Aalto University
School of Science

Decision making and problem solving – Lecture 4

- Risk measures
- Multiattribute value theory
- Axioms for preference relations
- Elicitation of attribute-specific value functions

Liesiö, Punkka, Salo, Vilkkumaa

Motivation

- **Last time we learned how :**
  - To model the DM's preferences over risk by eliciting her utility function
  - The shape (concave / linear / convex) of the utility function corresponds to the DM's risk attitude (risk averse / neutral / seeking)
  - Decision recommendations may be implied by stochastic dominance even if the utility function is not (completely) specified:
    - If the DM prefers more to less, she should not choose an FSD dominated alternative
    - If the DM is also risk averse, she should not choose an SSD dominated alternative
- **This time (Part A):**
  - We take a look at risk measures and examine how they can be used to describe alternatives' risks

Risk measures

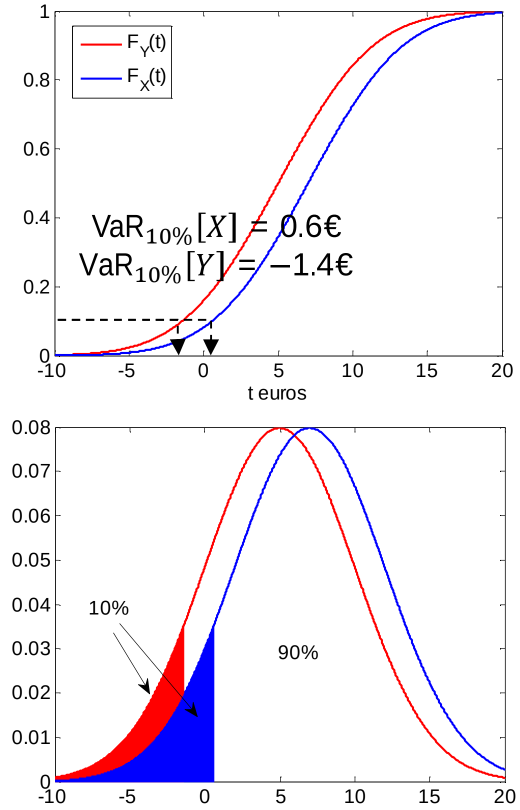
- Risk measure is a function that maps each decision alternative to a single number describing its risk
  - E.g., variance $Var[X] = E[(X - E[X])^2]$
    - The higher the variance, the higher the risk
- Risk measures are not based on EUT, but can be used together with expected values to produce decision recommendations
  - Risk constraint: Among alternatives whose risk is below some threshold, select the one with the highest expected value
  - Risk minimization: Among alternatives whose expected value is above some threshold, select the one with minimum risk
  - Efficient frontier: Select one of those alternative compared to which no other alternative yields higher expected value and smaller risk

Risk measures: Value-at-Risk (VaR)

- Value-at-Risk ($\text{VaR}_\alpha[X]$) is the outcome such that the probability of a worse or equal outcome is α:

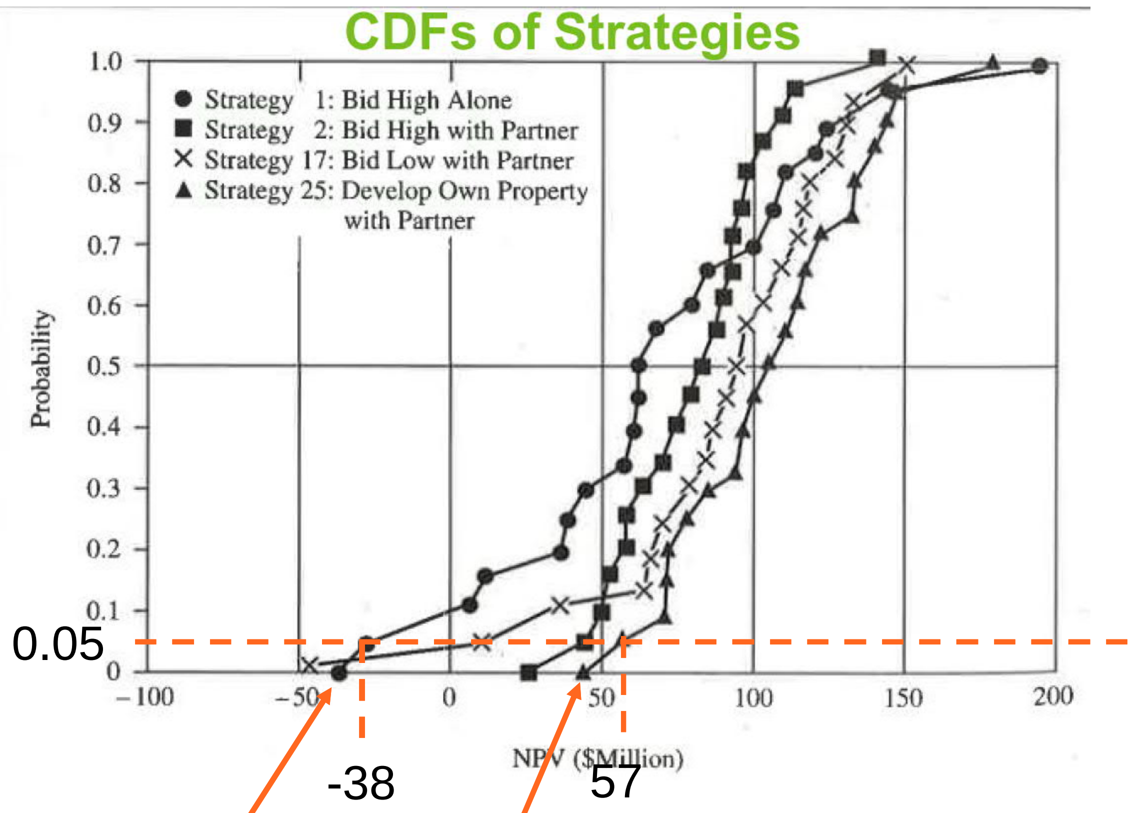
$$\int_{-\infty}^{\text{VaR}_\alpha[X]} f_X(t)dt = F_X(\text{VaR}_\alpha[X]) = \alpha.$$

- Higher VaR means smaller risk
  - Unless applied to a loss distribution
- Common values for α: 1%, 5%, and 10%
- Problem: the length/shape of the tail is not taken into account

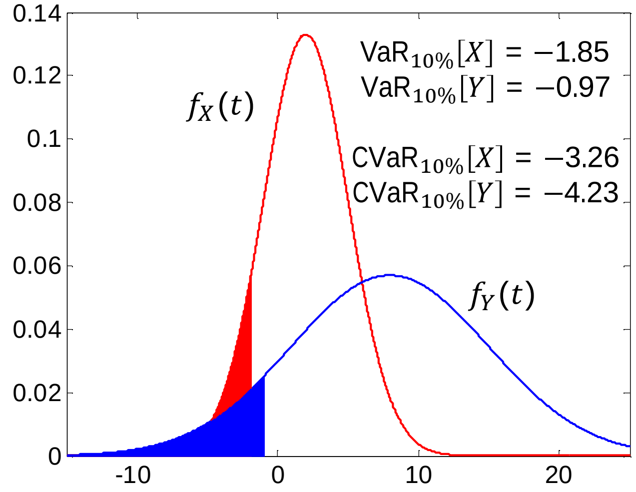


Mining example revisited

- Assess $VaR_{5\%}$ for strategies 1 and 25



Risk measures: Conditional Value-at-Risk (CVaR)



- Conditional Value-at-Risk ($\text{CVaR}_\alpha[X]$) is the expected outcome given that the outcome is at most VaR_α:

$$\text{CVaR}[X] = E[X|X \leq \text{VaR}_\alpha[X]]$$

- Higher CVaR means smaller risk (unless applied to losses)

- Computation of $\text{CVaR}[X]$ to discrete and continuous *X:*

$$E[X|X \leq \text{VaR}_\alpha[X]] = \sum_{t \leq \text{VaR}_\alpha[X]} t\,\frac{f_X(t)}{\alpha}, \qquad E[X|X \leq \text{VaR}_\alpha[X]] = \int_{-\infty}^{\text{VaR}_\alpha[X]} t\,\frac{f_X(t)}{\alpha}\,dt.$$

  – Note: $\alpha = P(X \leq \text{VaR}_\alpha[X])$; PMF/PDF $f_X(t)$ is scaled such that it sums/integrates up to 1.

Computation of VaR and CVaR

- If the inverse CDF of X is well-defined, VaR can be obtained from

$$\text{VaR}_{\alpha}[X] = F_X^{-1}(\alpha)$$

  - In Excel: norm.inv, lognorm.inv, beta.inv, binom.inv etc
  - In Matlab: norminv, logninv, betainv, binoinv etc

- CVaR can then be computed using the formulas on the previous slide
  - Sometimes an analytic solution can be obtained; if, e.g., $X \sim N(\mu, \sigma^2)$ and $\text{VaR}_{\alpha}[X] = \beta$, then

$$\text{CVaR}_{\alpha}[X] = \mu - \sigma \frac{\phi\left(\frac{\beta-\mu}{\sigma}\right)}{\Phi\left(\frac{\beta-\mu}{\sigma}\right)},$$

    where ϕ and Φ are the standard normal PDF and CDF, respectively.
  - Sometimes numerical integration is needed

Computation of VaR and CVaR

- With discrete random variables VaR and CVaR are not always well defined for small values of α
  - Example:

t	-10	-5	1	10	20
$f_X(t)$	0.06	0.02	0.02	0.5	0.4

  - $\text{VaR}_{10\%}[X]=1$
  - $\text{CVaR}_{10\%}[X] = \frac{0.06(-10)+0.02(-5)+0.02(1)}{0.06+0.02+0.02}=-6.8$
  - But what are $\text{VaR}_{5\%}[X]$, $\text{CVaR}_{5\%}[X]$?

VaR and CVaR with Monte Carlo - Excel

=AVERAGE(D12:D211)

	A	B	C	D	E	F
1						
8		Col.mean	Col.mean	CVaR-10%		
9		0.507501	1008.35	147.4443		VaR-10%
10						410.5591
11	Sample	u	x	Below VaR		
12	1	0.691314	1249.789	above		
13	2	0.603076	1130.659	above		
14	3	0.548331	1060.723	above		
15	4	0.058081	214.4534	214.4534		
16	5	0.442469	927.6436	above		
17	6	0.628886	1164.452	above		
18	7	0.157181	496.9445	above		
19	8	0.355657	814.9539	above		
20	9	0.545768	1057.488	above		
21	10	0.416183	894.1666	above		
22	11	0.879097	1585.243	above		
23	12	0.022042	-6.64468	-6.64468		
24	13	0.000927	-556.359	-556.359		
25	14	0.071391	267.2461	267.2461		

=PERCENTILE.INC(C12:C211;0.1)

=IF(C12<=F10;C12;"above")

Note! 200 samples is very little, because only 1/10=20 are used to estimate CVaR

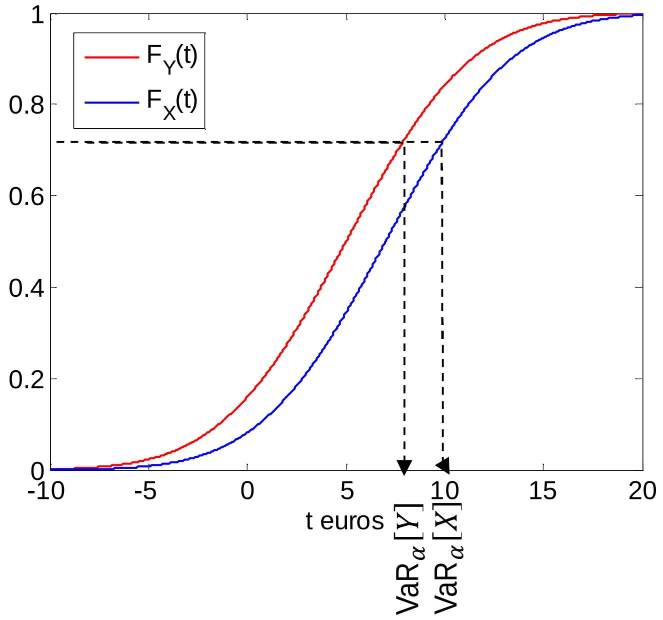
VaR and CVaR with Monte Carlo - Matlab

```
S=10^5;                          %Sample size 10,000
mu=1000;
sigma=500;
Sample=normrnd(mu,sigma,S,1);    %Generates 10^5 observations from N(mu,sigma)
VaR=prctile(Sample,10)           %Returns the 10% percentile of the sample
TailIndices=find(Sample<=VaR);   %Returns the indices of those elements
                                 %in the sample below or equal to VaR
CVaR=mean(Sample(TailIndices))   %Computes the arithmetic mean among those
                                 %elements in the sample belor or equal to VaR
```

Risk measures and stochastic dominance

- **Theorem:** $X \succcurlyeq_{\text{FSD}} Y$ if and only if $\text{VaR}_\alpha[X] \geq \text{VaR}_\alpha[Y]\ \forall\alpha \in [0,1]$

- **Theorem:** $X \succcurlyeq_{\text{SSD}} Y$ if and only if $\text{CVaR}_\alpha[X] \geq \text{CVaR}_\alpha[Y]\ \forall\alpha \in [0,1]$

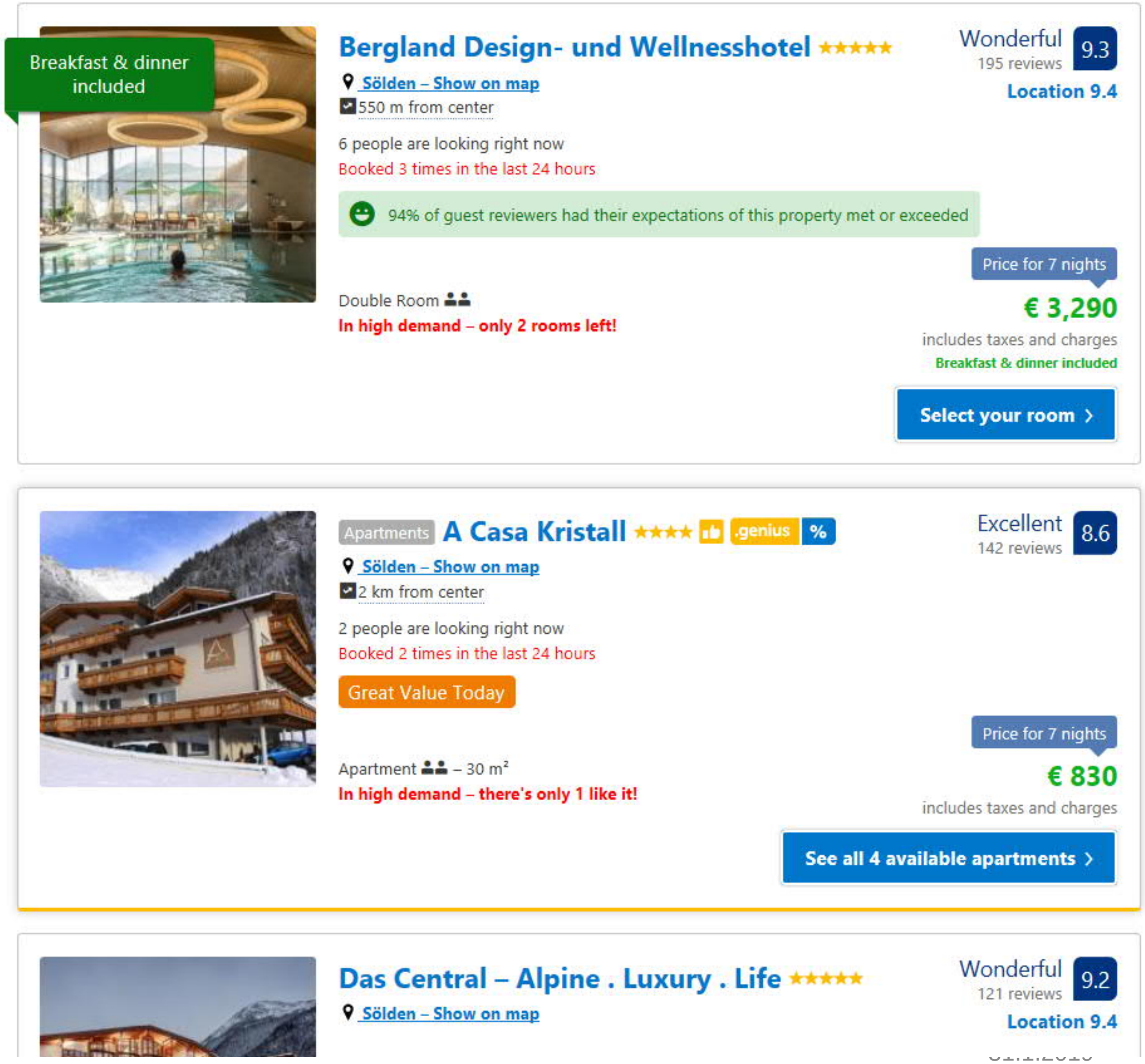


EUT vs. Risk measures

- EUT provides a more comprehensive way to capture the DM's preferences over uncertain outcomes
- With risk measures, one must answer questions such as
  - Which measure to use?
  - Which α to use in VaR and CVaR?
  - How to combine EV and the value of a risk measure into an overall performance measure?
- Yet, if answers to such questions are exogenously imposed, the use of risk measures can be easy
  - E.g., laws, regulations, industry standard etc.

Motivation

- ❑ **Consider yourself choosing accommodation for a (downhill) skiing vacation trip**
- ❑ **How do the accommodation alternatives differ from each other?**
  - ❑ What are the attributes that influence your decision?



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Motivation

- **So far:**
  - We have considered decision-making situations in which the DM has one objective (e.g., maximize the expected value/utility of a monetary payoff)

- **This time:**
  - We consider decision-making situations in which the DM has multiple objectives or, more precisely...
  - <u>Multiple attributes</u> with regard to which the achievement of some fundamental objective is measured

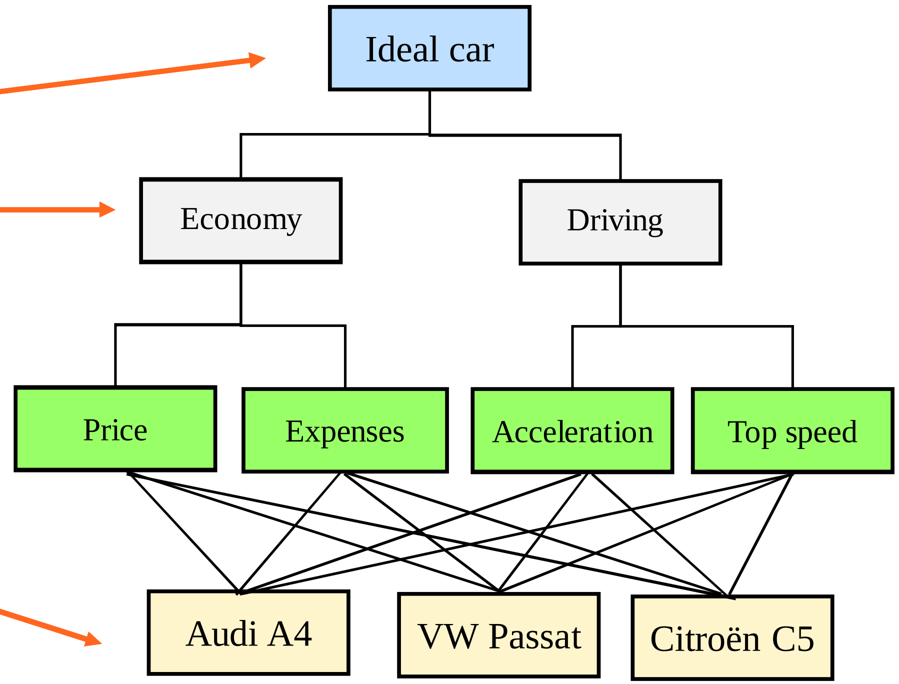
Multiattribute value theory

- Ralph Keeney and Howard Raiffa (1976): Decisions with Multiple Objectives: Preferences and Value Tradeoffs
- James Dyer and Rakesh Sarin (1979): Measurable multiattribute value functions, *Operations Research* Vol. 27, pp. 810-822

- Elements of MAVT
  - A value tree consisting of objectives, attributes, and alternatives
  - Preference relation over the alternatives' attribute-specific performances and differences thereof & their representation with an <u>attribute-specific value function</u>
  - Preference relation over the alternatives' overall performances and differences thereof & their representation with a <u>multiattribute value function</u>

Value tree: objectives, attributes, and alternatives

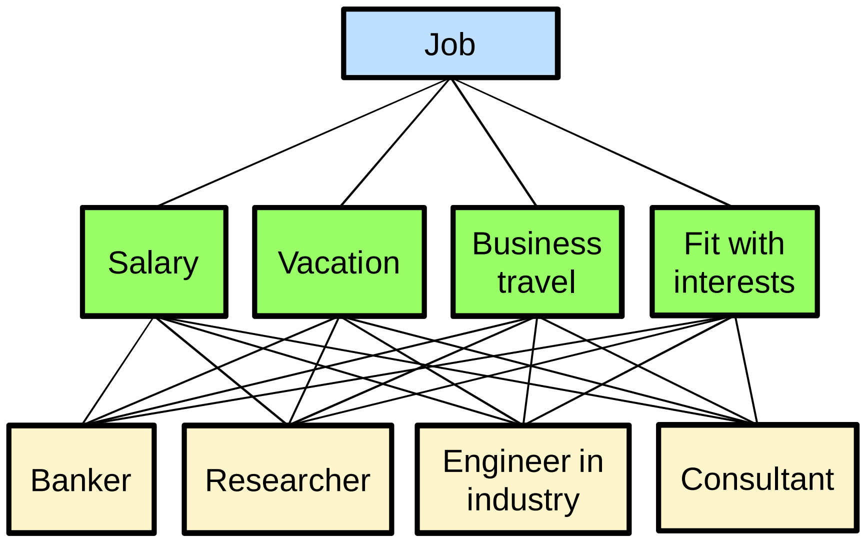
- A value tree consists of
  - A fundamental objective
  - Possible lower-level objectives
  - Attributes that measure the achievement of the objectives
  - Alternatives whose attribute-specific performances are being measured



Value tree: objectives, attributes and alternatives

- The attributes $a_1,\ldots, a_n$ have *measurement scales* X_i, i=1,…,n; e.g.,
  - X_1=[1000€/month, 6000€/month]
  - X_2 =[2 weeks/year, 8 weeks/year]
  - X_3 =[0 days/year, 200 days/year]
  - X_4 ={poor, fair, good, excellent}

- Alternatives $x = (x_1, x_2, \ldots x_n)$ are characterized by their performance w.r.t. the attributes; e.g.,
  - Banker=(6000€/month, 5 weeks/year, 40 days/year, fair) $\in X_1 \times X_2 \times X_3 \times X_4$.



Preference relation: attribute-specific performance

- Let $\succcurlyeq$ be preference relation among performance levels *a* and *b* on a given attribute

Preference $a \succcurlyeq b$: a **at least as preferable as** b

Strict preference $a \succ b$ **defined as** $\neg(b \succcurlyeq a)$

Indifference $a \sim b$ **defined as** $a \succcurlyeq b\ \wedge b \succcurlyeq a$

Axioms for preference relation

- **A1:** $\succcurlyeq$ is complete
  - For any $a, b \in X$, either $a \succcurlyeq b$ or $b \succcurlyeq a$ or both
- **A2:** $\succcurlyeq$ is transitive
  - If $a \succcurlyeq b$ and $b \succcurlyeq c$, then $a \succcurlyeq c$

Ordinal value function

Theorem: Let axioms **A1-A2** hold. Then, there exists an <u>ordinal</u> value function $v_i(\cdot)\colon X_i \to \mathbb{R}$ that represents preference relation $\succcurlyeq$ in the sense that

$$v_i(a) \geq v_i(b) \Leftrightarrow a \succcurlyeq b$$

- An ***ordinal*** value function does not describe strength of preference, i.e., it does not communicate much more an object is preferred to another

Ordinal value function

- Assume you have two mopeds A and B with top speeds of 30 and 35km/h, respectively
- You have two alternatives for upgrade
  - Increase top speed of moped A to 40
  - Increase top speed of moped B to 45
- Your prefer a higher top speed to a lower one
  - 45>40>35>30
  - $v(45)=1$, $v(40)=0.8$, $v(35)=0.5$, $v(30)=0.4$
  - $w(45)=0.9$, $w(40)=0.8$, $w(35)=0.6$, $w(30)=0.4$
- Both v and w are ordinal value functions representing your preferences but they do not describe your preferences between the two upgrade alternatives
  - $v(45)-v(35)=0.5 > v(40)-v(30)=0.4$, but $w(45)-w(35)=0.3 < w(40)-w(30) =0.4$

Ordinal value function

Theorem: Ordinal value functions $v_i(\cdot)$ and $w_i(\cdot)$ represent the same preference relation $\succcurlyeq$ **if and only if** there exists a strictly increasing function $\phi: \mathbb{R} \to \mathbb{R}$ such that $w_i(a) = \phi[v_i(\cdot)]\ \forall a \in A$**.**

Example: Let *consultant* $\succ$ *professor* $\succ$ *janitor* be Jim's preferences over these jobs and v(consultant) = 10 > v(professor) = 8 > v(janitor) = 7.

Then v' and v'' **both represent the same preferences as ordinal value function** v

	consultant	professor	janitor
v	10	8	7
v'	20	16	14
v''	20	16	8

The goal is to compare <u>multi-attribute</u> alternatives, wherefore ordinal value functions are not enough

- Let $\succcurlyeq_d$ be preference relation among <u>differences</u> in performance levels on a given attribute
  - Preference $(a \leftarrow b) \succcurlyeq_d (c \leftarrow d)$: a change from b to a is at least as preferable as a change from d to c
  - Strict preference $(a \leftarrow b) \succ_d \ (c \leftarrow d)$ defined as $\neg((c \leftarrow d) \succcurlyeq_d (a \leftarrow b))$
  - Indifference $(a \leftarrow b) \sim_d (c \leftarrow d)$ defined as $(a \leftarrow b) \succcurlyeq_d (c \leftarrow d) \wedge (c \leftarrow d) \succcurlyeq_d (a \leftarrow b)$

Axioms for preference relation (cont'd)

- **A3:** $\forall a, b, c \in X_i$: $a \succcurlyeq b \Leftrightarrow (a \leftarrow b) \succcurlyeq_d (c \leftarrow c)$
  - If a is preferred to b, then a change from b to a is preferred to no change
- **A4:** $\forall a, b, c, d \in X_i$: $(a \leftarrow b) \succcurlyeq_d (c \leftarrow d) \Leftrightarrow (d \leftarrow c) \succcurlyeq_d (b \leftarrow a)$
  - E.g., if an increase in salary from 1500€ to 2000€ is preferred to an increase from 2000€ to 2500€, then a decrease from 2500€ to 2000€ is preferred to a decrease from 2000€ to 1500€
- **A5:** $\forall a, b, c, d, e, f \in X_i$: $(a \leftarrow b) \succcurlyeq_d (d \leftarrow e) \wedge (b \leftarrow c) \succcurlyeq_d (e \leftarrow f) \Rightarrow (a \leftarrow c) \succcurlyeq_d (d \leftarrow f)$
  - If two incremental changes are both preferred to some other two, then the overall change resulting from the first two increments is also preferred.
- **A6:** $\forall b, c, d \in X_i\ \exists a \in X_i$ such that $(a \leftarrow b) \sim_d (c \leftarrow d)$ and $\forall b, c \in X_i\ \exists a \in X_i$ such that $(b \leftarrow a) \sim_d (a \leftarrow c)$
  - Equally preferred differences between attribute levels can always be constructed
  - There is always an attribute level a between b and c such that a change from c to a is equally preferred to a change from a to b.
- **A7:** The set (or sequence) $\{a_n | b \succ a_n\ where (a_n \leftarrow a_{n-1}) \sim_d (a_1 \leftarrow a_0)\}$ is finite for any b in X_i
  - The sequence of equally preferred differences over a fixed interval is finite
  - "No b can be infinitely better than other performance levels"

As French (1988) incorrectly puts it; the idea here is that it is possible to construct equally preferred changes in order to represent preferences

Cardinal value function

- **Theorem:** Let axioms **A1-A7** hold. Then, there exists a <u>cardinal</u> value function $v_i(\cdot)$: $X_i \rightarrow \mathbb{R}$ that represents preference relations $\succcurlyeq$ and $\succcurlyeq_d$ in the sense that

$$v_i(a) \geq v_i(b) \Leftrightarrow a \succcurlyeq b$$
$$v_i(a) - v_i(b) \geq v_i(c) - v_i(d) \Leftrightarrow (a \leftarrow b) \succcurlyeq_d (c \leftarrow d).$$

Note: A cardinal value function is unique up to positive affine transformations, i.e., $v_i(x)$ and $v_i'(x) = \alpha v_i(x) + \beta, \alpha > 0$ and represent the same preferences

Cardinal value function: positive affine transformations

Example: Let *consultant* ≻ *professor* ≻ *janitor* and $(consultant \leftarrow professor) \succcurlyeq_d (professor \leftarrow janitor)$ be Jim's preferences and v(consultant) = 10 > v(professor) = 8 > v(janitor) = 7.

Then v' and v'' **both represent same preferences as cardinal value function** v

	consultant	professor	janitor
v	10	8	7
$v' = 2v$	20	16	14
$v'' = v' - 10$	10	6	4

Attribute-specific value functions

- A value function maps the attribute-specific measurement scale onto a numerical scale in accordance with the DM's preferences

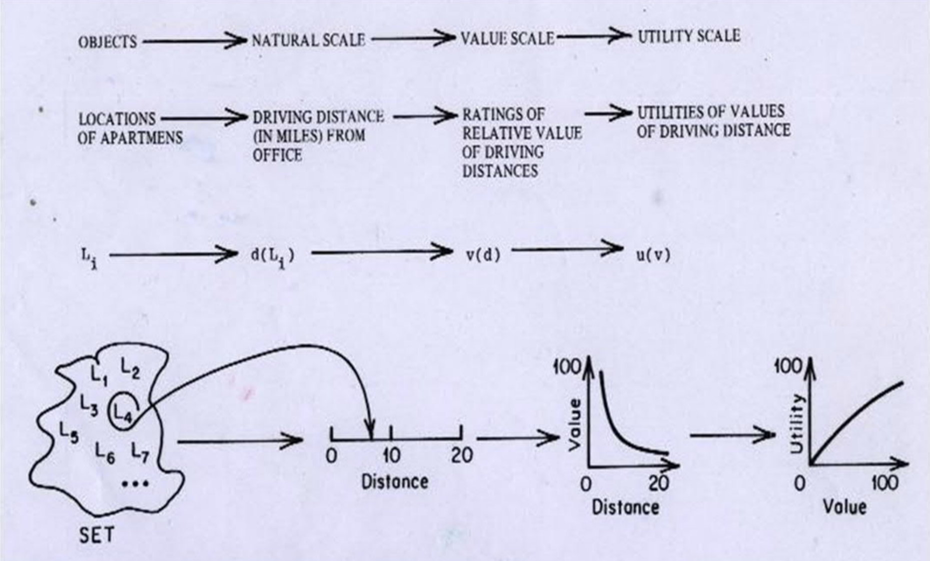


Figure 7.2. The four steps needed to construct value and utility functions.

- Value and utility:
  - Value is a measure of preference under certainty
  - Utility is a measure of preference under uncertainty

Elicitation of value functions

- **Phases:**
  - Define the measurement scale $X_i = [a_i^0, a_i^*]$ (or $[a_i^*, a_i^0]$)
  - Ask a series of eliciation questions
  - Check that the value function gives realistic results

Elicitation of value functions: Indifference methods

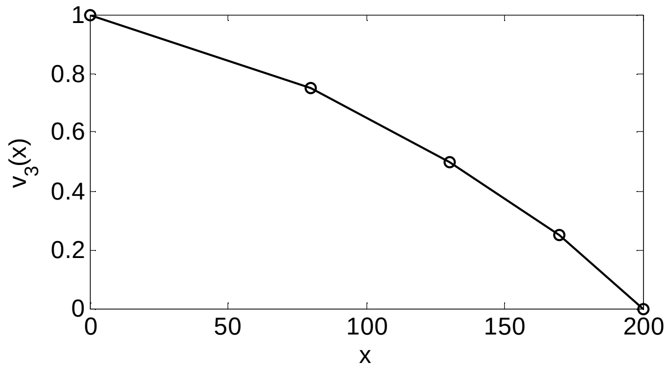
❑ Bisection method:

- Ask the DM to assess level $x_{0.5} \in [a_i^0, a_i^*]$ such that she is indifferent between change $x_{0.5} \leftarrow a^0$ and change $a^* \leftarrow x_{0.5}$.
- Then, ask her to assess levels $x_{0.25}$ and $x_{0.75}$ such that she is indifferent between
  - changes $x_{0.25} \leftarrow a^0$ and $x_{0.5} \leftarrow x_{0.25}$, and
  - changes $x_{0.75} \leftarrow x_{0.5}$ and $a^* \leftarrow x_{0.75}$.
- Continue until sufficiently many points have been obtained
  - Use, e.g, linear interpolation between elicited points if needed
- The value function can be obtained by fixing $v_i(a_i^0)$ and $v_i(a_i^*)$ at, e.g., 0 and 1

Elicitation of value functions: Indifference methods

❑ Example of the bisection method

- Attribute a_3 : Traveling days per year
- Measurement scale $[a_3^*, a_3^0]$, where $a_3^* = 0$ and $a_3^0 = 200$; fix $v_3(a_3^0) = 0$ and $v_3(a_3^*) = 1$
  - ”What would be the number $x_{0.5}$ of traveling days such that you would be indifferent between a decrease from 200 to $x_{0.5}$ days a year and a decrease from $x_{0.5}$ to zero days a year?” (Answer e.g., ”130”)
  - ”What would be the number $x_{0.25}$ of traveling days such that you would be indifferent between a decrease from 200 to $x_{0.25}$ days a year and a decrease from $x_{0.25}$ to 130 days a year?” (Answer e.g., ”170”)
  - ”What would be the number $x_{0.75}$ of traveling days such that you would be indifferent between a decrease from 130 to $x_{0.75}$ days a year and a decrease from $x_{0.75}$ to zero days a year?” (Answer e.g., ”80”)



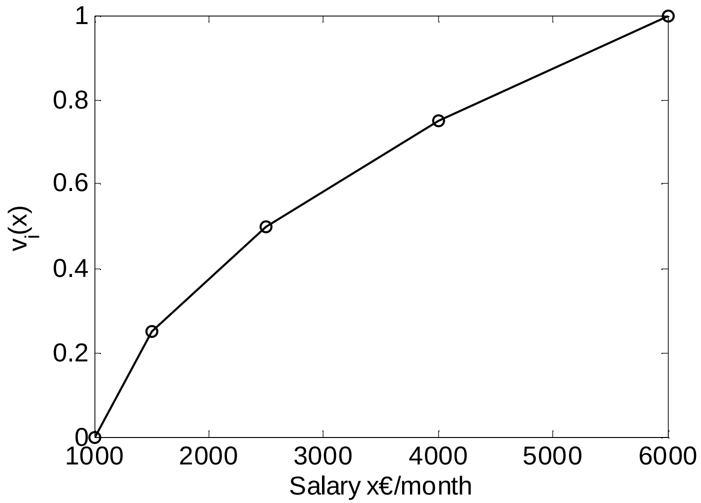
$$v_3(130) - v_3(200) = v_3(0) - v_3(130) \Rightarrow$$
$$v_3(130) = \frac{v_3(0) + v_3(200)}{2} = 0.5$$

$$v_3(170) - v_3(200) = v_3(130) - v_3(170) \Rightarrow$$
$$v_3(170) = \frac{v_3(130) + v_3(200)}{2} = 0.25$$

$$v_3(80) - v_3(130) = v_3(0) - v_3(80) \Rightarrow$$
$$v_3(80) = \frac{v_3(0) + v_3(130)}{2} = 0.75$$

Elicitation of value functions: Indifference methods

- **Sequence of equally preferred differences:**
  - Set $x_0 \in (a_i^0, a_i^*)$
  - Ask the DM to assess level $x_1 \in (x_0, a_i^*]$ such that he is indifferent between changes $x_0 \leftarrow a_i^0$ and $x_1 \leftarrow x_0$
    - $v_i(x_0) - v_i(a_i^0) = v_i(x_1) - v_i(x_0) \Rightarrow v_i(x_1)=2v_i(x_0)$
  - Then, ask him to assess level $x_2 \in (x_1, a_i^*]$ such that he is indifferent between change $x_1 \leftarrow x_0$ and $x_2 \leftarrow x_1$
    - $v_i(x_1) - v_i(x_0) = v_i(x_2) - v_i(x_1) \Rightarrow v_i(x_2)=3v_i(x_0)$
  - Continue until $x_N=a_i^*$ and solve the system of linear equations
    - $v_i(x_0) = \frac{v_i(x_N)}{N+1} = \frac{1}{N+1} \Rightarrow v_i(x_1)=\frac{2}{N+1}$ etc.
  - If $x_N>a_i^*$ (see the exercises!)
    - Change a_i^* to x_N and interpolate, or
    - Interpolate to get $v_i(a_i^*) - v_i(a_i^0)$



Example:

$[a_i^0, a_i^*] = [1000, 6000]$, $x_0 = 1500$

$x_1 = 2500, x_2 = 4000, x_3 = 6000 = a_i^* \Rightarrow$

$v_i(1500) = \frac{1}{4}, v_i(2500) = \frac{1}{2}, v_i(4000) = \frac{3}{4}$.

Elicitation of value functions: Indifference methods

- Indifference methods are likely to result in a cardinal value function that captures the DM's preferences
- Therefore, they should be used whenever possible

- Yet: indifference methods cannot be used when the measurement scale is discrete
  - E.g., Fit with interest: X_4 ={poor, fair, good, excellent}
  - Cf. Axiom A6

Elicitation of value functions: direct methods

- Direct rating
  - Ask the DM to directly attach a value to each attribute level
  - E.g. ”Assume that the value of poor fit with interests is 0 and the value of excellent fit with interests is 1. What is the value of fair fit with interests? How about good fit?”
- Class rating
  - Divide the measurement scale into classes and ask the DM to attach a value to these classes
- Ratio evaluation
  - Take one attribute level as a reference point and ask the DM to compare the other levels to this
  - E.g., ”How many times more valuable is 1000€ than 900€?”

- Direct methods should be avoided whenever possible
  - Usually do not result in a cardinal value function

Next time: Aggregation of values

- **Problem:** How to measure the overall value of alternative $x = (x_1, x_2, \dots x_n)$?

$$V(x_1, x_2, \dots x_n) = ?$$

- **Question:** Could the overall value be obtained by aggregating attribute-specific values?

$$V(x_1, x_2, \dots x_n) = f\big(v(x_1), \dots, v(x_n)\big)?$$

- **Answer:** Yes, if the attributes are
  – Mutually preferentially independent and
  – Difference independent

Summary

- Under certain axioms, the DM's preferences over changes on a measurement scale can be captured by a cardinal (measurable) value function
  - "I prefer a change from 0 euros to 10 euros to a change from 10 euros to 22 euros"

- Elicitation of the attribute-specific value functions
  - Use indifference methods if possible