Aalto University
School of Science

Decision making and problem solving – Lecture 5

- Preferential and difference independence
- Aggregation of values with an additive value function
- Interpretation and elicitation of attribute weights
- Trade-off methods
- SWING, SMART(S)

Liesiö, Punkka, Salo, Vilkkumaa

Last time

- Given certain axioms, a DM's preferences about a single attribute can be represented by a cardinal value function $v_i(x_i)$ such that

$$v_i(x_i) \geq v_i(y_i) \Leftrightarrow x_i \succcurlyeq y_i$$

$$v_i(x_i) - v_i(x_i') \geq v_i(y_i) - v_i(y_i') \Leftrightarrow (x_i \leftarrow x_i') \succcurlyeq_d (y_i \leftarrow y_i').$$

- Attribute-specific value functions are obtained by
  - Defining measurement scales $[x_i^0, x_i^*]$
  - Asking a series of elicitation questions through, e.g.,
    1. Bisection method
    2. Equally preferred differences
    3. Giving a functional form; e.g., $v_i(x_i)$ is linear and increasing
- Result: **<u>shape</u>** of the value function

- Value functions can be normalized such that $v_i(x_i^0)$ = 0 and $v_i(x_i^*)$ = 1.

This time

- How to measure the ***overall value*** of multi-attribute alternative $x = (x_1, x_2, \dots x_n)$?

$$V(x_1, x_2, \dots x_n) = ?$$

- Could the overall value be obtained by aggregating attribute-specific values?

$$V(x_1, x_2, \dots x_n) = f\big(v(x_1), \dots, v(x_n)\big) = \sum_{i=1}^{n} w_i v_i^N(x_i)\,?$$

- **Answer:** Yes, if the attributes are
  - Mutually preferentially independent and
  - Difference independent

- … But how to interpret and elicit ***attribute weights*** w_i?

Preferential independence

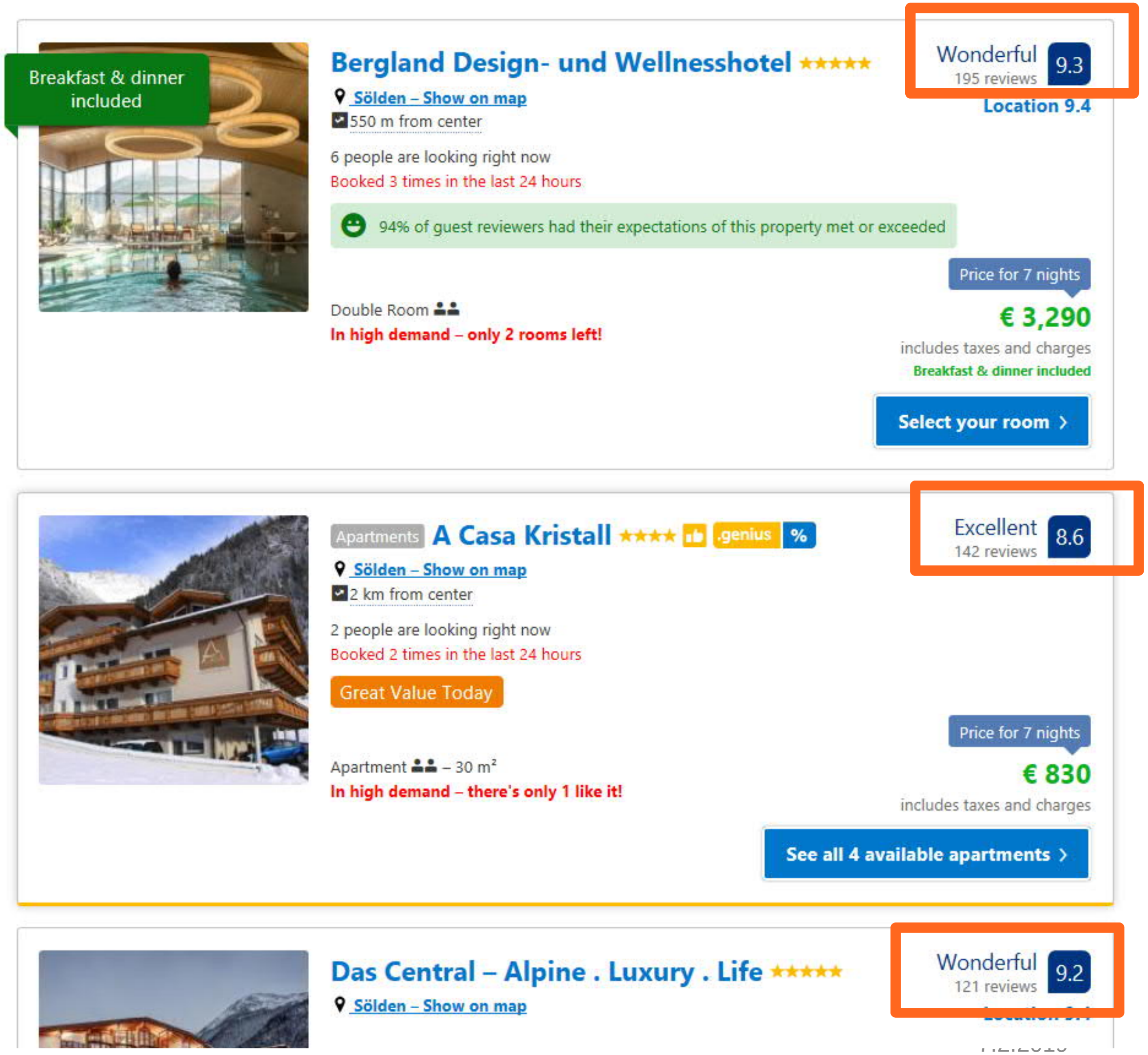
- **Definition:** Attribute X is preferentially independent of the other attributes $\boldsymbol{Y}$, if for all $x,x' \in X$

$$(x,\boldsymbol{y}') \succcurlyeq (x',\boldsymbol{y}') \Rightarrow (x,\boldsymbol{y}) \succcurlyeq (x',\boldsymbol{y}) \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}$$

- Interpretation: Preference over the level of attribute X does not depend on the levels of the other attributes, as long as they stay the same
  - "All other things Y being equal (no matter what they are), an alternative with performance level x w.r.t. X is preferred to an alternative with level x' ∈ X"

Last time

- ❑ **Consider yourself choosing accommodation for a (downhill) skiing vacation trip**
- ❑ **How do the accommodation alternatives differ from each other?**
  - ❑ What are the attributes that influence your decision?



Aalto University
School of Science

Preferential independence: example 1

- Attribute *X* is preferentially independent of the other attributes ***Y***, if for all $x, x' \in X$

$$(x, \boldsymbol{y}') \succcurlyeq (x', \boldsymbol{y}') \Rightarrow (x, \boldsymbol{y}) \succcurlyeq (x', \boldsymbol{y}) \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}$$

- 2 Attributes
  - X={1,...,500} number of reviews
  - Y=[1,10] average of reviews
- Is *X* preferentially independent of *Y*?
  - No: $(500,10) \succcurlyeq (5,10)$, but $(500,1) \prec (5,1)$
- Is *Y* preferentially independent of *X*?
  - Yes (if higher average is preferred independently of #reviews, as long there are equally many reviews): $(500,10) \succcurlyeq (500,9) \Rightarrow (x,10) \succcurlyeq (x,9)$ for any x

Preferential independence: example 2

- Consider choosing a meal using two attributes:
  1. Food ∈ {beef, fish}
  2. Wine ∈ {red, white}
- Preferences:
  1. Beef is preferred to fish (no matter what the wine is):
     - (beef, red) ≽ (fish, red)
     - (beef, white) ≽ (fish, white)
  2. White wine is preferred with fish and red wine with beef
     - (fish, white) ≽ (fish, red)
     - (beef, red) ≽ (beef, white)
- Food **is** preferentially independent of wine
  - Beef is preferred to fish, no matter what the wine is: $(x, \boldsymbol{y}') \succcurlyeq (x', \boldsymbol{y}') \Rightarrow (x, \boldsymbol{y}) \succcurlyeq (x', \boldsymbol{y})$ for all y ∈ Y
- Wine **is not** preferentially independent of food
  - Attribute-specific valuation of wine is not meaningful from the meal's perspective

Mutual preferential independence

- **Definition:** Attributes A are mutually perferentially independent, if any subset of attributes **X**⊂A is preferentially independent of the other attributes **Y**=A**X**. I.e., for any **X**⊂A**, Y**=A**X:**

$$(\boldsymbol{x},\boldsymbol{y}') \succcurlyeq (\boldsymbol{x}',\boldsymbol{y}') \Rightarrow (\boldsymbol{x},\boldsymbol{y}) \succcurlyeq (\boldsymbol{x}',\boldsymbol{y}) \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}$$

- Interpretation: Preference over the levels of attributes ***X*** does not depend on the levels of the other attributes, as long as they stay the same

Mutual preferential independence: example

- Consider choosing a meal using three attributes:
  1. Food ∈ {beef, fish}
  2. Side dish ∈ {potato, rice}
  3. Wine ∈ {red, white}
- Preferences:
  1. All other things being equal, red ≽ white, beef ≽ fish, potato ≽ rice
  2. Full meals:
     - (beef, rice, red)≽(beef, potato, white)
     - (fish, potato, white) ≽ (fish, rice, red)

  Each attribute is preferentially independent of the other two, but the attributes are not mutually preferentially independent:

  $$(\boldsymbol{y}', \boldsymbol{potato}, \boldsymbol{white}) \succcurlyeq (\boldsymbol{y}', \boldsymbol{rice}, \boldsymbol{red}) \not\Rightarrow (y, \boldsymbol{potato}, \boldsymbol{white}) \succcurlyeq (\boldsymbol{y}, \boldsymbol{rice}, \boldsymbol{red})$$

Mutual pref. independence: example 2

- Choosing a car w.r.t. attributes *A*={*top speed, price,* CO_2 *emissions*}
  - Attributes defined on continuous scales
- Are all *A*'s subsets (X) preferentially independent of the other attributes (Y=A\X)?
- Each single attribute is preferentially independent of the other attributes, because
  - Lower price is preferred to higher price independent of other attributes (if other attributes are equal)
  - Higher top speed is preferred to lower
  - Smaller emissions are preferred to bigger ones

Mutual pref. independence: example 2

- Is X={*price*, *CO_2 emissions*} pref. independent of Y={*top speed*}?
  - Consider two cars which differ in price (e.g., 30000 e, 25000 e) and emissions (150 g/km, 200 g/km) so that one of the alternatives is better in emissions and the other in price. Set the same top speed for the alternatives (e.g. 230 km/h). Which one is better?
    - DM says (230 km/h, 30000 e, 150 g/km) $\succ$ (230 km/h, 25000 e, 200 g/km)
    - = when top speed is 230 km/h, she is willing to pay extra 5000 € on top of 25000 € for this emission reduction
  - Change the top speed. Is the first car still preferred to the second? e.g. does (150 km/h, 30000 e, 150 g/km) $\succ$ (150 km/h, 25000 e, 200 g/km) hold?
    - "No matter what the top speed is, (30000 e, 150 g/km) $\succ$ (25000 e, 200 g/km)"
  - Consider other prices and emissions; does your preference hold for all top speeds?
  - If varying the top speed does not influence preference between alternatives, then {price, CO_2 emissions} is preference independent of {top speed}

Difference independence

- **Definition**: Attribute *X* is difference independent of the other attributes ***Y*** if for all *x*,*x*' ∈ X

$$(x, \boldsymbol{y}') \leftarrow (x', \boldsymbol{y}') \sim_d (x, \boldsymbol{y}) \leftarrow (x', \boldsymbol{y}) \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}$$

- Interpretation: The preference over a <u>change</u> in attribute X does not depend on the levels of the other attributes **Y**, as long as they stay the same

Difference independence: example

- Is {*top speed*} difference independent of the other attributes {*price, CO_2 emissions*}?
  - Construct y and y' from any two levels of price and CO_2 emissions; y=(25000 e, 150 g/km) and y'=(30000 e, 200 g/km)
  - Consider any two levels of top speed; x'=200 km/h, x=250 km/h
  - Does (250 km/h, 30000 e, 200 g/km) $\leftarrow$ (200 km/h, 30000 e, 200 g/km) $\sim_d$ (250 km/h, 25000 e, 150 g/km) $\leftarrow$ (200 km/h, 25000 e, 150 g/km) hold?
    - If yes (for all *x*,*x'*,***y***,***y'***), then difference independence holds
    - That is, does the value of increased top speed depend on the levels of other attributes or not?
    - Is the "amount of" value added by a fixed change in top speed independent of the other attributes?

Difference independence: example of implication

- We are choosing downhill skiing accommodation with regard to 6 attributes, which include *cost per night* (in €) and *possibility to go to sauna* (binary)
  - We think that (170 e, sauna, x_3, x_4, ...)~(145 e, no sauna, x_3, x_4, ...) with some $x_3,\dots,x_6$ = we would pay an additional 25 € on top of 145 € for the sauna, with some $x_3,\dots,x_6$
  - Then, if difference independence holds (for each attribute):

    (145 e, no sauna, x_3, x_4, ...) ← (170e, no sauna, x_3, x_4, ...) $\sim_d$

    (170 e, sauna, x_3, x_4, ...) ← (170 e, no sauna, x_3, x_4, ...) for any $x_3,\dots,x_6$
  - For any $x_3,\dots,x_6$ = ”No matter how close to nearest ski lifts , no matter how fancy the breakfast, how bad the reviews, etc.”

Implication: “the improvement needed in an attribute to compensate a loss in another attribute does not depend on the levels of other attributes”

Additive value function

Theorem: If all attributes are mutually preferentially independent and each attribute is difference independent of the others, then there exists an additive value function

$$V(x) = V(x_1, \dots, x_n) = \sum_{i=1}^{n} v_i(x_i)$$

which represents preference relations $\succcurlyeq$, $\succcurlyeq_d$ in the sense that

$$V(x) \geq V(y) \Leftrightarrow x \succcurlyeq y$$

$$V(x) - V(x') \geq V(y) - V(y') \Leftrightarrow (x \leftarrow x') \succcurlyeq_d (y \leftarrow y')$$

Note: The additive value function is unique up to positive affine transformations, i.e., V(x) and V'(x)=αV(x)+β, α>0 represent the same preferences

… But where are the attribute weights w_i?

Theorem: If all attributes are (…) , then there exists an additive value function

$$V(x) = V(x_1, \dots, x_n) = \sum_{i=1}^{n} v_i(x_i)$$

- **Slide 3:** Could the overall value be obtained by aggregating attribute-specific values?

$$V(x_1, x_2, \dots x_n) = f\big(v(x_1), \dots, v(x_n)\big) = \sum_{i=1}^{n} w_i v_i^N(x_i)\,?$$

Normalized form of the additive value function $V(x) = V(x_1, \dots, x_n) = \sum_{i=1}^{n} v_i(x_i)$

- **Denote**
  - x_i^0 = Least preferred level w.r.t to attribute i
  - x_i^* = Most preferred level w.r.t to attribute i
- **Then,**

$$V(x) = V(x) - V(x^0) + V(x^0)$$

$$= \sum_{i=1}^{n} v_i(x_i) - \sum_{i=1}^{n} v_i(x_i^0) + V(x^0) = \sum_{i=1}^{n} [v_i(x_i) - v_i(x_i^0)] + V(x^0)$$

$$= \sum_{i=1}^{n} \underbrace{[v_i(x_i^*) - v_i(x_i^0)]}_{W_i > 0} \frac{v_i(x_i) - v_i(x_i^0)}{v_i(x_i^*) - v_i(x_i^0)} + V(x^0)$$

$$= \sum_{i=1}^{n} W_i \left[\underbrace{\frac{1}{v_i(x_i^*) - v_i(x_i^0)}}_{\alpha_i > 0} v_i(x_i) + \underbrace{\frac{-v_i(x_i^0)}{v_i(x_i^*) - v_i(x_i^0)}}_{\beta_i} \right] + V(x^0) \dots$$

Normalized form of the additive value function (cont'd)

$$\ldots = \sum_{i=1}^{n} W_i \underbrace{[\alpha_i v_i(x_i) + \beta_i]}_{v_i^N \in [0,1]} + V(x^0)$$

Normalized attribute-specific value function $v_i^N(x_i) \in [0,1]$

$$= \sum_{i=1}^{n} \left[\left(\sum_{i=1}^{n} W_i\right) \cdot \underbrace{\frac{W_i}{\sum_{i=1}^{n} W_i}}_{= w_i > 0, \sum_{i=1}^{n} w_i = 1} \cdot v_i^N(x_i) \right] + V(x^0)$$

$$= \underbrace{\left(\sum_{i=1}^{n} W_i\right)}_{\chi > 0} \underbrace{\sum_{i=1}^{n} w_i v_i^N(x_i)}_{V^N(x)} + \underbrace{V(x^0)}_{\delta}$$

$$= \chi V^N(x) + \delta$$

Normalized additive value function $V^N(x) = \sum_{i=1}^{n} w_i v_i^N(x_i) \in [0,1]$

$V(x) = \chi V^N(x) + \delta$ is a positive affine transformation of $V^N(x)$; they represent the same preferences!

Interpretation of attribute weights

- By definition, $w_i = \frac{W_i}{\sum_{i=1}^{n} W_i} = \frac{v_i(x_i^*) - v_i(x_i^0)}{\sum_{i=1}^{n}(v_i(x_i^*) - v_i(x_i^0))} \propto v_i(x_i^*) - v_i(x_i^0)$

- Attribute weight w_i reflects the increase in overall value when the performance level on attribute a_i is changed from the worst level to the best – <u>relative to similar changes in other attributes</u>

- Weights thus reflect *trade-offs* between attributes; not their absolute ”importance”

- Elicitation of attribute weights without this interpretation is not meaningful
  - Do not ask: ”What is more important: environment or economy?”
  - Do ask: ”How much is society willing to pay to save an insect species?”

Interpretation of attribute weights

- Correct interpretation and hence application of the weights may lead to 'resistance'
  - Let the least preferred and the most preferred levels in
    - cost savings be 0 € and 1 B€ ("money")
    - the number of insect species saved from extinction in Finland be 0 and 1 ("environmental aspects")
    - Environmental aspects are likely to receive a small weight, as for example weighting (0.5, 0.5) would mean that we equally prefer saving 1 B€ and saving 1 species
  - Cf. .... Let the least preferred and the most preferred levels in
    - cost savings be 0 € and 1 B€
    - the number of insect species saved from extinction in Finland be 0 and 100

Conditions

- What if the conditions (mutual preferential independence and difference independence) do not hold?
  - Reconsider the attribute ranges $[a_i^0, a_i^*]$; conditions are more likely fulfilled when the ranges are small
  - Reconsider the attributes; are you using the right measures?

- Even if the conditions do not hold, additive value function is often used to obtain approximate results

Example (Ewing et al. 2006*): military value of an installation

- “How to realign US Army units and which bases to close in order to operate more cost-efficiently?”

- Many attributes, including ”total heavy maneuver area” (x_1) and ”largest contiguous area” (x_2; a measure of heavy maneuver area quality)

  - ”Total heavy maneuver area” is not difference independent of the other attributes $x_2 \cup \boldsymbol{y}''$ because (1000 ha, 100 ha, y'') $\leftarrow$ (100 ha, 100 ha, y'') $\sim_d$ (1000 ha, 10 ha, y'') $\leftarrow$ (100 ha, 10 ha, y'') as the ncrease from 100 to 1000 ha in total area is found quite useless, if total area consists of over 100 small isolated pieces of land

* Ewing, Tarantino, Parnell (2006): *Use of Decision Analysis in the Army Base Realignment and Closure (BRAC) 2005 Military Value Analysis*. Decision Analysis 3, 33-49

Example (Ewing et al. 2006*): military value of an installation

- Solution: unite the two attributes x_1 and x_2 into one attribute ”*heavy maneuver area*”
  - Then (1000 ha, 100 ha, Y) ← (100 ha, 100 ha, Y) $\succ_d$ (1000 ha, 10 ha, Y) ← (100 ha, 10 ha, Y) does not violate required difference independence conditions $(x, \boldsymbol{y}') \leftarrow (x', \boldsymbol{y}') \sim_d (x, \boldsymbol{y}) \leftarrow (x', \boldsymbol{y})$ for all $\boldsymbol{y} \in \boldsymbol{Y}$, because x_2 is no longer an element of y or y'
  - BUT we need to elicit preferences between different ’pairs’ (x_1, x_2)

Largest contiguous area (1,000s acres)	Total heavy maneuver area (1,000s acres)			
	≤10	>10 and ≤50	>50 and ≤100	>100
≤10	0.1	0.2	1.4	2.0
>10 and ≤50		3.2	4.3	5.2
>50 and ≤100			6.1	7.6
>100				10.0

Elicitation of attribute weights

- Attribute weights are derived from the DM's preference statements

- Approaches to eliciting attribute weights:
  - Trade-off weighting
  - ”Lighter” techniques: SWING, SMART(S), and ordinal methods

Trade-off weighting

- The DM is asked to
  1. Set the performance levels of two imaginary alternatives x and y such that they are equally preferred (x~y):

$$w_1 v_1^N(x_1) + \cdots + w_n v_n^N(x_n) = w_1 v_1^N(y_1) + \cdots + w_n v_n^N(y_n), \text{ or}$$

  2. Set the performance levels of four imaginary alternatives x, x', y, and y' such that changes x ← x' and y ← y' are equally preferred ($x \leftarrow x' \sim_d y \leftarrow y'$):

$$w_1(v_1^N(x_1) - v_1^N(x_1')) + \cdots + w_n(v_n^N(x_n) - v_n^N(x_n')) = w_1(v_1^N(y_1) - v_1^N(y_1')) + \cdots + w_n(v_n^N(y_n) - v_n^N(y_n'))$$

Trade-off weighting

- n-1 pairs of equally preferred alternatives/changes → n-1 linear constraints + 1 normalization constraint

- If the pairs are suitably selected (no linear dependencies), the system of n linear constraints has a unique solution
  - E.g., select a reference attribute and compare the other attributes against it
  - E.g., compare the ”most important” attribute to the second most important, the second most important to the third most important etc

Trade-off weighting: example (1/7)

- Consider two magazines A and B reporting a comparison of cars x^1, x^2, and x^3, **based on the same expert appraisal**, using the **same attributes**:

	a_1: Top speed km/h	a_2: Acceleration 0-100 km/h	a_3: CO_2 emissions g/km	a_4: Maintenance costs €/year
x^1	192 km/h	12.0 s	120 g/km	400 €/year
x^2	200 km/h	10.4 s	140 g/km	500 €/year
x^3	220 km/h	8.2 s	150 g/km	600 €/year

Trade-off weighting: example (2/7)

- Consider changing top speed (reference attribute) **from 150 to 250 km/h**. All other things being equal, what would be an equally preferred change in
  - Acceleration time? Expert's answer: from 14 to 7 s ⇒

$$w_1\left(v_1^N(250)-v_1^N(150)\right)=w_2\left(v_2^N(7)-v_2^N(14)\right)\Rightarrow\frac{w_1}{w_2}=\frac{v_2^N(7)-v_2^N(14)}{v_1^N(250)-v_1^N(150)}$$

  - CO_2 emissions? Expert's answer: from 100 to 0 g/km ⇒

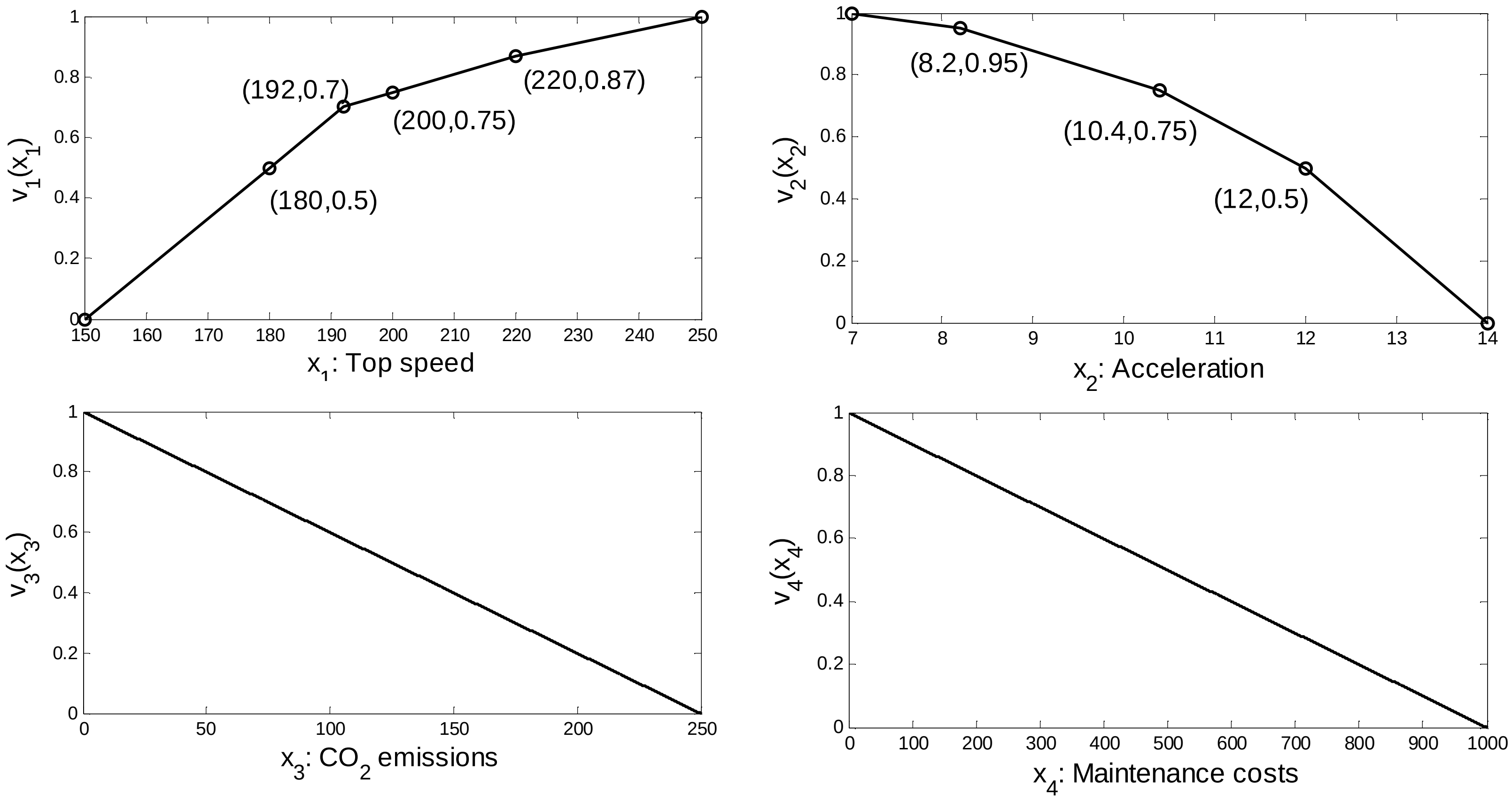
$$w_1\left(v_1^N(250)-v_1^N(150)\right)=w_3\left(v_3^N(0)-v_3^N(100)\right)\Rightarrow\frac{w_1}{w_3}=\frac{v_3^N(0)-v_3^N(100)}{v_1^N(250)-v_1^N(150)}$$

  - Maintenance costs? Expert's answer: from 800 to o €/year ⇒

$$w_1\left(v_1^N(250)-v_1^N(150)\right)=w_4\left(v_4^N(0)-v_4^N(800)\right)\Rightarrow\frac{w_1}{w_4}=\frac{v_4^N(0)-v_4^N(800)}{v_1^N(250)-v_1^N(150)}$$

Trade-off weighting: example (3/7)

- Attribute-specific value functions according to the expert:



Trade-off weighting: example (4/7)

- **Magazine A** uses the following measurement scales:

Attribute	Measurement scale	v_i^N
a_1: Top speed (km/h)	[150, 250]	$v_1^N(180) = 0.5$, $v_1^N(192) = 0.7$, $v_1^N(200) = 0.75$, $v_1^N(220) = 0.87$
a_2: Acceleration time (s)	[7, 14]	$v_2^N(12) = 0.5$, $v_2^N(10.4) = 0.75$, $v_2^N(8.2) = 0.95$
a_3: CO_2 emissions (g/km)	[120, 150]	$5 - x_3/30$
a_4: Maintenance costs (€/year)	[400,600]	$3 - x_4/200$

– $\frac{w_1}{w_2} = \frac{v_2^N(7)-v_2^N(14)}{v_1^N(250)-v_1^N(150)} = 1$

– $\frac{w_1}{w_3} = \frac{v_3^N(0)-v_3^N(100)}{v_1^N(250)-v_1^N(150)} = \frac{\frac{100}{30}(v_3^N(120) - v_3^N(150))}{1} = \frac{10}{3}$

– $\frac{w_1}{w_4} = \frac{v_4^N(0)-v_4^N(800)}{v_1^N(250)-v_1^N(150)} = \frac{\frac{800}{200}(v_3^N(400) - v_3^N(600))}{1} = 4$

- The three equalities and $\sum_{i=1}^4 w_i = 1$ give $w_1 = w_2 = 0.39$, $w_3 = 0.12$, $w_4 = 0.10$.

Trade-off weighting: example (5/7)

- **Magazine A** reports the alternatives' attribute-specific values multiplied by 10 (i.e., scaled to interval [0,10]) and the attribute weights:

	v_1 : Top speed	v_2 : Acceleration	v_3 : CO_2	v_4 : Maintenance	Overall value
x^1	7	5	10	10	6.86
x^2	7.5	7.5	3.3	5	6.76
$\boldsymbol{x^3}$	**8.7**	**9.5**	**0**	**0**	**7.14**
Weights w_i	39%	39%	12%	10%	

- Possible (mis)interpretations / "headlines":
  - "Only power matters – minor emphasis on costs and environment"
  - "Car x^3 terrible w.r.t. CO_2 emissions and maintenance costs – yet, it's the expert's choice!"
  - "No significant differences in top speed – differences are in CO_2 emissions and maintenance costs"

Trade-off weighting: example (6/7)

- **Magazine B** uses the following measurement scales:

Attribute	M. scale	v_i^N
a_1: Top speed	[192, 220]	$v_1^N(150) = -4.12$, $v_1^N(180) = -1.18$, $v_1^N(192) = 0$, $v_1^N(200) = 0.29$, $v_1^N(220) = 1$, $v_1^N(250) = 1.76$
a_2: Acceleration	[8.2, 12]	$v_2^N(14) = -1.11$, $v_2^N(12) = 0$, $v_2^N(10.4) = 0.56$, $v_2^N(8.2) = 1$, $v_2^N(7) = 1.11$
a_3: CO_2 emissions	[0, 250]	$1 - x_3/250$
a_4: Maintenance	[0,1000]	$1 - x_4/1000$

– $w_1\left(v_1^N(250) - v_1^N(150)\right) = w_2\left(v_2^N(7) - v_2^N(14)\right) \Rightarrow \frac{w_1}{w_2} = \frac{v_2^N(7)-v_2^N(14)}{v_1^N(250)-v_1^N(150)} = \frac{1.11+1.11}{1.76+4.12} = 0.378$

– $\frac{w_1}{w_3} = \frac{v_3^N(0)-v_3^N(100)}{v_1^N(250)-v_1^N(150)} = \frac{1-\frac{150}{250}}{1.76+4.12} = 0.068$

– $\frac{w_1}{w_4} = \frac{v_4^N(0)-v_4^N(800)}{v_1^N(250)-v_1^N(150)} = \frac{1-\frac{200}{1000}}{1.76+4.12} = 0.136$

- The three equalities and $\sum_{i=1}^4 w_i = 1$ give $w_1 = 0.039$, $w_2 = 0.103$, $w_3 = 0.572$, $w_4 = 0.286$.

Trade-off weighting: example (7/7)

- **Magazine B** reports the alternatives' attribute-specific values multiplied by 10 (i.e., scaled to interval [0,10]) and the attribute weights:

	v_1 : Top speed	v_2 : Acceleration	v_3 : CO_2	v_4 : Maintenance	Overall value
x^1	0	0	5.2	6	4.7
x^2	2.9	5.6	4.4	5	4.6
$\boldsymbol{x^3}$	**10**	**10**	**4**	**4**	**4.9**
Weights w_i	3.9%	10.3%	57.2%	28.6%	

- Possible (mis)interpretations:
  - ”Emphasis on costs and environmental issues”
  - ”x^3 wins only on the least important attributes – yet, it's the expert's choice!”
  - ”Car x^1 terrible w.r.t. top speed and acceleration time”

Trade-off weighting

- Weights reflect value differences over the measurement scales → changing the measurement scales changes the weights
- The attribute-specific values used in trade-off weighting take the measurement scales explicitly into account → weights represent the DM's preferences regardless of the measurement scales
- Trade-off weighting has a solid theoretical foundation and requires thinking; use whenever possible

SWING

- **Swing-weighting process:**
  1. Consider alternative $x^0 = (x_1^0, \ldots, x_n^0)$ (each attribute on the worst level).
  2. Choose the attribute a_j that you would first like to change to its most preferred level x_j^* (i.e., the attribute for which such a change is the most valuable). Give that attribute a (non-normalized) weight $W_j = 100$.
  3. Consider x^0 again. Choose the next attribute a_k that you would like to change to its most preferred level. Give it weight $W_j \in (0,100]$ that reflects this improvement relative to the first one.
  4. Repeat step 3 until all attributes have been weighted.
  5. Obtain weights w_j by normalizing W_j.

SWING: example

- **Magazine A**'s measurement scales
  - Alternative $x^0 = \left(150\frac{km}{h}, 14s, 150\frac{g}{km}, 600\frac{€}{year}\right)$
  - The first attribute to be changed from the worst to the best level: $a_1 \rightarrow W_1 = 100$
  - The second attribute: $a_2 \rightarrow W_2 = 100$
  - The third attribute: $a_3 \rightarrow W_3 = 30$
  - The fourth attribute: $a_4 \rightarrow W_4 = 20$
  - Normalized weights: $w_1 = w_2 = 40\%$ $w_3 = 12\%$, $w_4 = 8\%$.

Attribute	Measurement scale
a_1: Top speed	[150, 250]
a_2: Acceleration	[7, 14]
a_3: CO_2 emissions	[120, 150]
a_4: Maintenance	[400,600]

About SWING weighting

- **The mode of questioning explicitly (but only) considers the least and most preferred levels of the attributes**
  - Assumes that the DM can directly numerically assess the strength of preference of changes between these levels

- **NOTE that we only have two preference relations: $\succcurlyeq$ and $\succcurlyeq_d$**
  - For example preference statement $W_1 = 100$, $W_4 = 20$ is equal to $v_1(x_1^*) - v_1(x_1^0) = 5[v_4(x_4^*) - v_4(x_4^0)]$, which assumes that there exist levels $x_1^{0.2}$, $x_1^{0.4}$, $x_1^{0.6}$, $x_1^{0.8}$ so that $(x_1^{0.2} \leftarrow x_1^0) \sim_d (x_1^{0.4} \leftarrow x_1^{0.2}) \sim_d \ldots \sim_d (x_1^* \leftarrow x_1^{0.8})$
    - Then $v_1(x_1^*) - v_1(x_1^0) = 5[v_1(x_1^{0.2}) - v_1(x_1^0)] = 5[v_4(x_4^*) - v_4(x_4^0)]$ if $(x_1^{0.2}, x_2, x_3, x_4) \leftarrow (x_1^0, x_2, x_3, x_4) \sim_d (x_1, x_2, x_3, x_4^*) \leftarrow (x_1, x_2, x_3, x_4^0)$

SMART

- Simple Multi-Attribute Rating Technique process:
  1. Select the least important attribute and give it a weight of 10 points.
  2. Select the second least important attribute and give it a weight (≥10 points) that reflects its importance compared to the least important attribute.
  3. Go through the remaining attributes in ascending order of importance and give them weights that reflect their importance compared to the less important attributes.
  4. Normalize the weights.

- This process does not consider the measurement scales at all → interpretation of weights is questionable

SMARTS

SMARTS = SMART using <u>S</u>wings

1. Select the attribute corresponding to the least preferred change from worst to best level and give it a weight of 10 points.
2. Go through the remaining attributes in ascending order of preference over changing the attribute from the worst to the best level, and give them weights that reflect their importance compared to the less preferred changes.
3. Normalize the weights.

SMARTS: example

- **Magazine A**’s measurement scales
  - Alternative $x^0 = \left(150\,\frac{km}{h}, 14s, 150\,\frac{g}{km}, 600\,\frac{€}{year}\right)$
  - Least preferred change from the worst to the best level: $a_4 \rightarrow W_4 = 10$
  - The second least preferred change: $a_3 \rightarrow W_3 = 20$
  - The third least preferred change : $a_2 \rightarrow W_2 = 40$
  - The fourth least preferred change: $a_1 \rightarrow W_1 = 40$
  - Normalized weights: $w_1 = w_2 = 36\%$, $w_3 = 18\%$, $w_4 = 9\%$.

Attribute	Measurement scale
a_1: Top speed	[150, 250]
a_2: Acceleration	[7, 14]
a_3: CO_2 emissions	[120, 150]
a_4: Maintenance	[400,600]

Empirical problems related to SWING & SMARTS

- People tend to use only multiples of 10 when assessing the weights, e.g.,
  - SWING: $W_1 = W_2 = 100, W_3 = 30, W_4 = 20 \rightarrow w_1 = w_2 = 0.40, w_3 = 0.12, w_4 = 0.08$
  - SMARTS: $W_1 = W_2 = 40, W_3 = 20, W_4 = 10 \rightarrow w_1 = w_2 = 0.36, w_3 = 0.18, w_4 = 0.09$
  - SWING and SMARTS typically produce different weights

- Assessments may reflect only ordinal, not cardinal information about the weights
  - E.g., SMARTS weights $W_4 = 10$ and $W_3 = 20$ only imply that $W_4 < W_3$, not that $W_3 / W_4 = 2$

Summary

- Additive value function describes the DM's preferences if and only if the attributes are mutually preferentially independent and each attribute is difference independent of the others
- The <u>only</u> meaningful interpretation for attribute weight w_i:

The improvement in overall value when attribute a_i is changed from its worst level to its best **relative to** similar changes in other attributes

- In trade-off weighting, attribute weights are elicited by specifying equally preferred alternatives (or changes in alternatives), which differ from each other on at least two attributes
  - Use trade-off weighting whenever possible