CS-E4500 Advanced Course in Algorithms (5 cr)

Problem Set 3 *Spring 2019*

1. Arithmetic with rational numbers in radix-point representation.

   (a) Let us work in base $B = 3$. Multiply 22122.21201 and 22121.22001. Present the result in radix-point representation.

   (b) Let us work in base $B = 7$. Add 145.2332632 and 1345053.103. Present the result in radix-point representation.

   Hint: Reduce to integer multiplication and addition in base B.

2. Polynomial division by reversal. For a polynomial $f = \varphi_0 + \varphi_1 x + \ldots + \varphi_n x^n \in R[x]$ of degree at most $n \in \mathbb{Z}_{\geq 0}$ over a ring R, let the *n-reversal* of f be the polynomial $\operatorname{rev}_n f = \varphi_n + \varphi_{n-1} x + \ldots + \varphi_0 x^n \in R[x]$. Let $a, b \in R[x]$ be polynomials with b monic and $n = \deg a \geq \deg b = m$. Show that the quotient $q \in R[x]$ and the remainder $r \in R[x]$ with $a = qb + r$ and $\deg r \leq m - 1$ satisfy the reversal identity

$$\operatorname{rev}_n a = (\operatorname{rev}_{n-m} q)(\operatorname{rev}_m b) + x^{n-m+1} \operatorname{rev}_{m-1} r\,.$$

   Hint: Observe that for all $i = 0, 1, \ldots, n - m$ and $j = 0, 1, \ldots, m$ we have $x^{n-i-j} = x^{n-m-i} x^{m-j}$.

3. Closure under addition. Let

$$\alpha = sB^e \sum_{i=0}^{d-1} \alpha_i B^{-i} \in \mathbb{Q}_B$$

   and

$$\tilde{\alpha} = \tilde{s}B^{\tilde{e}} \sum_{i=0}^{\tilde{d}-1} \tilde{\alpha}_i B^{-i} \in \mathbb{Q}_B$$

   be two rational numbers in normal radix-point representation in base B with $s, \tilde{s} \in \{-1, 1\}$, $d, \tilde{d} \in \mathbb{Z}_{\geq 1}$, and $e, \tilde{e} \in \mathbb{Z}$. Show that the rational number $\alpha + \tilde{\alpha}$ admits a radix-point representation in base B. Assuming that the representation of $\alpha + \tilde{\alpha}$ is normal, give an upper bound as a function of $d, e, \tilde{d}, \tilde{e}$ for the number of digits in the representation. Your upper bound should be tight up to a constant independent of $d, e, \tilde{d}, \tilde{e}$.

   Hint: By padding with zero-digits, reduce to integer addition in base B. We allow the constant slack because of carries in addition, so be careful to make your upper bound accommodate any and all digits resulting from carries. Observe that you need to consider subtraction as well since $s, \tilde{s} \in \{-1, 1\}$.

4. Recovering the integer quotient. Fix the base $B \in \mathbb{Z}_{\geq 2}$. Suppose you have available a subroutine that, given a $\nu \in \mathbb{Q}_B$ with $B^{-1} \leq \nu < 1$ and $t \in \mathbb{Z}_{\geq 1}$ as input, in time $O(M(t))$ outputs a $(t + O(1))$-digit $\mu \in \mathbb{Q}_B$ with $|1 - \mu\nu| \leq B^{-t}$.

   Present an algorithm that, given as input two at-most-d-digit integers $\alpha, \beta \in \mathbb{Z}_{\geq 1}$ in base B, in time $O(M(d))$ outputs the quotient $\eta \in \mathbb{Z}_{\geq 0}$ and the remainder $\rho \in \mathbb{Z}_{\geq 0}$ with $\alpha = \eta\beta + \rho$ and $0 \leq \rho \leq \beta - 1$. Carefully justify the correctness of your algorithm and its running time.

Here $M(d)$ is a function such that two at-most-d-digit integers in base B can be multiplied in time $O(M(d))$. For example, we can take $M(d) = d \log d \log \log d$.

Hints: Observe that the subroutine needs $B^{-1} \leq \nu < 1$, whereas β is an integer, so you need to transform β into an appropriate ν. Remember to reverse this transform once you obtain a μ from the subroutine. Use your control on t to compute a rational number $\tilde{\eta}$ in radix point representation that is close enough to the quotient $\eta = \lfloor \alpha/\beta \rfloor$. To use your control on t, it may be a good idea to assume that α has exactly n digits and β has exactly m digits in base B, with $1 \leq n, m \leq d$. What inequalities (lower and upper bounds) for the values of α and β do you obtain with such an assumption? Observe that you can easily check that you have the correct η by recovering an associated remainder $0 \leq \rho \leq \beta - 1$ with $\alpha = \eta\beta + \rho$. Be very careful in justifying that you always find the correct η.

Deadline and submission instructions. This problem set is due no later than Sunday 3 February 2019, 20:00 (8pm), Finnish time. Please submit your solutions as a single PDF file via e-mail to the lecturer (`petteri.kaski(atsymbol)aalto.fi`). Please use the precise title

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CS-E4500 Problem Set 3: [your-student-number]
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with “`[your-student-number]`” replaced by your student number. For example, assuming that my student number is 123456, I would carefully title my e-mail

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CS-E4500 Problem Set 3: 123456
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and attach to the e-mail a single PDF file containing my solutions. Please note that the submissions are automatically processed and archived, implying that failure to follow these precise instructions may result in your submission not being graded.