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Brief Recap of Chapter 3 of Brown et al. (2014)

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Contents of Chapter 3 ”Rotating Reference Frames and Resonance”

Introduction

- **Interaction of magnetic moment with external magnetic field can be seen as rotation about the field**
- **In static field, it is constant precession with Larmor frequency**
- **We consider adding a perpendicular, radiofrequency (rf) field**
- **The rf field tips the magnetic moment away from the static field**
- **After tipping, the magnetic moments precess at an angle w.r.t. static field.**
- **The rf field (pulse) is typically tuned to the Larmor frequency**
- **Useful to work in rotating frame**

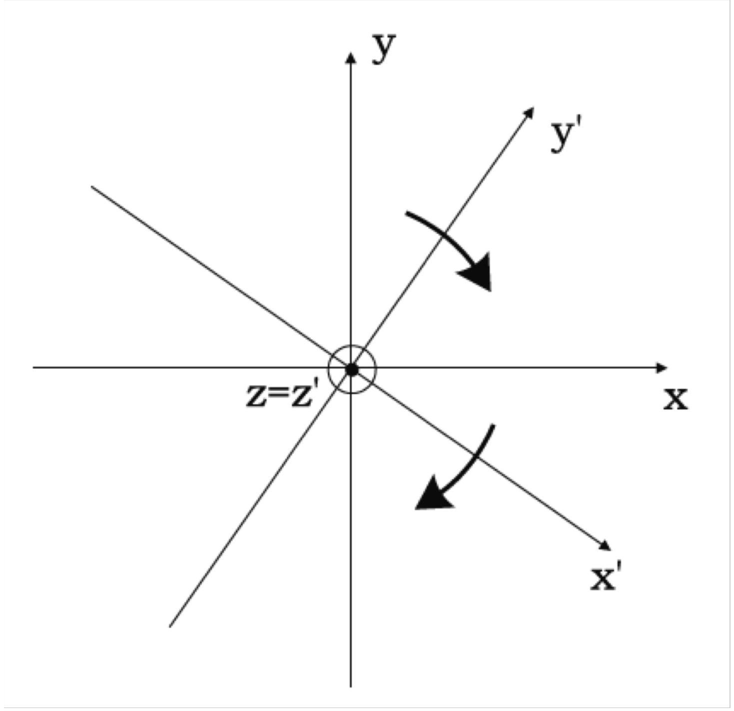
Rotating Reference Frames

- **Consider a reference frame (*x'*,*y'*,*z'*) rotating with Larmor frequency**
- **The rotation is clockwise, around *z*-axis**
- **In this frame the spin appears static**
- **Mathematically, rotation is described as**

$$\frac{d\vec{C}}{dt} = \vec{\Omega} \times \vec{C}$$

- **In our case we have**

$$\vec{\Omega} = -\gamma B_0 \hat{z}$$



Rotating Reference Frames

- **Recall that in the laboratory frame we have**

$$\frac{d\vec{\mu}}{dt} = \gamma\vec{\mu} \times \vec{B}$$

- **For the rotating frame we have**

$$\frac{d\vec{\mu}}{dt} = \left(\frac{d\vec{\mu}}{dt}\right)' + \vec{\Omega} \times \vec{\mu}$$

- **This gives the equation for the magnetic moment in rotating frame:**

$$\left(\frac{d\vec{\mu}}{dt}\right)' = \gamma\vec{\mu} \times \vec{B}_{eff}$$

- **The 'effective magnetic field' is**

$$\vec{B}_{eff} = \vec{B} + \frac{\vec{\Omega}}{\gamma}$$

- **When** $\vec{\Omega} = -\gamma B_0 \hat{z}$ **this gives** $(d\vec{\mu}/dt)' = 0$

Linearly and circularly polarized rf fields

- Assume that a proton spin is aligned with B_0 z'.
- We wish to use an rf field B_1 to tip the spin.
- A linearly polarized field has the form

$$\vec{B}_1^{lin} = b_1^{lin} \cos\omega t \; \hat{x}$$

- In rotating coordinate system this is

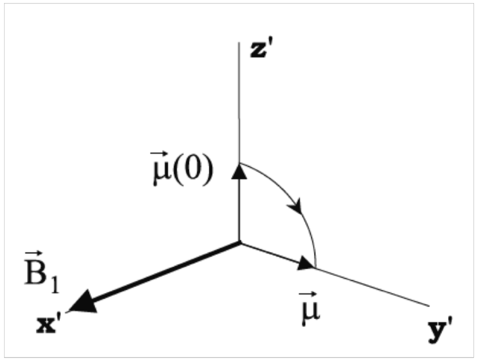
$$\vec{B}_1^{lin} = \frac{1}{2} b_1^{lin} [\hat{x}'(1 + \cos 2\omega t) + \hat{y}' \sin 2\omega t]$$

- Bad, because only half of energy is available for tipping (see the book): $< \vec{B}_1^{lin} >_{\text{primed}} = \frac{1}{2} b_1^{lin} \hat{x}'$
- A left-circularly polarized (quadrature) field has the form

$$\vec{B}_1^{cir} = B_1(\hat{x} \; \cos\omega t - \hat{y} \; \sin\omega t)$$

- In primed frame this is the following (which is good):

$$\vec{B}_1^{cir} = B_1 \hat{x}'$$



Resonance Condition and the RF Pulse

- **Assume a circularly polarized rf pulse with angular rate ω.**
- **The magnetic moment equation is then**

$$\begin{aligned}\left(\frac{d\vec{\mu}}{dt}\right)' &= \vec{\mu} \times [\hat{z}'(\omega_0 - \omega) + \hat{x}'\omega_1] \\ &= \gamma\vec{\mu} \times \vec{B}_{eff}\end{aligned}$$

- **Where $\omega_0 = \gamma B_0$ and $\omega_1 \equiv \gamma B_1$**
- **The effective magnetic field is**

$$\vec{B}_{eff} \equiv [\hat{z}'(\omega_0 - \omega) + \hat{x}'\omega_1]/\gamma$$

- **When rf pulse is in resonance, we have $\omega = \omega_0$**
- **Then the *z'*-term above disappears.**

Flip-Angle Formula and Illustration

- **The cornerstone equation of motion:**

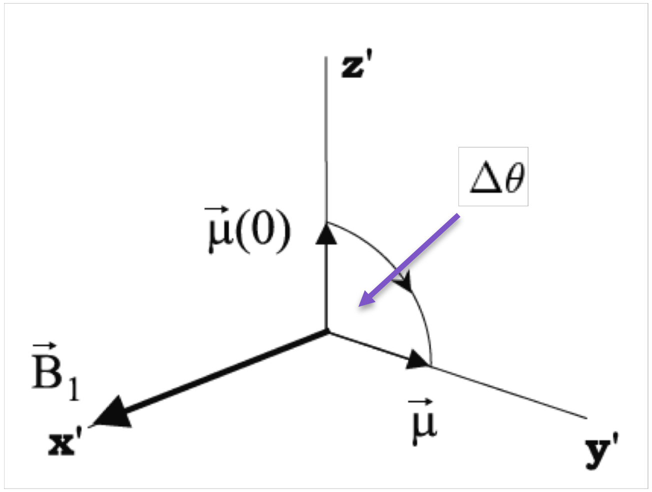
$$\left(\frac{d\vec{\mu}}{dt}\right)' = \omega_1 \vec{\mu} \times \hat{x}' \qquad \text{(when } \omega = \omega_0\text{)}$$

- **Rotates with angular velocity** $\omega_1 \equiv \gamma B_1$ **around** ***x'*-axis.**
- **When applied for time** τ **we get the flip angle**

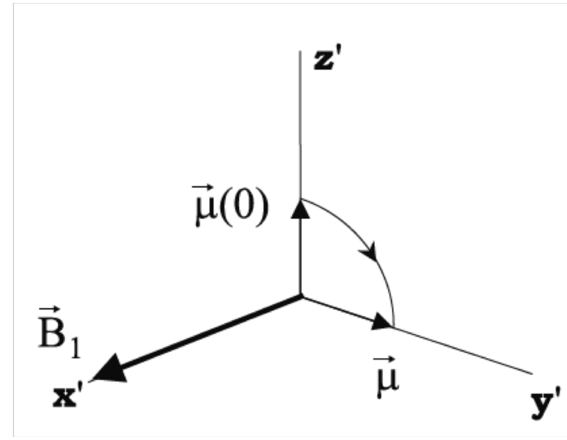
$$\Delta\theta = \gamma B_1 \tau$$

- **For example, 1.0 ms pulse of strength 5.9 μT gives approximately 90o flip angle:**

2.675e8 * 5.9e-6 * 1e-3 * 180 / pi = 90.4



RF Solutions



- In rotating frame we now have

$$\vec{B}_1 = B_1 \hat{x}'$$

- Thus the magnetic moment is given by

$$\left(\frac{d\vec{\mu}}{dt}\right)' = \omega_1 \vec{\mu} \times \hat{x}'$$

- The solution to this equation can be expressed in forms

$$\begin{aligned} \mu_{x'}(t) &= \mu_{x'}(0) \\ \mu_{y'}(t) &= \mu_{y'}(0)\cos\phi_1(t) + \mu_{z'}(0)\sin\phi_1(t) \\ \mu_{z'}(t) &= -\mu_{y'}(0)\sin\phi_1(t) + \mu_{z'}(0)\cos\phi_1(t) \end{aligned}$$

$$\phi_1(t) = \omega_1 t$$

$$\vec{\mu}(t) = R_{x'}(\phi_1(t))\vec{\mu}(0)$$

$$R_{x'}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

- More generally we have $\phi_1(t) = \int_{t_0}^{t} dt' \omega_1(t')$ and $\omega_1(t) = \gamma B_1(t)$

Home Work Problem 1

Problem 3.1

A static field points uniformly along the positive z-axis. A classical spinning particle, with positive gyromagnetic ratio γ and fixed magnetic moment magnitude μ, has its spin initially in the direction of the static field. A circularly polarized rf field points along the $\hat{y}'$ axis with time-dependent amplitude $B_{1y'}(t)$ (e.g., the rf field may be turned off at a later time) applied on-resonance starting at $t = 0$.

a) Give expressions analogous to Equation (3.33) on p. 46 for all three magnetic-moment vector components in the rotating (prime) reference frame for $t > 0$. Your answer will be in terms of a definite integral.

b) Show that the equation of motion (2.24) on p. 28 is satisfied by your answer in (a) for $\vec{B} \to B_{1y'}\hat{y}'$.

c) Find the generalization of Equation (2.35) on p. 33 needed for this time-dependent case.

Home Work Problem 1 (Eqs.)

$$
\begin{aligned}
\mu_{x'}(t) &= \mu_{x'}(0) \\
\mu_{y'}(t) &= \mu_{y'}(0)\cos\phi_1(t) + \mu_{z'}(0)\sin\phi_1(t) \\
\mu_{z'}(t) &= -\mu_{y'}(0)\sin\phi_1(t) + \mu_{z'}(0)\cos\phi_1(t)
\end{aligned} \tag{3.33}
$$

with

$$\phi_1(t) = \omega_1 t \tag{3.34}$$

$$\frac{d\vec{\mu}}{dt} = \gamma\vec{\mu} \times \vec{B} \tag{2.24}$$

$$
\begin{aligned}
\frac{d^2\mu_x}{dt^2} &= -\omega_0^2\mu_x \\
\frac{d^2\mu_y}{dt^2} &= -\omega_0^2\mu_y
\end{aligned} \tag{2.35}
$$

Different Polarization Bases and Representations

- **The left-circular polarization that we have is**

$$\hat{x}^{left} = \hat{x}\ \cos\omega t - \hat{y}\ \sin\omega t = \hat{x}'$$

- **We could also consider right-circular version:**

$$\hat{x}^{right} = \hat{x}\ \cos\omega t + \hat{y}\ \sin\omega t$$

- **This turns out to average to zero and is thus useless.**
- **We can also express these in complex form:**

$$B_1^{left} \equiv B_1^L = B_1 e^{-i\omega t}$$

$$B_1^{right} \equiv B_1^R = B_1 e^{i\omega t}$$

- **The linear polarization is then given as**

$$B_1^{lin} = B_1^L + B_1^R = 2B_1 \cos\omega t$$

Laboratory Angle of Precession

- **For off-resonance we have**

$$\vec{B}_{eff} = (B_0 - \frac{\omega}{\gamma})\hat{z} + B_1\hat{x}'$$

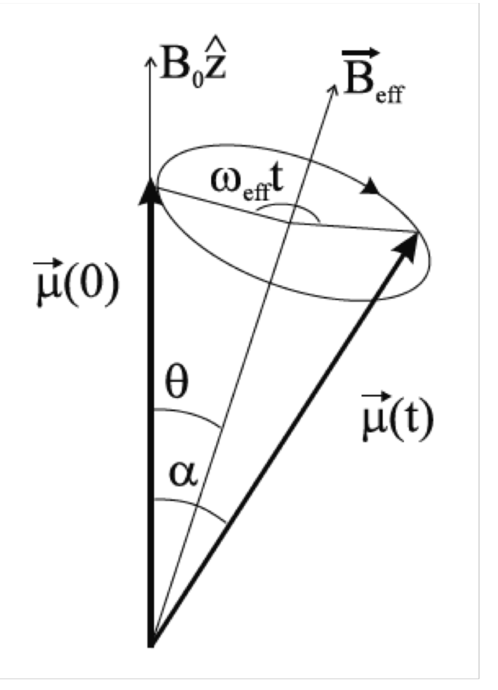
- **The angle between B_{eff} and B_0:**

$$\cos\theta = \frac{B_0 - \omega/\gamma}{B_{eff}} = \frac{\omega_0 - \omega}{\omega_{eff}}$$

$$\sin\theta = \frac{\omega_1}{\omega_{eff}}$$

$$\omega_{eff} = \gamma B_{eff} = \gamma\sqrt{(B_0 - \omega/\gamma)^2 + B_1^2} = \sqrt{(\omega_0 - \omega)^2 + \omega_1^2}$$

- **The magnetic moment precesses around B_{eff} with angular frequency ω_{eff}**



Home Work Problem 2

Problem 3.2

Show that

$$\hat{x}^{right} = \hat{x}' \cos 2\omega t + \hat{y}' \sin 2\omega t$$

using steps like those used in deriving (3.21). Also show that the time average

$$\frac{1}{T}\int_0^T \hat{x}^{right}(t)\mathrm{d}t$$

approaches zero as $T \to \infty$.