Def. Let A, B be groups with identities e and f, respectively.
On the cartesian product $A \times B = \{(a,b) \mid a \in A, b \in B\}$ we define a binary operation $*$ by

$$(a,b) * (a',b') = (aa', bb')$$

for all $(a,b), (a',b') \in A \times B$.

We easily verify that $*$ is associative because it is so on the components, and the identity is given by (e,f). We ~~denote~~ leave the inverse $(a,b)^{-1}$ to be (a^{-1}, b^{-1}) and denote the resulting group by the symbol $A \boxplus B$, calling it the outer direct product of A and B.

As a matter of fact, $A \boxplus B$ contains isomorphic copies of A and B as subgroups, namely the groups $\tilde{A} := A \times \{f\}$ and $\tilde{B} := \{e\} \times B$.

<u>Proposition</u>: Let $A \boxplus B$ and $\tilde{A}, \tilde{B}$ be given as above

a) $\tilde{A} \cap \tilde{B} = \{(e,f)\}$

b) $x * y = y * x$ for all $x \in \tilde{A}$ and $y \in \tilde{B}$

c) $\tilde{A}$ and $\tilde{B}$ are normal subgroups of $A \boxplus B$

d) $\tilde{A} * \tilde{B} = \tilde{B} * A = A \boxplus B$.

Proof. a) $\tilde{A} \cap \tilde{B} = \{(a,b) \mid a \in A, b \in B \text{ with } a = e, b = f\}$

$= \{(e,f)\}$.

b) $x = (a,f)$ and $y = (e,b)$, then we have

$x * y = (a,f) * (e,b) = (a,b) = (e,b) * (a,f) = y * x$.

c) $x^{-1} * \tilde{A} * x = (a^{-1}, b^{-1}) * (A \times \{f\}) * (a,b) = (a^{-1} A a) \times (b^{-1}\{f\} b)$

$= A \times \{f\} = \tilde{A}$

similar for $\tilde{B}$ and hence $\tilde{A}, \tilde{B} \trianglelefteq A \boxplus B$

d) $A \boxplus B = \{(a,b) \mid a \in A, b \in B\} \overset{(b)}{=} \{x * y \mid x \in \tilde{A}, y \in \tilde{B}\}$

$= \tilde{A}\tilde{B} \overset{(b)}{=} \tilde{B}\tilde{A}$

We will now consider groups G with two subgroups $A, B \leq G$ such that a), c) and finally also d) of the previous proposition are satisfied. We will see, that b) is then a consequence, and even more...

<u>Lemma</u>. Assume $A, B \leq G$ such that the identity of G is d and

(i) $A \cap B = \{d\}$

(ii) $A, B \trianglelefteq G$

Then $ab = ba$ for all $a \in A$ and $b \in B$

<u>Proof</u>. $\left.\begin{array}{l} ab a^{-1} b^{-1} = \underbrace{a}_{\in A} \underbrace{(b a^{-1} b^{-1})}_{\in A} \in A \\ = \underbrace{(a b a^{-1})}_{\in B} \underbrace{b^{-1}}_{\in B} \in B \end{array}\right\} A \cap B = \{d\}$

$\rightarrow ab a^{-1} b^{-1} = d \longrightarrow ab = ba$

proposition. Let G be a group with identity d. (41)

Let A, B normal subgroups of G with

(i) $A \cap B = \{d\}$

(ii) $AB = G$ $(= BA)$

Then there is an isomorphism between $A \boxplus B$ and G.

proof: We will need the following fact: If $ab = d$, then $a = d = b$. In fact, if we have $ab = d$ for some $a \in A$ and $b \in B$, then $a = b^{-1}$, which is in A and also in B, ie $\in A \cap B = \{d\}$. This shows $a = d$ and $b^{-1} = d$ and hence $b = d$.

Now consider the mapping $A \boxplus B \longrightarrow G$,
$(a,b) \mapsto ab$.

(i) φ is a homomorphism. $\varphi((a,b) * (a',b')) =$
$\varphi(aa', bb') = aa'bb' \underset{\substack{\uparrow \\ \text{lemma!}}}{\overset{!}{=}} aba'b' = \varphi(a,b) \cdot \varphi(a',b')$

(ii) φ is injective. $\varphi(a,b) = d$ means $ab = d$ and hence $a = d$ and $b = d$, so $(a,b) = (d,d)$ so $\ker(\varphi) = \{(d,d)\}$.

(iii) φ is surjective $\varphi(A \boxplus B) = \{\varphi(a,b) \mid (a,b) \in A \times B\}$
$= \{ab \mid a \in A, b \in B\} = AB = G$.

So we finally have $A \boxplus B \cong G$

Def: Whenever we have a group G and two normal subgroups A, B such that (i) $A \cap B = \{id\}$ and (ii) $AB = G$ $(= BA)$, then we will call G the inner direct sum of A and B and write $G = A \oplus B$.

The foregoing proposition therefore says that $A \oplus B \cong A \boxplus B$ in this case, and of course, we can revisit our initial observation about the role of $\tilde{A} = A \times \{f\}$ and $\tilde{B} = \{e\} \times B$ in the outer direct product $A \boxplus B$.
We then see that $\tilde{A} \oplus \tilde{B} = A \boxplus B$. The latter is not an isomorphism, but true equality.

Remark a) Things can be extended in obvious ways. For $A_1, A_2, \ldots A_n$ given groups we form $A_1 \boxplus A_2 \boxplus \cdots \boxplus A_n$ by componentwise operation, to arrive at the direct product of these groups.

b) If we have normal subgroups $A_1, \ldots, A_n \trianglelefteq G$ then we need the conditions (i) $\prod_{i=1}^{n} A_i = G$ and (ii) $A_i \cap \prod_{j \neq i} A_j = \{id\}$ for all $i = 1, \ldots, n$.

We then can write $G = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ 43
and we will find an isomorphism φ that
relates $A_1 \boxplus \cdots \boxplus A_n$ and $A_1 \oplus \cdots \oplus A_n$.