Nonlinear dynamics & chaos
Limit cycles

Lecture VI

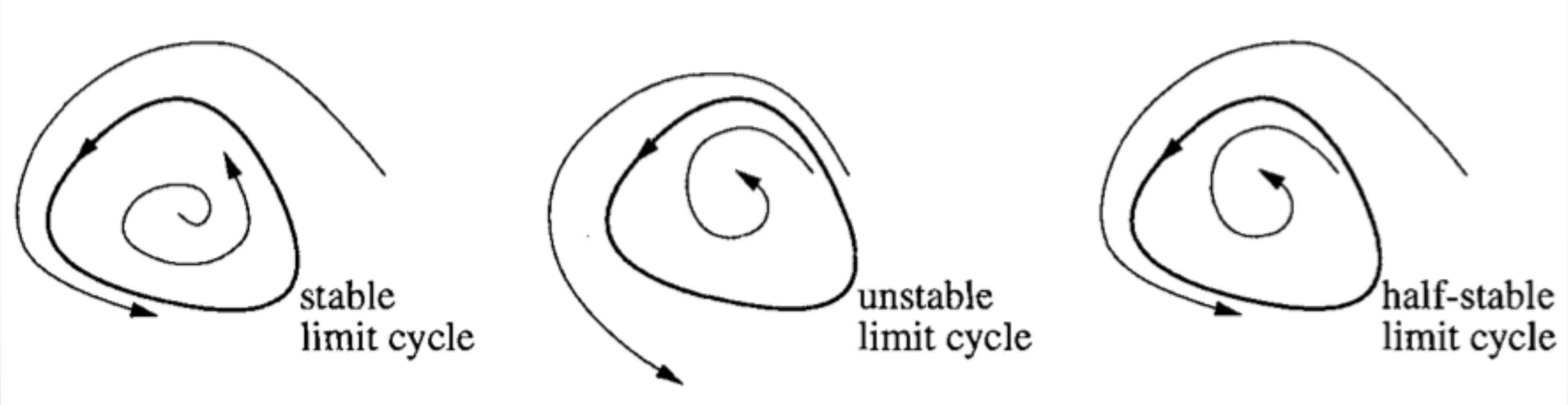
Recap

Last time:
2D nonlinear systems
Conservative systems
Reversible systems
Possibility/impossibility of closed orbits

Limit cycles

A limit cycle is an isolated closed trajectory: neighbouring trajectories either spiral away from it or towards it.

If neighbouring trajectories tend towards the limit cycle, the latter is called stable or attracting, otherwise unstable, in exceptional cases it may be half-stable .



Limit cycles

Stable limit cycles model systems/phenomena that exhibit self-sustained oscillations, like:

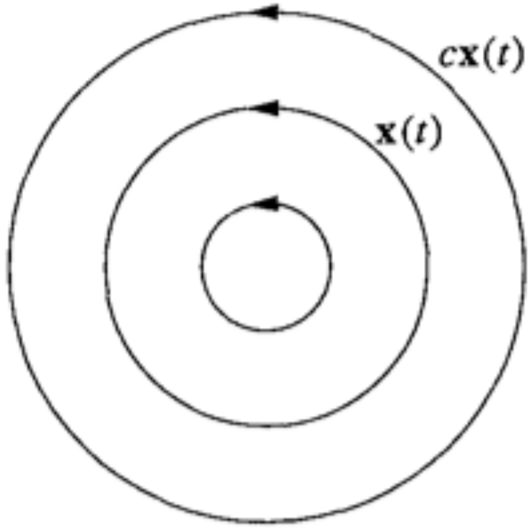
1) The beating of a heart
2) The periodic firing of a pacemaker neuron
3) Daily rhythms in human body temperature and hormone secretion
4) Chemical reactions that oscillate spontaneously
5) Dangerous self-excited vibrations in bridges and airplane wings.

A slight perturbation makes the system return to the cycle.

Limit cycles

Limit cycles are typical features of nonlinear systems: in linear systems there are periodic orbits, but they are not isolated!

If $\mathbf{x}(t)$ is a periodic solution of a linear system, $c\mathbf{x}(t)$ is also a solution, for any value of c.

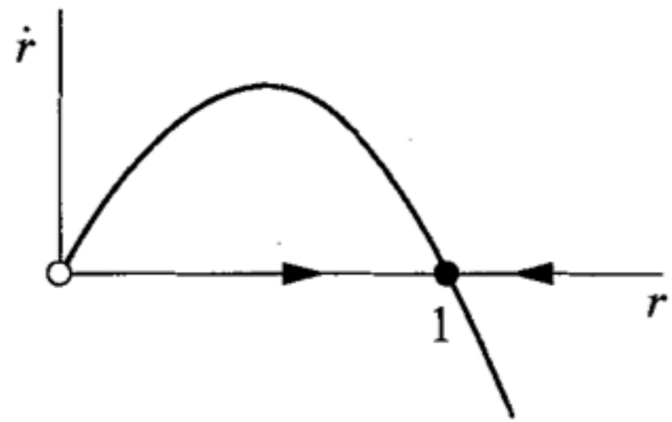


The amplitude of a linear oscillation is set by its initial condition: any perturbation will persist forever. Limit cycles are determined by the structure of the system itself.

Example I

In polar coordinates
$$\begin{aligned}\dot{r} &= r(1-r^2)\\ \dot{\theta} &= 1\end{aligned}$$
, where $r \geq 0$.

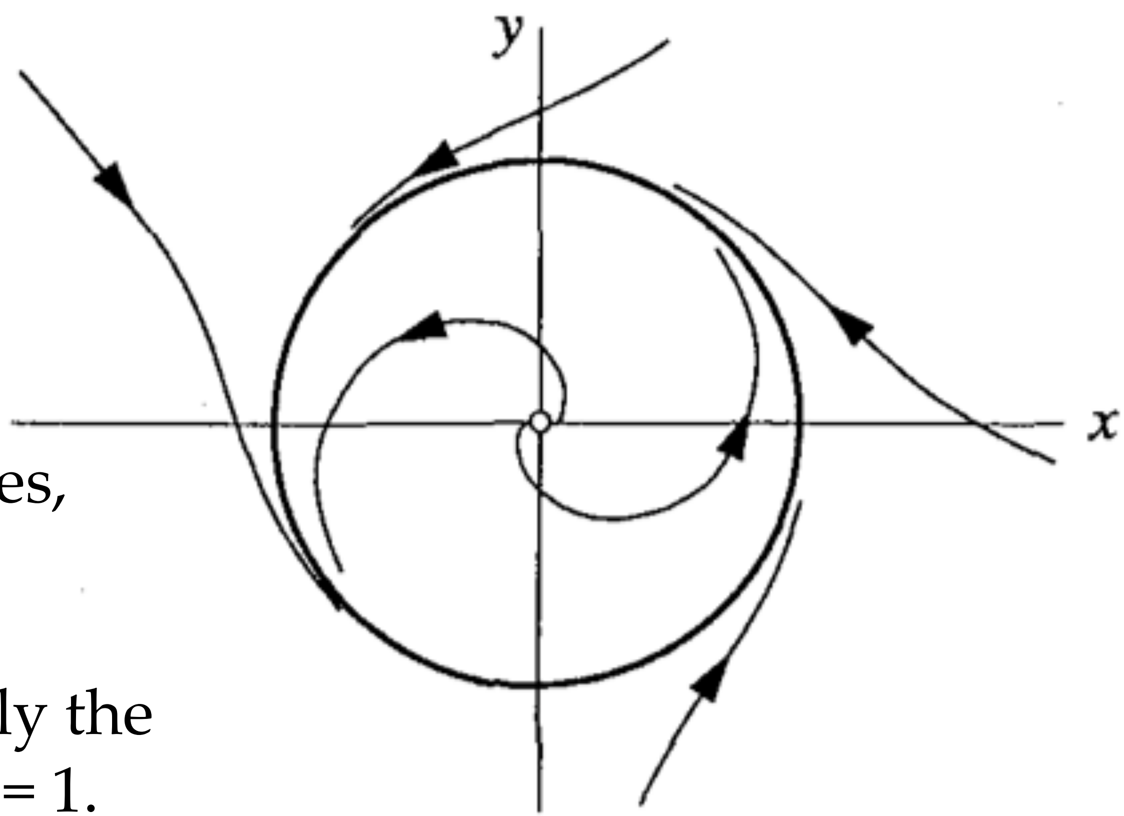
Radial and angular dynamics are decoupled, they can be treated separately.



$r^* = 0$ unstable fixed point, $r^* = 1$ stable: all trajectories approach the unit circle asymptotically.

Example I

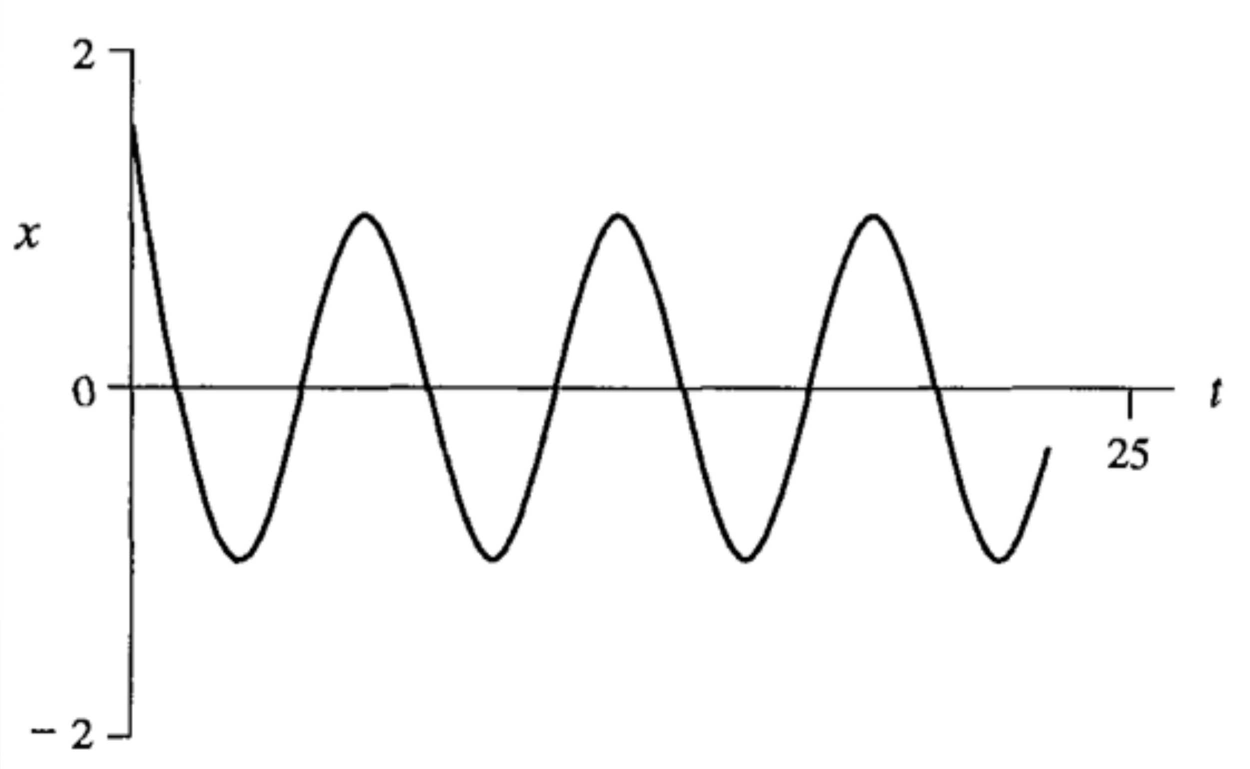
Phase plane:
$$\begin{aligned}\dot{r} &= r(1-r^2)\\ \dot{\theta} &= 1\end{aligned}$$



All trajectories, except $r^* = 0$, approach monotonically the unit circle $r^* = 1$.

Example I

$$\begin{aligned}\dot{r} &= r(1-r^2)\\ \dot{\theta} &= 1\end{aligned}$$



$x(t) = r(t)\cos\theta(t)$ ➡ Sinusoidal oscillation of constant amplitude

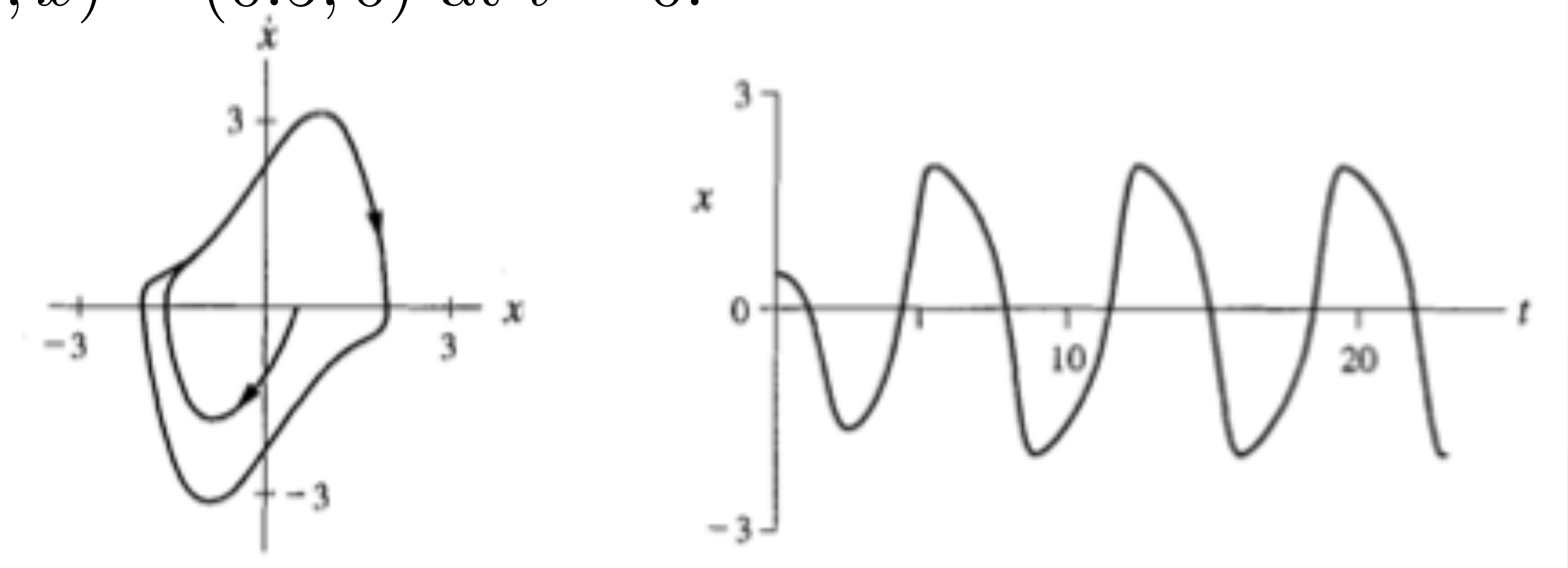
Example II

van der Pol Oscillator

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \qquad \mu \geq 0$$

Nonlinear damping term: if $|x| > 1$ oscillations are damped; if $|x| < 1$ oscillations are enhanced.

It can be shown that the van der Pol equation has a unique stable limit cycle for each μ. Shown here $\mu = 1.5$ and $(x, \dot{x}) = (0.5, 0)$ at $t = 0$.



Ruling out closed orbits

Gradient systems

$$\dot{\mathbf{x}} = -\nabla V$$

$V(\mathbf{x})$ = single-valued scalar function (Note that this really is potential, since in the above equation friction coefficient can be thought of as 1, and the l.h.s. of eq. is frictional force.)

Theorem: Closed orbits are impossible for gradient systems.

Proof: Let us assume that there is a closed orbit. A change in V after one circuit should be $\Delta V = 0$ since V is single-valued. On the other hand,

$$\Delta V = \int_0^T \frac{dV}{dt} dt = \int_0^T (\nabla V \cdot \dot{\mathbf{x}}) dt$$

$$= -\int_0^T \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} \; dt = -\int_0^T ||\dot{\mathbf{x}}||^2 < 0$$

Contradiction ➔ There can't be closed orbits in gradient systems.

Ruling out closed orbits

Gradient systems

$$\dot{\mathbf{x}} = -\nabla V$$

Usefulness of this theorem is a bit limited, since most two-dimensional systems are not gradient systems.

However, all vector fields on the line are gradient systems, which is another explanation for the impossibility of oscillations in one-dimensional systems.

Example I

No closed orbits for the system, because it is a gradient system.

$$\begin{aligned}\dot{x} &= \sin y \\ \dot{y} &= x\cos y\end{aligned}$$

$$V(x,y) = -x\sin y \quad \rightarrow \quad \begin{aligned}\dot{x} &= -\partial V/\partial x \\ \dot{y} &= -\partial V/\partial y\end{aligned}$$

Note: For a general gradient system

$$-\nabla V = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}\right) = (f(x,y), g(x,y)) \Leftrightarrow -\frac{\partial V}{\partial x} = f(x,y), \ -\frac{\partial V}{\partial y} = g(x,y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = -\frac{\partial^2 V}{\partial y \partial x} = -\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial g}{\partial x}$$

Sufficient condition for a system $\dot{x} = f(x,y), \ \dot{y} = g(x,y)$ **to be gradient**:

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial x} \quad !$$

Example II

Similar techniques as in the Theorem can be used to exclude closed orbits even in non-gradient systems.

Nonlinearly damped oscillator $\ddot{x} + (\dot{x})^3 + x = 0$

Energy function $E(x, \dot{x}) = \frac{1}{2}(x^2 + \dot{x}^2)$

After one cycle, position and velocity take the same value as at the starting point, so $\Delta E = 0$.

$$\dot{E}(x, \dot{x}) = \dot{x}(x + \ddot{x}) = \dot{x}(-\dot{x}^3) = -\dot{x}^4 \leq 0$$

$$\Delta E = \int_0^T \dot{E} dt = -\int_0^T (\dot{x})^4 dt \leq 0$$

$\Delta E = 0$ only when the velocity dx/dt is zero, which means a fixed point. Contradiction ➔ no closed orbits.

Lyapunov functions

Energy-like functions that decrease along trajectories

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

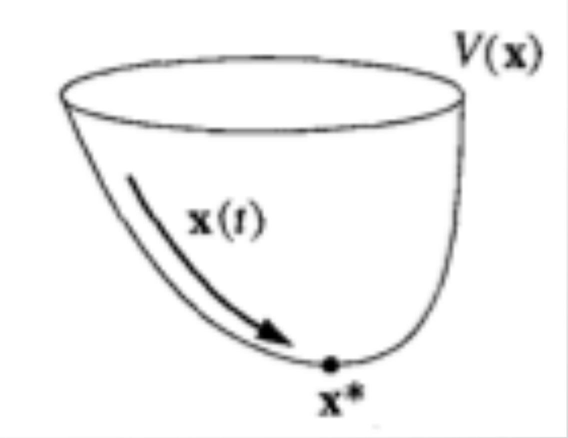
$\mathbf{x}^*$ is a fixed point. A Lyapunov function is a continuously differentiable function $V(\mathbf{x})$ such that:

1) $V(\mathbf{x}) > 0$, $\mathbf{x} \neq \mathbf{x}^*$ and $V(\mathbf{x}^*) = 0$ (V is positive definite).

2) $\dot{V} < 0$, for all $\mathbf{x} \neq \mathbf{x}^*$. (All trajectories flow towards $\mathbf{x}^*$.)

If there is such a function, $\mathbf{x}^*$ is globally asymptotically stable: for all initial conditions $\mathbf{x}(t) \rightarrow \mathbf{x}^*$ when $t \rightarrow \infty$. Consequently, the system has no closed orbits.

Solutions cannot get stuck anywhere else, because if they did, V would stop changing, which would contradict 2).



Example I

There is no systematic way of constructing Liapunov functions; sometimes sums of squares work.

$$\begin{aligned} \dot{x} &= -x + 4y \\ \dot{y} &= -x - y^3 \end{aligned}$$

Consider $$V(x, y) = x^2 + ay^2$$

$$\dot{V} = 2x\dot{x} + 2ay\dot{y} = 2x(-x + 4y) + 2ay(-x - y^3) = -2x^2 + (8 - 2a)xy - 2ay^4$$

Choosing $a = 4$,

$$\dot{V} = -2x^2 - 8y^4 \leq 0$$

$$\dot{V} = 0 \text{ only at the origin}$$

So V is a Lyapunov function and there are no closed orbits: all trajectories will approach the origin when $t \to \infty$.

Dulac's criterion

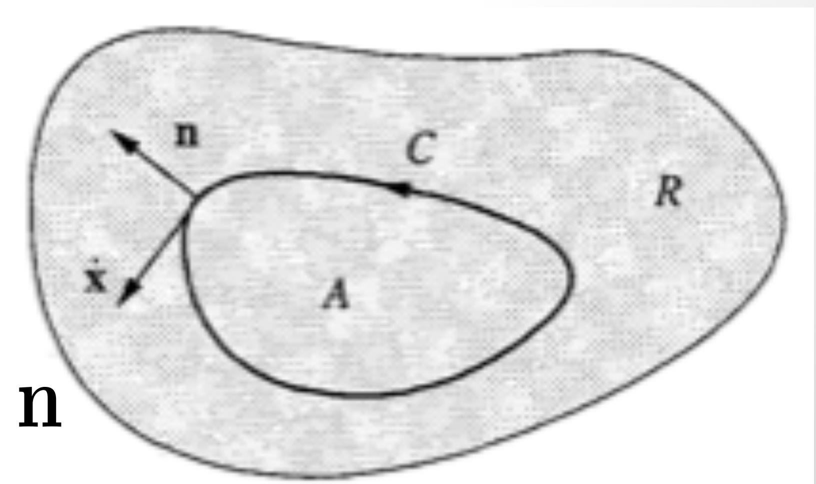
Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a continuously differentiable vector field on some simply connected subset *R* of the plane. If there exists a continuously differentiable, real-valued function $g(\mathbf{x})$ such that $\nabla \cdot (g\dot{\mathbf{x}})$ has one sign throughout *R*, there are no closed orbits lying entirely in *R*.

Proof: Let us assume that there is a closed orbit *C* lying entirely in *R*. A is the area within *C*.

Green's theorem

$$\iint_A \nabla \cdot (g\dot{\mathbf{x}}) dA = \oint_C g\dot{\mathbf{x}} \cdot \mathbf{n}\, d\ell$$

$\Downarrow$ Non-zero $\qquad$ $\Downarrow$ Zero, since $\dot{\mathbf{x}} \perp \mathbf{n}$



Contradiction → no closed orbits.

Example I

There is no general procedure to find the function $g(\mathbf{x})$. Candidates: $1/x^a y^b$, e^{ax}, e^{ay}.

Show that there are no closed orbits in $x, y > 0$ for the system

$$\begin{aligned}\dot{x} &= x(2 - x - y)\\ \dot{y} &= y(4x - x^2 - 3)\end{aligned}$$

Try $g(\mathbf{x}) = 1/(xy)$,

$$\nabla \cdot (g\dot{\mathbf{x}}) = \frac{\partial}{\partial x}(g\dot{x}) + \frac{\partial}{\partial y}(g\dot{y}) = \frac{\partial}{\partial x}\left(\frac{2-x-y}{y}\right) + \frac{\partial}{\partial y}\left(\frac{4x-x^2-3}{x}\right) = -\frac{1}{y}$$

$\nabla \cdot (g\dot{\mathbf{x}}) < 0$ for $x, y > 0$. ➔ Dulac: No closed orbits in $x, y > 0$.

Example II

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x - y + x^2 + y^2\end{aligned}$$

Let $g(\mathbf{x}) = e^{-2x}$.

$$\nabla \cdot (g\dot{\mathbf{x}}) = -2e^{-2x}y + e^{-2x}(-1 + 2y) = -e^{-2x} < 0$$

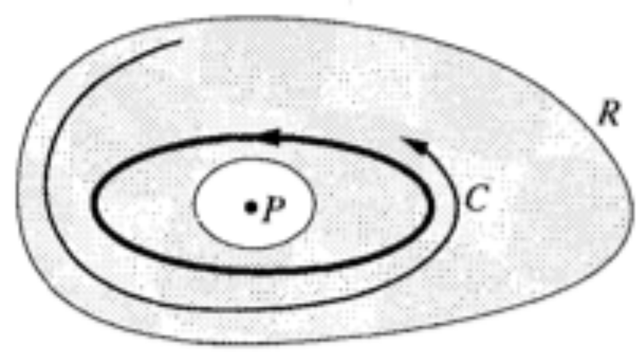
No closed orbits because of Dulac's criterion.

Poincaré-Bendixson Theorem

A criterion to establish that closed orbits exist.

Theorem: Suppose that:

1) R is a closed, bounded subset of the plane;
2) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R;
3) R does not contain any fixed points;
4) There exists a trajectory C that is *confined* in R, i.e. it starts in R and stays in R for all future time .



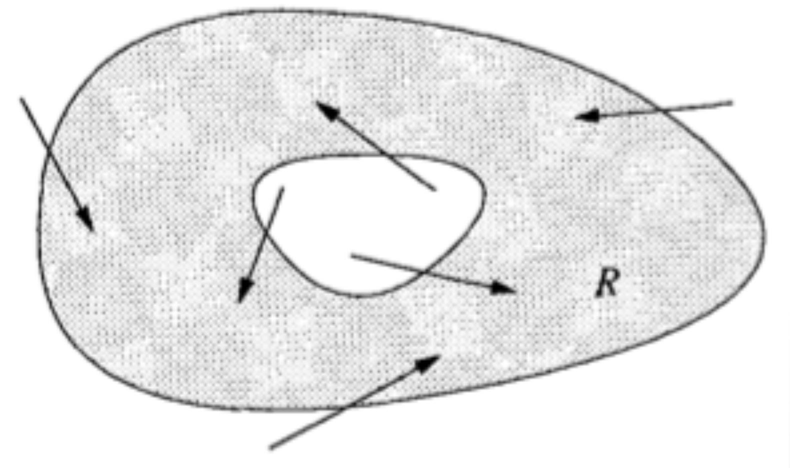
Then either C is a closed orbit or it spirals towards a closed orbit as $t \rightarrow \infty$. ➔ Chaos cannot occur in the phase plane.

Poincaré-Bendixson Theorem

Ring-shaped region R because any closed orbit must encircle a fixed point!

How can we ensure condition 4 that there is a confined trajectory C?

Standard trick: Construct a trapping region, i.e. a region on whose boundary the vector field points inwards.



Example I

$$
\begin{aligned}
\dot{r} &= r(1-r^2) + \mu r \cos\theta \\
\dot{\theta} &= 1
\end{aligned}
$$

For $\mu = 0$ there's a limit cycle at $r = 1$. What happens when $\mu > 0$?

Search for a trapping region: annulus between two circles of radii r_{min} and r_{max} with $\dot{r} < 0$ on the outer circle and $\dot{r} > 0$ on the inner circle.

No fixed points inside because angular velocity is always positive!

Example I

$$\begin{aligned} \dot{r} &= r(1-r^2) + \mu r \cos\theta \\ \dot{\theta} &= 1 \end{aligned}$$

Search for $r_{\min}$

$$\dot{r} = r(1-r^2) + \mu r \cos\theta > 0$$

$$\cos\theta > -1, \forall\theta \quad \rightarrow \quad r(1-r^2) - \mu r > 0 \quad \rightarrow \quad r^2 + \mu - 1 < 0$$

Any $r_{min} < \sqrt{1-\mu}$ would do the job (for $\mu < 1$).

To come close to the limit cycle we pick a value close to the limit, like $r_{min} = 0.999\sqrt{1-\mu}$

Similarly, require $\dot{r} < 0$ to get $r_{max} = 1.001\sqrt{1+\mu}$

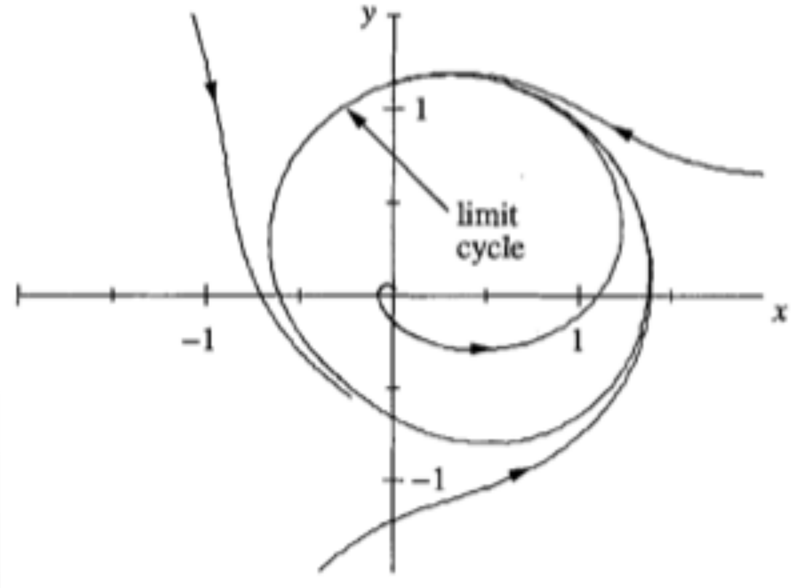
Example I

$$\begin{aligned}\dot{r} &= r(1-r^2) + \mu r\cos\theta \\ \dot{\theta} &= 1\end{aligned}$$

Poincaré-Bendixson theorem: there is a limit cycle in the annulus

$$0.999\sqrt{1-\mu} < r < 1.001\sqrt{1+\mu}$$

True also for $\mu \geq 1$



Example II

Glycolysis

Fundamental biochemical process, where living cells obtain energy by breaking down sugar.

In intact yeast cells as well as in yeast or muscle extracts, glycolysis proceeds in an *oscillatory* fashion: concentrations of intermediate products wax and wane within a few minutes.

Sel'kov model (1968) (in a dimensionless form)

$$
\begin{aligned}
\dot{x} &= -x + ay + x^2 y \\
\dot{y} &= b - ay - x^2 y
\end{aligned}
$$

x = concentration of adenosine diphosphate (ADP)
y = concentration of fructose-6-phosphate (F6P)
$a, b > 0$ are kinetic parameters

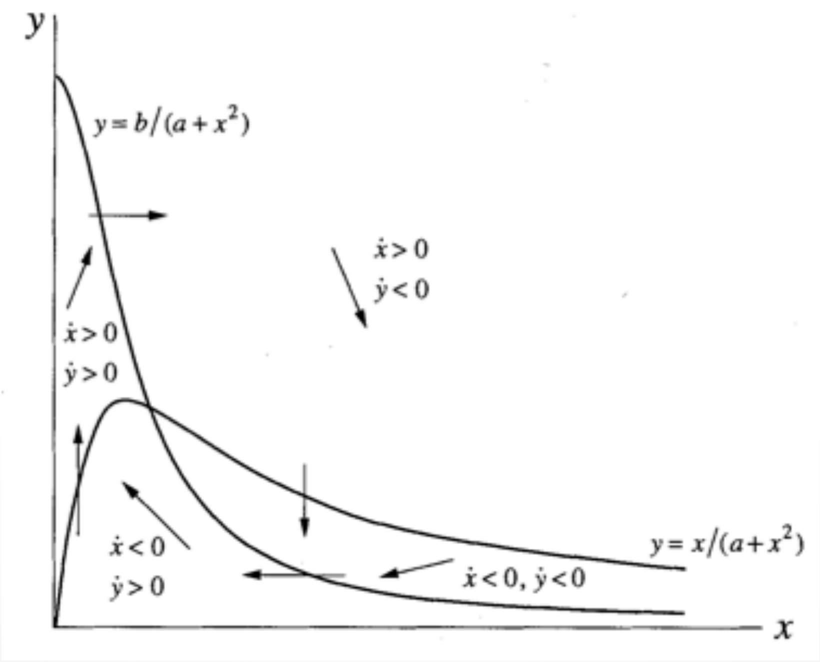
Example II

Glycolysis

$$\begin{aligned}\dot{x} &= -x + ay + x^2 y \\ \dot{y} &= b - ay - x^2 y\end{aligned}$$

Nullclines

$$y = \frac{x}{a+x^2}\;;\quad y = \frac{b}{a+x^2}$$



Example II

Glycolysis

$$\begin{aligned}\dot{x} &= -x + ay + x^2 y \\ \dot{y} &= b - ay - x^2 y\end{aligned}$$

Trapping region?

In the limit of very large *x*

$$\dot{x} \approx x^2 y\,;\quad \dot{y} = -x^2 y \quad \rightarrow \quad \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} \approx -1$$

Vector field at large *x* is roughly parallel to the line *y* = -*x*.

$$\dot{x} - (-\dot{y}) = -x + ay + x^2 y + (b - ay - x^2 y) = b - x$$

$$-\dot{y} > \dot{x} \;\; \text{if} \;\; x > b$$

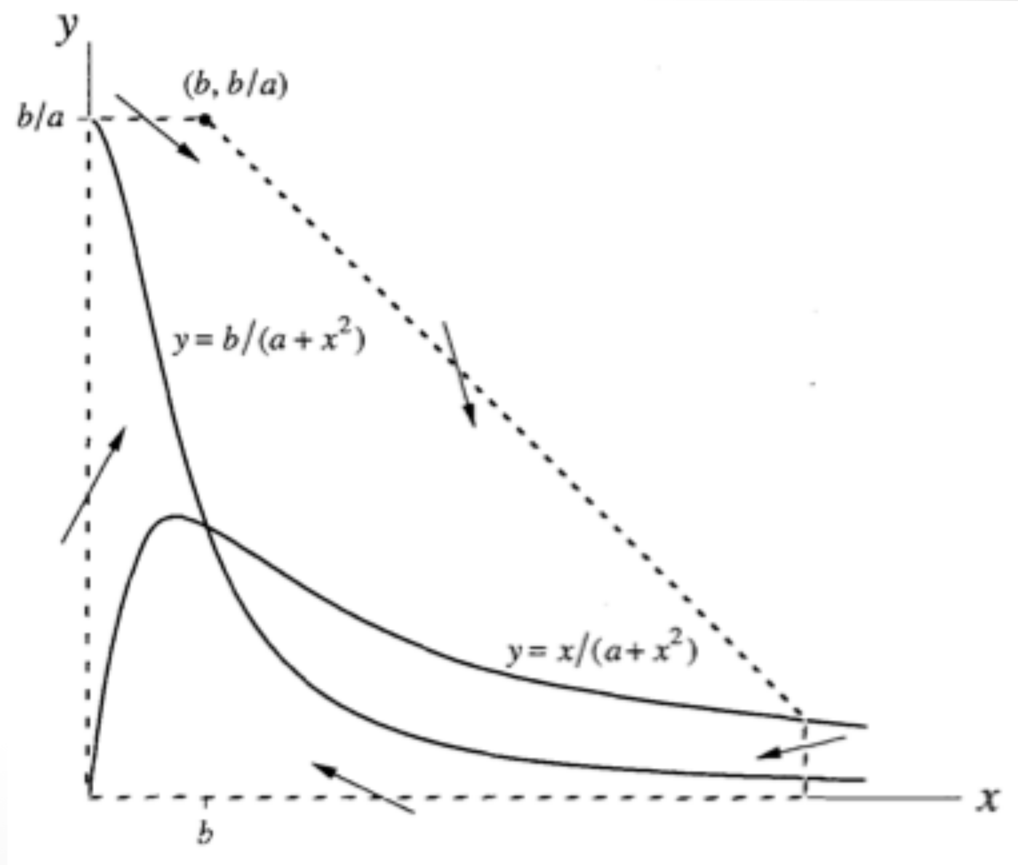
Example II

Glycolysis

$-\dot{y} > \dot{x}$ if $x > b$: the vector field points inwards on the diagonal line because dy/dx < -1.

Dashed contour encloses a trapping region!

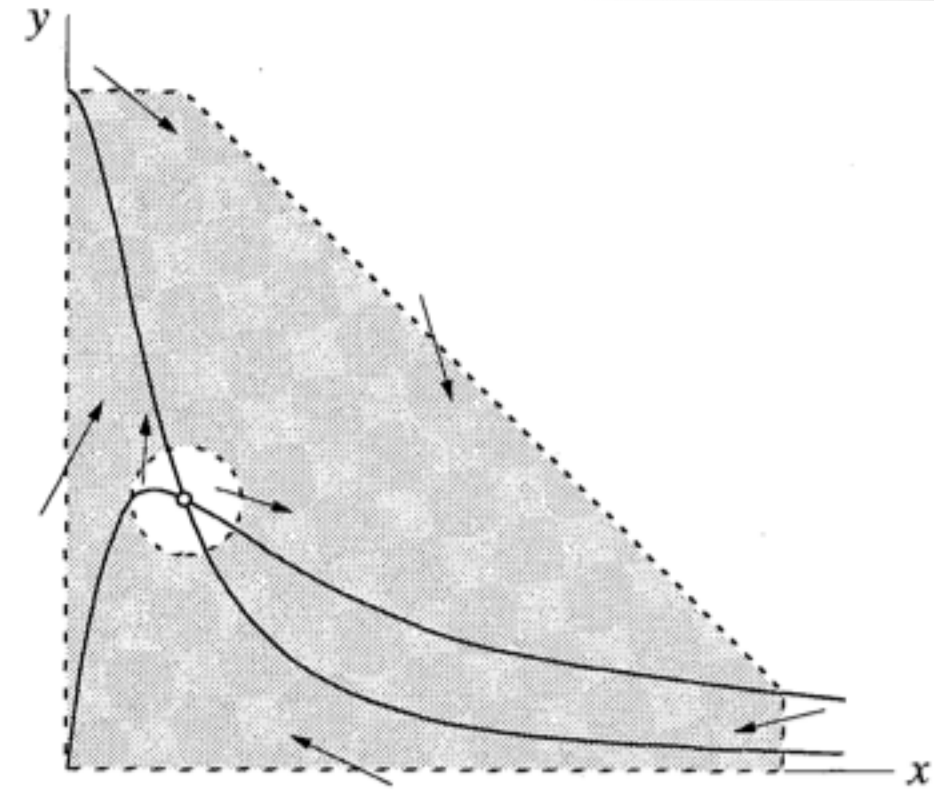
Problem: There is a fixed point inside at the intersection of the nullclines.



Example II

Glycolysis

Solution: if the fixed point is a repeller then the theorem still holds.



The repeller drives all neighboring trajectories into the shaded region.

Question: So, is the fixed point a repeller?

Example II

Glycolysis

$$\begin{aligned} \dot{x} &= -x + ay + x^2 y \\ \dot{y} &= b - ay - x^2 y \end{aligned}$$

Jacobian

$$A = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -a - x^2 \end{pmatrix}$$

Fixed point

$$x^* = b, \quad y^* = \frac{b}{a + b^2}$$

$$\Delta = a + b^2 > 0\,; \quad \tau = -\frac{b^4 + (2a-1)b^2 + (a + a^2)}{a + b^2}$$

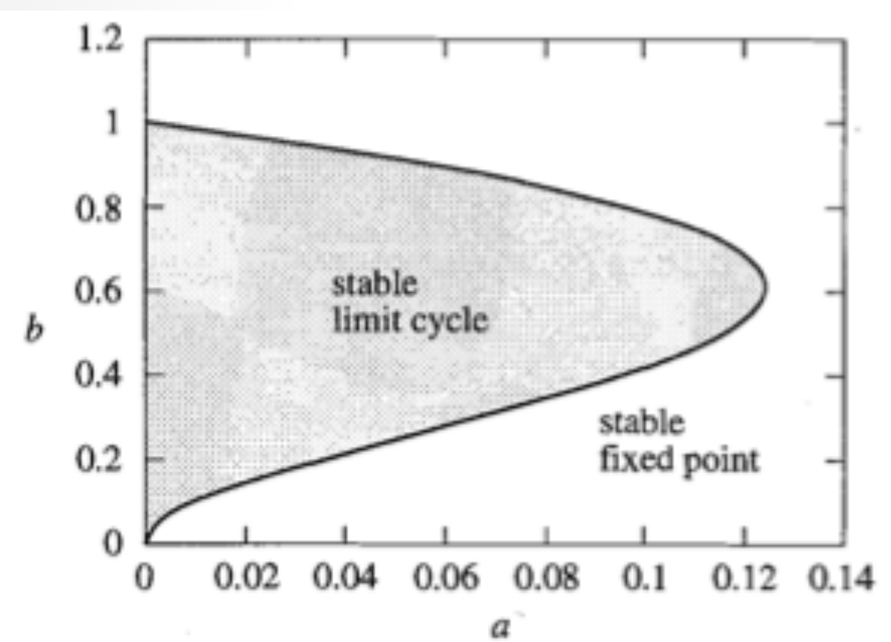
The fixed point is stable for $\tau < 0$ and unstable for $\tau > 0$.

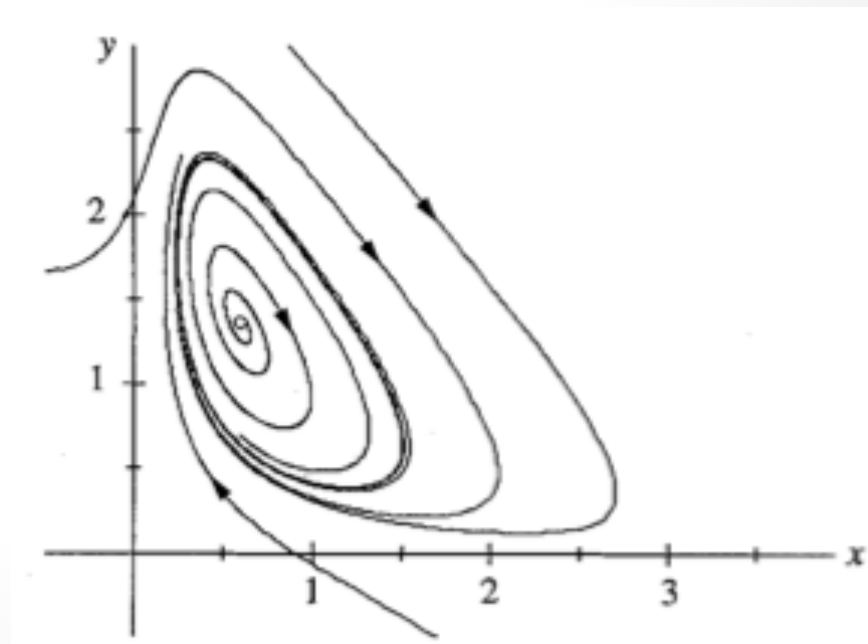
Example II

Glycolysis

Dividing line $\tau = 0$ occurs when

$$b^2 = \frac{1}{2}\left(1 - 2a \pm \sqrt{1 - 8a}\,\right)$$





No chaos in the phase plane

Poincaré-Bendixson theorem is valid only in two dimensions, not in three and higher dimensions!

It states that if a trajectory is confined in a closed region without fixed points it must approach a closed orbit.

So, in two dimensions, a trajectory can either approach a fixed point, diverge to infinity, or approach and follow a closed orbit. Poincaré-Bendixson theorem: no chaos in 2D phase plane.

In dimensions three and higher, instead, a particle might wander around forever *in a bounded region without settling down to a fixed point or a closed orbit*. In some cases trajectories are attracted to a complex geometric (fractal) object → chaos!

Liénard Systems

Many oscillating circuits & systems can be modelled by

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

This the ***Liénard's equation***.

Van der Pol oscillator

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \qquad \mu \geq 0$$

is a special case of the Liénard's equation.

Equivalent form for Liénard:

$$\dot{x} = y$$

$$\dot{y} = -g(x) - f(x)y.$$

Liénard Systems

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

Liénard's theorem

Suppose:

(1) $f(x)$ and $g(x)$ are continuously differentiable for all x;

(2) $g(-x) = - g(x)$ for all x;

(3) $g(x) > 0$ for $x > 0$;

(4) $f(-x) = f(x)$ for all x;

(5) The odd function $F(x) = \int_0^x f(u)du$ has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$.

Then the system $\begin{aligned}\dot{x} &= y \\ \dot{y} &= -g(x) - f(x).\end{aligned}$ has a **unique, stable limit cycle surrounding the origin in the phase plane**.

Liénard Systems

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

$g(x)$: the restoring force acts like an ordinary spring
$f(x)$: damping at small $|x|$, positive feedback at large $|x|$.
→ Small oscillations are pumped up and large oscillations are damped: the system tends to settle into a self-sustained oscillation at an intermediate amplitude.

van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \qquad \mu \geq 0$$

Here, $f(x) = \mu(x^2 - 1), \; g(x) = x$

Conditions (1)-(4) are satisfied. $F(x) = \mu(\frac{1}{3}x^3 - x) = \frac{1}{3}\mu x(x^2 - 3) \Rightarrow$
Condition (5) is satisfied for $a = \sqrt{3}$. vdP equation has a unique, stable limit cycle.

Relaxation Oscillations

These oscillations are characterised by repetitious slow build up and sudden discharge.

Example: van der Pol in the strongly nonlinear limit

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \;\; \mu >> 1.$$

Note: Standard trick leads to a system that's hard to interpret.

$$\dot{x} = y$$

$$\dot{y} = -\frac{1}{\mu}(x^2 - 1) - x$$

$\rightarrow$ introduce different variables.

Relaxation Oscillations

vdP: $$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \;\; \mu >> 1.$$

Notice that $$\ddot{x} + \mu\dot{x}(x^2 - 1) = \frac{d}{dt}\left[\dot{x} + \mu\left(\frac{1}{3}x^3 - x\right)\right]$$

So, let $F(x) = \dfrac{1}{3}x^3 - x, \;\; w = \dot{x} + \mu F(x).$

$$\Rightarrow \dot{w} = \ddot{x} + \mu\dot{x}(x^2 - 1) = -x.$$

$$\Rightarrow \dot{x} = w - \mu F(x)$$ Finally, let $y = \dfrac{w}{\mu}.$

$$\dot{w} = -x$$

We obtain

$$\dot{x} = \mu[y - F(x)]$$

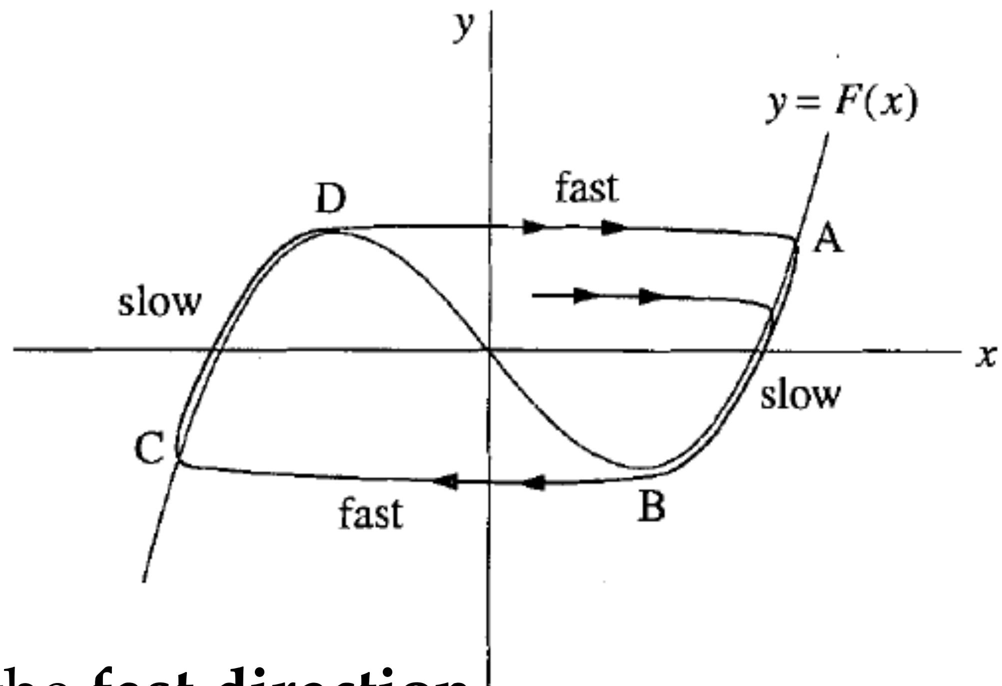
$$\dot{y} = -\frac{1}{\mu}x.$$

Relaxation Oscillations

vdP: $$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \;\; \mu >> 1.$$

$$\dot{x} = \mu[y - F(x)]$$

$$\dot{y} = -\frac{1}{\mu}x.$$



Cubic nullcline

$$\dot{x} = 0: \;\; y = F(x).$$

is the **slow direction.**

Nullcline $\dot{y} = 0: \;\; x = 0$ is the **fast direction.**

To see this, suppose $y - F(x) \sim O(1)$.

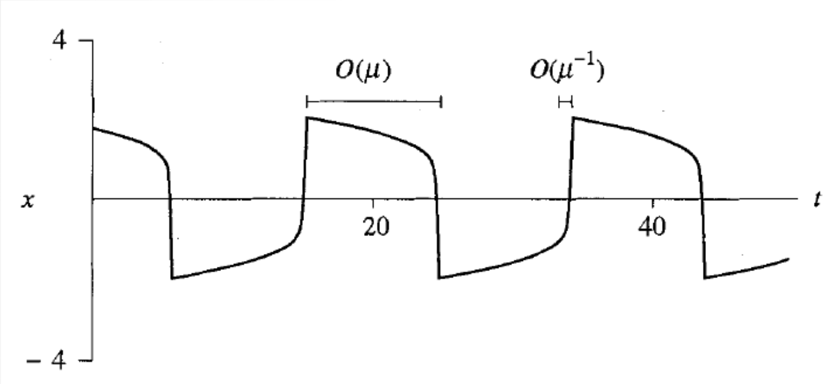
Then $|\dot{x}| \sim O(\mu) >> 1, \;\; |\dot{y}| \sim O(\mu^{-1}) << 1.$

If $y - F(x) \sim O(\mu^{-2}): \;\; |\dot{x}| \sim |\dot{y}| \sim O(\mu^{-1}).$

So, what makes the trajectory jump at "knees"?

Relaxation Oscillations

vdP: $$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \;\; \mu >> 1.$$



The period of vdP oscillator, when $\mu \gg 1$:

$$T \approx 2 \int_{t_A}^{t_B} dt$$

On slow branches $y \approx F(x)$: $$\frac{dy}{dt} \approx F'(x)\frac{dx}{dt} = (x^2 - 1)\frac{dx}{dt}$$

$$\dot{x} = \mu[y - F(x)]$$

$$\dot{y} = -\frac{1}{\mu}x.$$

$$\Rightarrow \frac{dy}{dt} = \frac{-x}{\mu} \Rightarrow \frac{dx}{dt} = \frac{-x}{\mu(x^2 - 1)}$$

On slow branches: $$dt \approx \frac{\mu(x^2 - 1)}{x}dx$$

The positive branch: x_A=2, x_B=1 (exercise 6.4.);

$$T \approx 2\int_2^1 \frac{-\mu}{x}(x^2 - 1)dx = 2\mu\left[\frac{x^2}{2} - \ln x\right]_1^2 = \mu[3 - 2\ln 2]$$

Weakly Nonlinear Oscillators

In the strongly nonlinear oscillators two time scales operated *sequentially.*

In weakly nonlinear oscillators time scales operate *concurrently.*

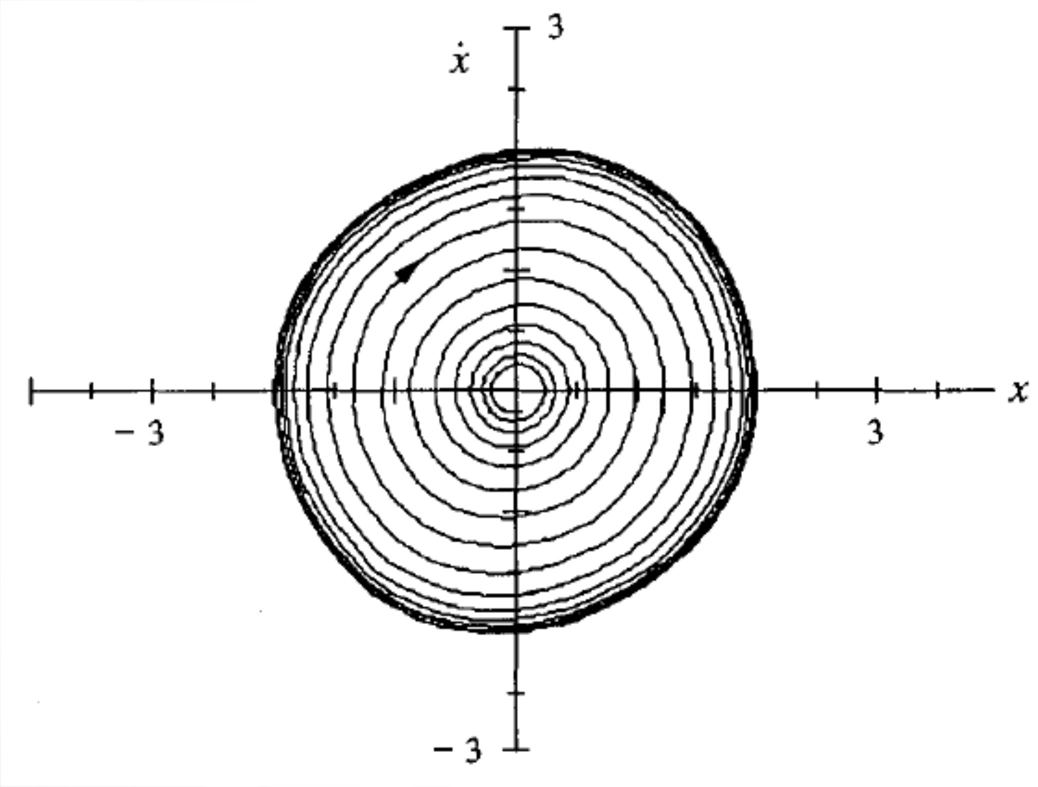
General form: $\ddot{x} + x + \epsilon h(x, \dot{x}) = 0$, where $0 \leq \epsilon << 1$

Two fundamental cases:

1. Van der Pol $\quad \ddot{x} + x + \epsilon(x^2 - 1)\dot{x} = 0$
2. Duffing $\quad \ddot{x} + x + \epsilon x^3 = 0$

Weakly Nonlinear Oscillators

van der Pol, $\epsilon = 0.1$; initial condition close to (0, 0)



We've seen this before!

Weakly Nonlinear Oscillators

Example $\ddot{x} + 2\epsilon\dot{x} + x = 0, \;\; x(0) = 0, \dot{x}(0) = 1.$

$$\dot{x} = y \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & -2\epsilon \end{pmatrix} \text{ stable spiral}$$

$$\dot{y} = -2\epsilon y - x$$

$$\text{Det} \rightarrow \quad \lambda^2 + 2\epsilon\lambda + 1 = 0$$

$$\lambda = -\epsilon \pm \sqrt{\epsilon^2 - 1} = -\epsilon \pm i\sqrt{1 - \epsilon^2}$$

$$\mathbf{x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2}$$

$$x(t) = c_1 e^{[-\epsilon - i\sqrt{1-\epsilon^2}]t} = c_1 e^{-\epsilon t}[\cos(\sqrt{1-\epsilon^2}t) - i\sin(\sqrt{1-\epsilon^2}t)]$$

$$x(0) = 0, \;\; \dot{x}(0) = 1: \quad x(t) = \frac{1}{\sqrt{1-\epsilon^2}} e^{-\epsilon t} \sin(\sqrt{1-\epsilon^2}t)$$

Weakly Nonlinear Oscillators

$$x(t) = \frac{1}{\sqrt{1-\epsilon^2}} e^{-\epsilon t} \sin(\sqrt{1-\epsilon^2}\, t)$$

Two concurrent time scales; slowly decaying oscillations.

Generally, it's hard to solve this way completely, so…
Perturbation technique, that is, looking for a solution in the form

$$x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) \ldots$$

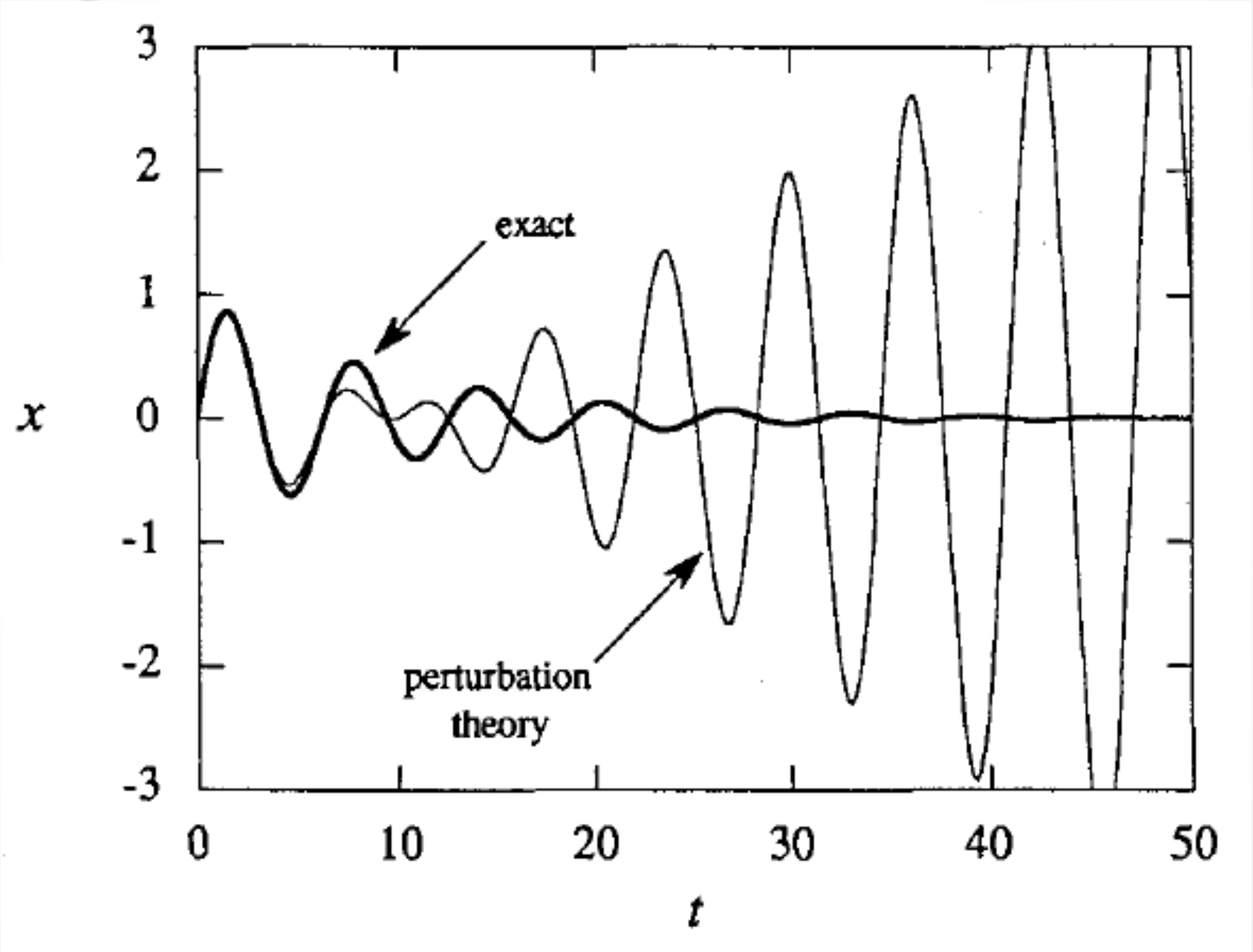
By substituting $x(t,\varepsilon)$ for $x(t)$ in the original differential equation, solving for terms of $O(1)$ and $O(\varepsilon)$ separately gives a good approximation for $x(t)$ up to $t \ll 1/\varepsilon$.

$$[\ddot{x}_0 + x_0] + \epsilon[\ddot{x}_1 + 2\dot{x}_0 + x_1] + O(\epsilon^2) = 0$$

$$O(1): \ \ddot{x}_0 + x_0 = 0$$
$$O(\epsilon): \ \ddot{x}_1 + 2\dot{x}_0 + x_1 = 0$$

$$\Rightarrow x(t,\epsilon) = \sin t - \epsilon t \sin t + O(\epsilon^2)$$

Weakly Nonlinear Oscillators



Weakly Nonlinear Oscillators

The book introduces a method called two-timing, where an ansatz of the form $x(t,\epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + O(\epsilon^2)$ is made. Here, $T = \varepsilon t$ and $\tau = t$.

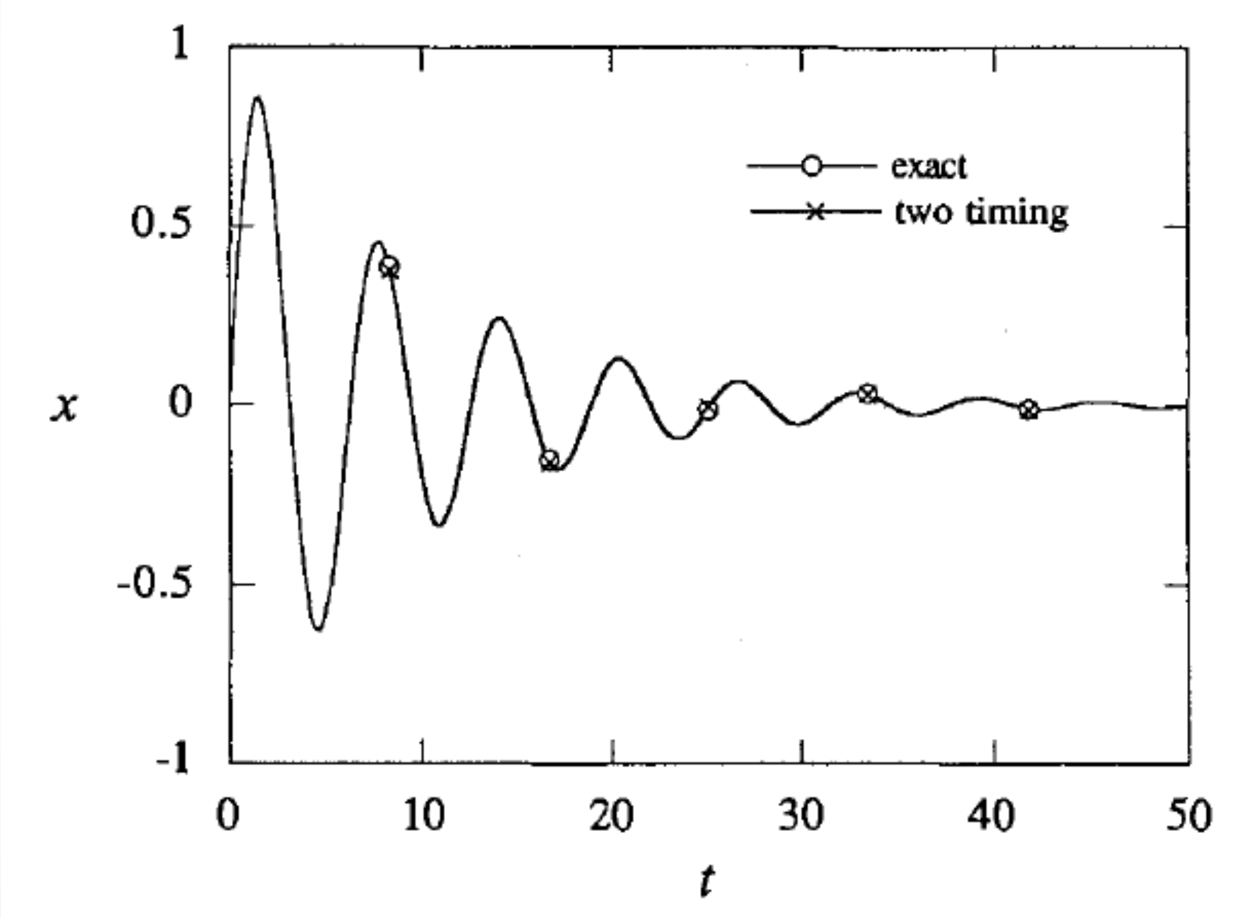
Then, time derivatives are taken as:

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial \tau} + \frac{\partial x}{\partial T}\frac{\partial T}{\partial t} = \frac{\partial x}{\partial \tau} + \epsilon \frac{\partial x}{\partial T}$$

This is technical, we won't go into more details. Suffice it to say that two-timing works incredibly well.

Weakly Nonlinear Oscillators

Two-timing solution: $x = e^{-T}\sin\tau + O(\epsilon) = e^{-\epsilon t}\sin t + O(\epsilon)$



Strongly & Weakly Nonlinear Oscillators

Sequentially and concurrently operating different time scales.

Even approximate solutions are hard.

Dump that! The important thing is to have a qualitative understanding of fast and slow modes in oscillating solutions.

Note: Next time is Wed. 27th February:
Bifurcations in the phase plane.