = -\frac{L^3}{24}\rho g \delta u_{Z3}.$$

- Finally, principle of virtual work $\delta W = 0$ with $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ implies the equilibrium equation

$$\frac{L^2}{6}\frac{C(1-\nu)}{(1+\nu)(1-2\nu)}(1+\mathrm{a})(\mathrm{a}+\frac{1}{2}\mathrm{a}^2)+\frac{L^3}{24}\rho g = 0.$$

- In terms of $F = \frac{1}{4}\frac{1-\nu-2\nu^2}{1-\nu}\frac{\rho g L}{C}$ the physically meaningful solution is given by

$$\mathrm{a} = \frac{1}{3^{1/3}\alpha} + \frac{\alpha}{3^{2/3}} - 1 \quad \text{where} \quad \alpha = (-9F + \sqrt{3}\sqrt{-1+27F^2})^{1/3}.$$

THIN SLAB

Virtual work densities of plate combine the thin-slab and plate bending modes. Assuming that the two modes de-couple and the bending mode can be omitted

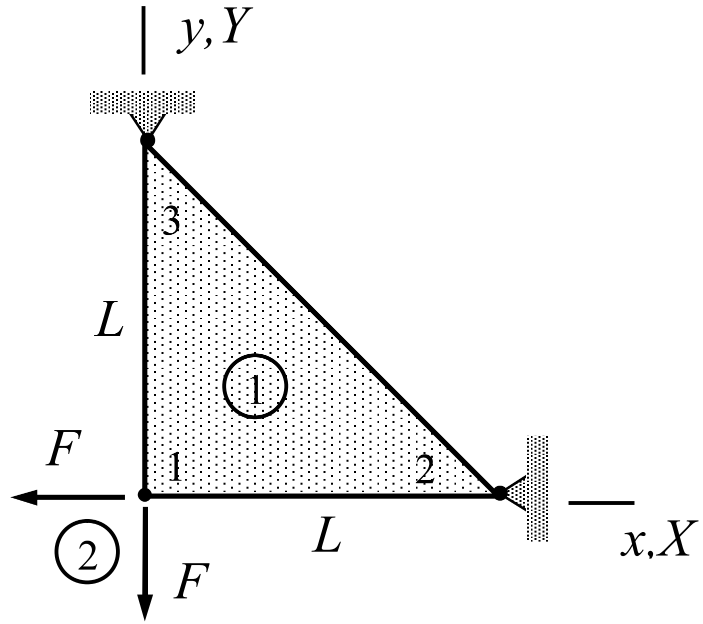
$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ 2\delta E_{xy} \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix}, \quad \delta w_{\Omega^\circ}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \rho^\circ t \begin{Bmatrix} g_x \\ g_y \end{Bmatrix} \quad \text{where}$$

$$\begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} + \begin{Bmatrix} (\partial u / \partial x)^2 / 2 + (\partial v / \partial x)^2 / 2 \\ (\partial u / \partial y)^2 / 2 + (\partial v / \partial y)^2 / 2 \\ (\partial u / \partial x)(\partial u / \partial y) + (\partial v / \partial x)(\partial v / \partial y) \end{Bmatrix}.$$

The planar solution domain Ω° (reference-plane of the initial geometry) can be represented by triangular or rectangular elements.

EXAMPLE 4.5. Consider the thin triangular structure shown. Assuming plane-stress conditions and $xy-$plane deformation, determine the equation for the displacement $u_{X1} = \mathrm{a}L$ and $u_{Y1} = \mathrm{a}L$ of node 1 according to the large displacement theory. Young's modulus E, Poisson's ratio ν, and thickness t are constants and distributed external force vanishes.

Answer: $(-1+2\mathrm{a})L\dfrac{tE}{1-\nu^2}\mathrm{a}(-1+\mathrm{a})-F=0$



- Nodes 2 are 3 are fixed and the non-zero displacement/rotation components are $u_{X1} = \mathrm{a}L$ and $u_{Y1} = \mathrm{a}L$. Linear shape functions $N_1 = (L - x - y)/L$, $N_2 = x/L$ and $N_3 = y/L$ are easy to deduce from the figure. Therefore $u = v = (L - x - y)\mathrm{a}$ and

$$\begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix}(-\mathrm{a} + \mathrm{a}^2), \begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ 2\delta E_{xy} \end{Bmatrix} = \delta \mathrm{a}(-1 + 2\mathrm{a})\begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix}.$$

- Virtual work density of internal forces simplifies to

$$\delta w_{\Omega^\circ}^{\mathrm{int}} = -\begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ 2\delta E_{xy} \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix} = -\delta \mathrm{a}(-1 + 2\mathrm{a})\frac{4tE}{1 - \nu^2}(-\mathrm{a} + \mathrm{a}^2).$$

- Integration over the triangular domain gives (integrand is constant)

$$\delta W^1 = -\delta a(-1+2a)L^2 \frac{2tE}{1-\nu^2}(-a+a^2).$$

- Virtual work expression for the point forces follows from the definition of work

$$\delta W^2 = -2\delta a L F \ .$$

- Principle of virtual work in the form $\delta W = \delta W^1 + \delta W^2 = 0 \;\; \forall \delta a$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta a L[(-1+2a)L\frac{2tE}{1-\nu^2}(-a+a^2) - 2F] = 0 \quad \forall \delta a \quad \Leftrightarrow$$

$$(-1+2a)L\frac{2tE}{1-\nu^2}(-a+a^2) - 2F = 0. \quad \Leftarrow$$

The point forces acting on a thin slab should be considered as "equivalent nodal forces" i.e. just representations of tractions acting on some part of the boundary. Under the action of an actual point force, displacement becomes non-bounded. In practice, numerical solution to the displacement at the point of action increases when the mesh is refined.

- In the Mathematica code of the course, the problem description is given by

	type	properties	geometry
1	PLANE	{{E, ν}, {t}, {0, 0, 0}}	Polygon[{1, 2, 3}]
2	FORCE	{-F, -F, 0}	Point[{1}]

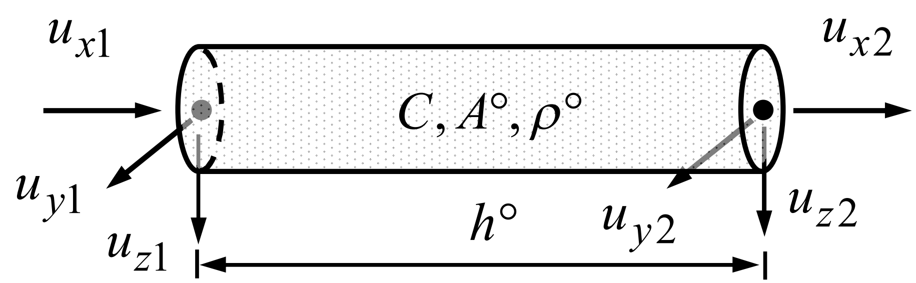
	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{L a[1], L a[1], 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, 0}	{0, 0, 0}
3	{0, L, 0}	{0, 0, 0}	{0, 0, 0}

$$\delta W = -(\delta a[1])^{T}\left(-\frac{2L\left(F - F\nu^{2} + L t \Xi a[1]\left(1 - 3a[1] + 2a[1]^{2}\right)\right)}{-1+\nu^{2}}\right)$$

BAR MODE

With the assumptions of the bar model $\vec{u} = u(x)\vec{i} + v(x)\vec{j} + w(x)\vec{k}$, $\overleftrightarrow{S} = S_{xx}\vec{i}\vec{i}$ etc. in the generic expressions for large displacement analysis for the solid model simplify to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\delta E_{xx} A^\circ C E_{xx},$$



$$\delta w_{\Omega^\circ}^{\text{ext}} = A^\circ \rho^\circ (\delta u g_x + \delta v g_y + \delta w g_z),$$

where $E_{xx} = \dfrac{du}{dx} + \dfrac{1}{2}(\dfrac{du}{dx})^2 + \dfrac{1}{2}(\dfrac{dv}{dx})^2 + \dfrac{1}{2}(\dfrac{dw}{dx})^2$.

In FEA, the solution domain (a line segment) is represented by line elements and the displacement components $u(x)$, $v(x)$, $w(x)$ by their interpolants.

- Let us start with the kinematical assumption $\vec{u} = u(x)\vec{i} + v(x)\vec{j} + w(x)\vec{k}$. The kinetic assumption is $\overleftrightarrow{S} = S_{xx}\vec{i}\vec{i}$. Green-Lagrange strain and its variation are

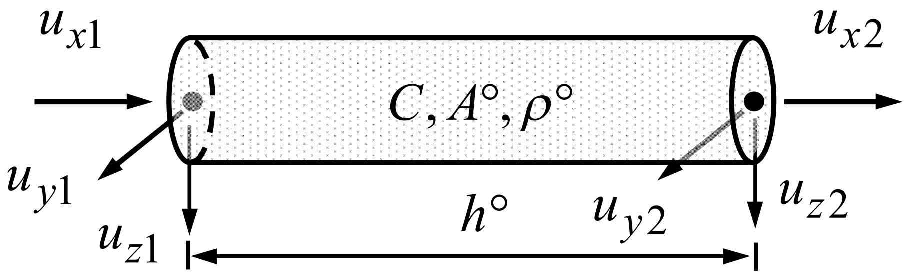
$$E_{xx} = \frac{du}{dx} + \frac{1}{2}(\frac{du}{dx})^2 + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2, \;\; \delta E_{xx} = \frac{d\delta u}{dx} + \frac{d\delta u}{dx}\frac{du}{dx} + \frac{d\delta v}{dx}\frac{dv}{dx} + \frac{d\delta w}{dx}\frac{dw}{dx}.$$

- Assuming the constitutive equation $S_{xx} = CE_{xx}$, virtual work densities of the internal and external forces per unit length of the initial domain become (expression is integrated over the cross section of the initial geometry)

$$\delta w_{\Omega^\circ}^{\text{int}} = -\delta E_{xx} A^\circ C E_{xx} \quad \text{and} \quad \delta w_{\Omega^\circ}^{\text{ext}} = A^\circ \rho^\circ (\delta u g_x + \delta v g_y + \delta w g_z). \quad \leftarrow$$

BAR MODE

Virtual work expression can be expressed in a concise form in terms of initial and deformed lengths of a bar element



$$\delta W^{\mathrm{int}} = -\delta h \frac{h}{h^\circ} C A^\circ \frac{1}{2}[(\frac{h}{h^\circ})^2 - 1],$$

$$\delta W^{\mathrm{ext}} = \begin{Bmatrix} g_x \delta u_{x1} + g_y \delta u_{y1} + g_z \delta u_{z1} \\ g_x \delta u_{x2} + g_y \delta u_{y2} + g_z \delta u_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho^\circ A^\circ h^\circ}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

Length squared $h^2 = (h^\circ + u_{x2} - u_{x1})^2 + (u_{y2} - u_{y1})^2 + (u_{z2} - u_{z1})^2$ of the deformed element depends also on the nodal displaments in the $y-$ and $z-$directions. Transformation into the components of the structural system follows the lines of the linear theory.

- Linear approximations to the displacement components give constant values to the derivatives du/dx, dv/dx, and dw/dx and the Green-Lagrange strain component E_{xx} is simply the relative difference in the squares of lengths:

$$E_{xx} = \frac{1}{2}\frac{h^2 - (h^\circ)^2}{(h^\circ)^2} = \frac{1}{2}[(\frac{h}{h^\circ})^2 - 1] \quad \text{and} \quad \delta E_{xx} = \frac{\delta h}{h^\circ}\frac{h}{h^\circ}.$$

- As virtual work density of internal forces is constant and the approximation linear Virtual works of internal and external forces become

$$\delta W^{\text{int}} = \delta w_{\Omega^\circ}^{\text{int}} h^\circ = -\delta h \frac{h}{h^\circ} CA^\circ \frac{1}{2}[(\frac{h}{h^\circ})^2 - 1], \quad \Leftarrow$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} g_x \delta u_{x1} + g_y \delta u_{y1} + g_z \delta u_{z1} \\ g_x \delta u_{x2} + g_y \delta u_{y2} + g_z \delta u_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{1}{2} \rho^\circ h^\circ A^\circ \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \quad \Leftarrow$$

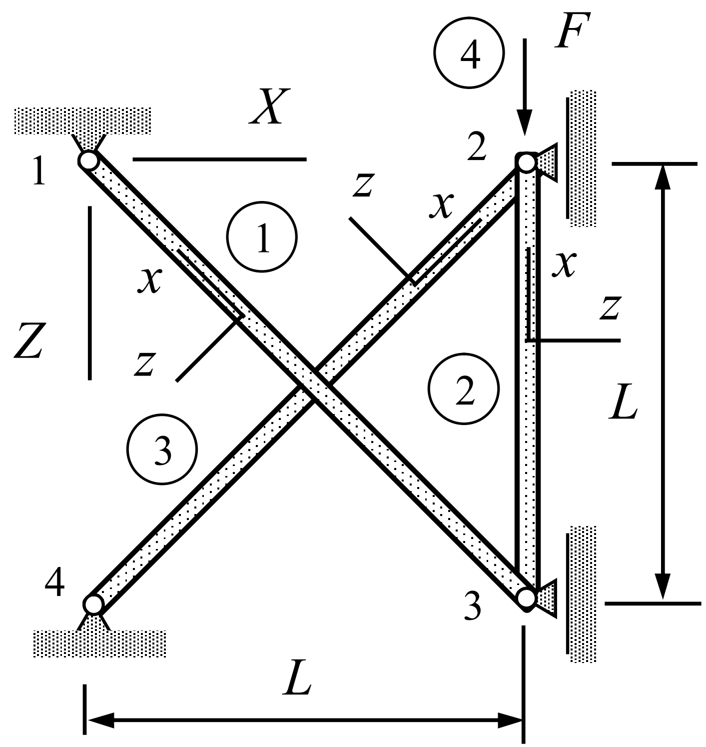
- It is noteworthy that PK2 does not represent the true stress in bar. The constitutive equation for the (true) axial force in terms of Green-Lagrange strain follows from the relationship between the Cauchy stress and PK2 stress. Here the relationship $J\vec{\sigma} = \vec{F} \cdot \vec{S} \cdot \vec{F}_{\mathrm{c}}$ simplifies to $J\sigma = FSF$ in which $J = V / V^{\circ} = hA / h^{\circ} A^{\circ}$ and $F = h / h^{\circ}$ giving

$$S = \frac{h^{\circ}}{h}\sigma\frac{h^{\circ}}{h}\frac{hA}{h^{\circ}A^{\circ}} = \frac{h^{\circ}}{hA^{\circ}}(\sigma A) = \frac{h^{\circ}}{hA^{\circ}}N \quad \Rightarrow \quad N = \frac{h}{h^{\circ}}A^{\circ}S = \frac{h}{h^{\circ}}A^{\circ}CE\,.$$

- Using the axial force N and the variation δh (at deformed geometry)

$$\delta W^{\mathrm{int}} = -N\delta h = -\delta h\frac{h}{h^{\circ}}CA^{\circ}\frac{1}{2}[(\frac{h}{h^{\circ}})^2 - 1] \quad \text{(the same as earlier).}$$

EXAMPLE 4.6. Write the virtual work expression of the structure shown in terms of the nodal displacement u_{Z2} and u_{Z3}. Use non-linear bar elements and linear approximations. Solve for the nodal displacement when the cross-sectional areas and material properties are $L = 1\text{m}$, $A^{\circ} = 1/100\,\text{m}^2$, $C = 100\,\text{Nm}^{-2}$ and $F = 1/20\,\text{N}$.



Answer $u_{Z2} = 0.085\text{m}$ and $u_{Z3} = 0.061\text{m}$

- For bar 1, the nodal displacement components of material coordinate system are $u_{x1} = u_{z1} = 0$, $u_{x2} = -u_{Z3}/\sqrt{2}$, and $u_{z1} = u_{Z3}/\sqrt{2}$. As approximations are linear, derivatives are

$$\frac{du}{dx} = (0 + \frac{u_{Z3}}{\sqrt{2}})\frac{1}{\sqrt{2}L} = \frac{u_{Z3}}{2L},\ \frac{dv}{dx} = 0,\quad \frac{dw}{dx} = (0 - \frac{u_{Z3}}{\sqrt{2}})\frac{1}{\sqrt{2}L} = -\frac{u_{Z3}}{2L}.$$

$$E_{xx} = \frac{1}{2}\frac{u_{Z3}}{L}(1 + \frac{1}{2}\frac{u_{Z3}}{L}) \quad \Rightarrow \quad \delta E_{xx} = \frac{1}{2}\frac{\delta u_{Z3}}{L}(1 + \frac{u_{Z3}}{L}).$$

When the approximations are substituted there, virtual work expression of internal forces simplifies to (density is constant)

$$\delta W^1 = -\delta u_{Z3}(1 + \frac{u_{Z3}}{L})\frac{CA^{\circ}}{4\sqrt{2}}\frac{u_{Z3}}{L}(2 + \frac{u_{Z3}}{L}).$$

- For bar 2, the nodal displacement components are $u_{x3}=-u_{Z3}$, $u_{x2}=-u_{Z2}$ and $u_{z1}=u_{z2}=0$. As approximations are linear, derivatives and the Green-Lagrange strains take the forms

$$\frac{du}{dx}=\frac{u_{x2}-u_{x3}}{L}=\frac{u_{Z3}-u_{Z2}}{L},\ \frac{dv}{dx}=0,\text{ and }\frac{dw}{dx}=0.$$

$$E_{xx}=\frac{u_{Z3}-u_{Z2}}{L}(1+\frac{1}{2}\frac{u_{Z3}-u_{Z2}}{L})\quad\Rightarrow\quad\delta E_{xx}=\frac{\delta u_{Z3}-\delta u_{Z2}}{L}(1+\frac{u_{Z3}-u_{Z2}}{L}).$$

When the approximations are substituted there, virtual work expression of internal forces simplifies to

$$\delta W^2=-(\delta u_{Z3}-\delta u_{Z2})(1+\frac{u_{Z3}-u_{Z2}}{L})CA^\circ\frac{u_{Z3}-u_{Z2}}{L}(1+\frac{1}{2}\frac{u_{Z3}-u_{Z2}}{L}).$$

- For bar 3, the nodal displacement components are $u_{x4}=u_{z4}=0$, $u_{x2}=-u_{Z2}/\sqrt{2}$, and $u_{z2}=-u_{Z2}/\sqrt{2}$. As approximations are linear, derivatives and the Green-Lagrange strain take the forms

$$\frac{du}{dx}=(-\frac{u_{Z2}}{\sqrt{2}})\frac{1}{\sqrt{2}L}=-\frac{u_{Z2}}{2L},\ \frac{dv}{dx}=0,\ \text{and}\ \frac{dw}{dx}=(-\frac{u_{Z2}}{\sqrt{2}})\frac{1}{\sqrt{2}L}=-\frac{u_{Z2}}{2L}.$$

$$E_{xx}=\frac{1}{2}\frac{u_{Z2}}{L}(-1+\frac{1}{2}\frac{u_{Z2}}{L})\quad\Rightarrow\quad\delta E_{xx}=\frac{1}{2}\frac{\delta u_{Z2}}{L}(-1+\frac{u_{Z2}}{L}).$$

When the approximations are substituted there, virtual work expression of internal forces simplifies to (density is constant)

$$\delta W^3=-\frac{CA^\circ}{4\sqrt{2}}\delta u_{Z2}(-1+\frac{u_{Z2}}{L})\frac{u_{Z2}}{L}(-2+\frac{u_{Z2}}{L}).$$

- Virtual work expression is sum of the element contributions. By taking into account also the point force contribution $\delta W^4 = \delta u_{Z2} F$

$$\delta W = -\delta u_{Z3}(1+\frac{u_{Z3}}{L})\frac{CA^\circ}{4\sqrt{2}}\frac{u_{Z3}}{L}(2+\frac{u_{Z3}}{L}) - (\delta u_{Z3}-\delta u_{Z2})(1+\frac{u_{Z3}-u_{Z2}}{L})\times$$

$$CA^\circ\frac{u_{Z3}-u_{Z2}}{L}(1+\frac{1}{2}\frac{u_{Z3}-u_{Z2}}{L}) - \delta u_{Z2}(-1+\frac{u_{Z2}}{L})\frac{CA^\circ}{4\sqrt{2}}\frac{u_{Z2}}{L}(-2+\frac{u_{Z2}}{L}) + \delta u_{Z2}F\,.$$

- Principle of virtual work and the fundamental lemma of variation calculus give a non-linear algebraic equation system for the non-zero displacement components u_{Z2} and u_{Z3}. However, finding an analytical solution in terms of the parameters of the problem is not possible. Mathematica code of the course gives a real valued solution with the minimal norm for selections $L = 1\,\text{m}$, $A^\circ = 1/100\,\text{m}^2$, $E = C = 100\,\text{Nm}^{-2}$ and

$F = 1/20\,\mathrm{N}$ (that is likely to be the physically meaningful solution when the initial displacement is zero)

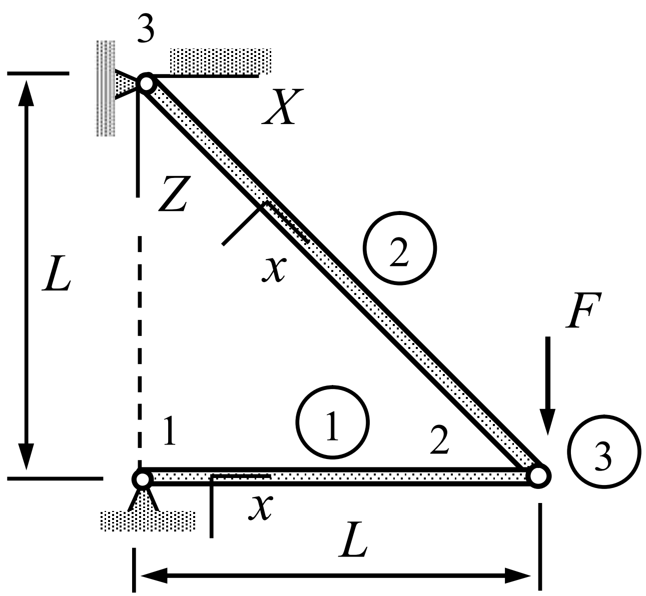
	type	properties	geometry
1	BAR	{{E}, {A, {0, 1, 0}}, {0, 0, 0}}	Line[{3, 1}]
2	BAR	{{E}, {A, {0, 1, 0}}, {0, 0, 0}}	Line[{3, 2}]
3	BAR	{{E}, {A, {0, 1, 0}}, {0, 0, 0}}	Line[{4, 2}]
4	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, 0, 0}
3	{L, 0, L}	{0, 0, uZ[3]}	{0, 0, 0}
4	{0, 0, L}	{0, 0, 0}	{0, 0, 0}

{uZ[2] → 0.0854082, uZ[3] → 0.0609567}

EXAMPLE 4.7. A bar truss is loaded by a point force having magnitude F as shown in the figure. Determine the equilibrium equations according to the large displacement theory. At the initial (non-loaded) geometry, cross-sectional area of bar 1 is A° and that for bar 2 $A^\circ/\sqrt{2}$. Find also the solution for $L = 1\,\text{m}$, $A^\circ = 1/100\,\text{m}^2$, $C = 100\,\text{Nm}^{-2}$ and $F = 1/20\,\text{N}$.

Answer $u_{X2} = -0.085\text{m}$ and $u_{Z2} = 0.25\text{m}$



- For bar 1, the nodal displacement components of material coordinate system are $u_{x1} = u_{z1} = 0$, $u_{x2} = u_{X2}$, and $u_{z2} = u_{Z2}$. As the approximations are linear

$$\frac{du}{dx} = \frac{u_{X2}}{L}, \frac{dv}{dx} = 0, \text{ and } \frac{dw}{dx} = \frac{u_{Z2}}{L}$$

and the virtual work expression (density is constant) of internal forces simplifies to

$$\delta W^1 = -(\delta u_{X2} + \delta u_{X2}\frac{u_{X2}}{L} + \delta u_{Z2}\frac{u_{Z2}}{L})CA°[\frac{u_{X2}}{L} + \frac{1}{2}(\frac{u_{X2}}{L})^2 + \frac{1}{2}(\frac{u_{Z2}}{L})^2].$$

- For bar 2, the nodal displacement components of material coordinate system are $u_{x3} = u_{z3} = 0$, $u_{x2} = (u_{X2} + u_{Z2})/\sqrt{2}$ and $u_{z2} = (-u_{X2} + u_{Z2})/\sqrt{2}$ (notice the use of initial geometry). As the approximations are linear

$$\frac{du}{dx} = \frac{u_{X2} + u_{Z2}}{L}, \quad \frac{dw}{dx} = \frac{u_{Z2} - u_{X2}}{L}$$

and the virtual work expression (density is constant) of internal forces simplifies to

$$\delta W^2 = -[\delta u_{X2} + \delta u_{Z2} + (\delta u_{X2} + \delta u_{Z2})(\frac{u_{X2} + u_{Z2}}{L}) + (\delta u_{Z2} - \delta u_{X2})(\frac{u_{Z2} - u_{X2}}{L})] \times$$

$$CA^{\circ}[(\frac{u_{X2} + u_{Z2}}{L}) + \frac{1}{2}(\frac{u_{X2} + u_{Z2}}{L})^2 + \frac{1}{2}(\frac{u_{Z2} - u_{X2}}{L})^2].$$

- Virtual work expression of the point follows from definition of work

$$\delta W^3 = F \delta u_{Z2}.$$

- Virtual work expression is sum of the element contributions. After a considerable amount of manipulations, the standard form with notations $a_1 = u_{X2} / L$ and $a_2 = u_{Z2} / L$

$$\delta W = -\frac{EA}{8}\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -(1+\mathrm{a}_1)(10\mathrm{a}_1 + 5\mathrm{a}_1^2 + 2\mathrm{a}_2 + 5\mathrm{a}_2^2) \\ 8\dfrac{F}{EA} - [2\mathrm{a}_1(1+5\mathrm{a}_2) + \mathrm{a}_1^2(1+5\mathrm{a}_2) + \mathrm{a}_2(2+3\mathrm{a}_2+5\mathrm{a}_2^2)] \end{Bmatrix} = 0.$$

- Principle of virtual work and the fundamental lemma of variation calculus give a non-linear algebraic equation system ($\mathrm{a}_1 = u_{Z3}/L$ and $\mathrm{a}_2 = u_{Z2}/L$)

$$\begin{Bmatrix} -(1+\mathrm{a}_1)(10\mathrm{a}_1 + 5\mathrm{a}_1^2 + 2\mathrm{a}_2 + 5\mathrm{a}_2^2) \\ 8\dfrac{F}{EA} - [2\mathrm{a}_1(1+5\mathrm{a}_2) + \mathrm{a}_1^2(1+5\mathrm{a}_2) + \mathrm{a}_2(2+3\mathrm{a}_2+5\mathrm{a}_2^2)] \end{Bmatrix} = 0. \quad \Leftarrow$$

- It is obvious that finding an analytical solution in terms of the parameters of the problem becomes impossible even when the truss is very simple when the number of non-zero displacement components exceeds one. Mathematica code of the course gives the real valued solution with the minimal norm (that is likely to be the physically meaningful solution when the initial displacement is zero) ($L = 1\,\mathrm{m}$, $A^\circ = 1/100\,\mathrm{m}^2$, $E = C = 100\,\mathrm{Nm}^{-2}$ and $F = 1/20\,\mathrm{N}$.

	type	properties	geometry
1	BAR	{{E}, {A, {0, 1, 0}}, {0, 0, 0}}	Line[{1, 2}]
2	BAR	$\{\{E\}, \{\frac{A}{\sqrt{2}}, \{0, 1, 0\}\}, \{0, 0, 0\}\}$	Line[{3, 2}]
3	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, L}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}

{uX[2] → -0.0848497, uZ[2] → 0.25}