Taylor series for $x \in \mathbb{R}$ at $x = m$

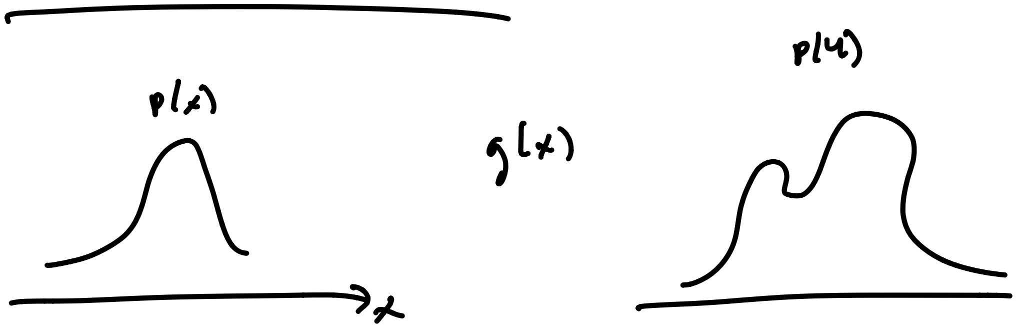
$$f(x) = f(m) + f'(m)(x-m) + \frac{1}{2!} f''(m)(x-m)^2 + \dots$$

Taylor series for $x \in \mathbb{R}^n$ $f: \mathbb{R}^n \to \mathbb{R}^m$, $x = m$:

$$\vec{f}_i(\vec{x}) = \vec{f}_i(\vec{m}) + (\nabla f_i)^T (x-m) + \frac{1}{2}(x-m)^T (\nabla \nabla^T f_i)(x-m) + \dots$$

Linear approximation: Jacobian $F_x = \begin{pmatrix} \nabla f_1^T \\ \nabla f_2^T \\ \vdots \\ \nabla f_m^T \end{pmatrix}$

$$\vec{f}(\vec{x}) \approx \vec{f}(\vec{m}) + F_x(m)(\vec{x} - \vec{m})$$



$x \sim N(m, P)$

$$g(x) \approx g(m) + G_x(m)(x-m)$$

$$E[g(x)] \approx E[g(m) + G_x(m)(x-m)]$$

$$= g(m) + G_x(m)\underbrace{E[(x-m)]}_{=0}$$

$$= g(m)$$

$$\mathrm{Cov}[g(x)] = E[(g(x) - E[g(x)])\,(\cdot)^T]$$

$$\approx E[(g(x) - g(m))(g(x) - g(m))^T]$$

$$\approx E[(\cancel{g(m)} + G_x(m)(x-m) - \cancel{g(m)})$$

$$(g(m) + G_x(m)(x-m) - g(m))^T]$$

$$= E[G_x(m)(x-m)(x-m)^T G_x^T(m)]$$

$x \sim N(m, P)$

$$= G_x(m)\underbrace{E[(x-m)(x-m)^T]}_{P}G_x^T(m)$$

$$= G_x(m)\,P\,G_x^T(m)$$

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linear: $x \sim N(m, P)$

$y = Hx + q, \quad q \sim N(0, Q)$

non-linear:

$x \sim N(m, P)$

$q \sim N(0, Q)$

$y_1 = x$

$y_2 = g(x) + q$

$$E\left[\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] = E\left[\begin{pmatrix} x \\ g(x)+q \end{pmatrix}\right] \approx \begin{pmatrix} m \\ g(m) \end{pmatrix}$$

$$\mathrm{Cov}\left[\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] = E\left[\begin{pmatrix} x - E(x) \\ g(x)+q - E[g(x)+q] \end{pmatrix}\begin{pmatrix} x - E(x) \\ g(x)+q - E[g(x)+q] \end{pmatrix}^T\right]$$

$$\approx E\left[\begin{pmatrix} x-m \\ g(x)+q-g(m)\end{pmatrix}\begin{pmatrix} x-m \\ g(x)+q-g(m)\end{pmatrix}^T\right]$$

$$= E\left[\begin{pmatrix} (x-m)(x-m)^T & (x-m)(g(x)+q-g(m))^T \\ (g(x)+q-g(m))(x-m)^T & (g(x)+q-g(m))(\cdot)^T \end{pmatrix}\right]$$

$$\Rightarrow \quad E[(x-m)(x-m)^T] = P$$

$$E[(g(x)+q-g(m))(x-m)^T]$$

$$\approx E[(\cancel{g(m)} + \underline{G_x(m)(x-m)} + q - \cancel{g(m)})\underline{(x-m)}^T]$$

$$= E[G_x(m)(x-m)(x-m)^T]$$

$$+ \underbrace{E[q(x-m)^T]}_{=0} = G_x E[(x-m)(x-m)^T]$$

$$= G_x(m) P$$

$$E[(g(x)+q-g(m))(g(x)+q-g(m))^T]$$

$$\approx E[(\cancel{g(m)} + G_x(m)(x-m) + q - \cancel{g(m)})$$

$$(\cancel{g(m)} + G_x(m)(x-m) + q - \cancel{g(m)})^T$$

$$= E[G_x(m)\underline{(x-m)(x-m)}^T G_x^T(m) + q q^T]$$

$$= G_x(m) P G_x^T(m) + Q$$

$$x_{k-1} \sim N(m_{k-1}, P_{k-1})$$

$$x_k = f(x_{k-1}) + q_{k-1}, \quad q_{k-1} \sim N(0, Q_{k-1})$$

$$\begin{pmatrix} x_{k-1} \\ x_k \end{pmatrix} \sim N\left(\begin{pmatrix} m_{k-1} \\ f(m_{k-1})\end{pmatrix}, \begin{pmatrix} P_{k-1} & P_{k-1} F_x^T(m_{k-1}) \\ F_x(m_{k-1}) P_{k-1} & \end{pmatrix}\right)$$

$$F_x(m_{k-1}) P_{k-1} \cdot F_x^T(m_{k-1})$$

marginal of x_k: $\qquad\qquad + Q_{k-1}$

$$x_k \sim N\Big(\underbrace{f(m_{k-1})}_{m_k^-},\ \underbrace{F_x(\cdot)\, P_{k-1} F_x^T(\cdot) + Q_{k-1}}_{P_k^-}\Big)$$

$$x_k \sim N(m_k^-, P_k^-)$$
$$y_k = h(x_k) + r_k, \quad r_k \sim N(0, R_k)$$

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} \sim N\left(\begin{pmatrix} m_k^- \\ h(m_k^-) \end{pmatrix}, \begin{pmatrix} P_k^- & P_k^- H_x^T(m_k^-) \\ H_x(m_k^-) P_k^- & H_x(\cdot) P_k^- H_x^T(\cdot) + R_k \end{pmatrix} \right)$$

$$x_k | y_k \sim N\Big(m_k^- + \underbrace{P_k^- H_x^T(m_k^-)(\quad)^{-1}}_{K_k}(y_k - h(m_k^-)),$$
$$P_k^- - P_k^- H_x^T(m_k^-)(\quad)^{-1} H(m_k^-) P_k^- \Big)$$

$$S_k = H_x(m_k^-)\, P_k^-\, H^T(m_k^-) + R_k$$
$$K_k = P_k^- H_x^T(m_k^-)\, S_k^{-1}$$
$$m_k = m_k^- + K_k(y_k - h(m_k^-))$$
$$P_k = P_k^- - K_k S_k K_k^T$$