GIS-E3010 Least-Squares Methods in Geoscience

Lecture 12a

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Learning objectives

- To be able to solve homogeneous system of equations

Homogeneous system of equations

- A homogeneous system (using homogeneous coordinates and projective geometry) of equations looks (e.g.) like

$$\begin{cases} a_1 x_1 + a_2 y_1 + \cdots = 0 \\ a_1 x_2 + a_2 y_2 + \cdots = 0 \\ a_1 x_3 + a_2 y_3 + \cdots = 0 \end{cases}$$

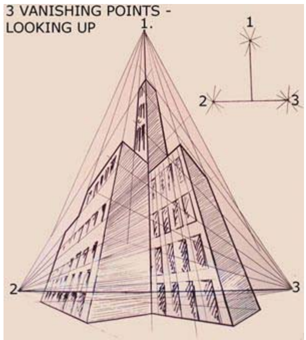
$$0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix} \quad x = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}$$

$$Ax = 0$$

$$A = \begin{bmatrix} x_1 & y_1 & \cdots \\ x_2 & y_2 & \cdots \\ x_3 & y_2 & \cdots \end{bmatrix}$$

Applications of homogeneous systems

- Solving vanishing points, an absolute conic and an interior orientation of a camera (computer vision approach) from perpendicular object lines



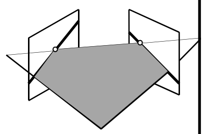
$$v_1^T \omega v_2 = 0$$

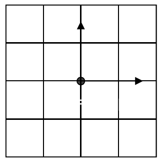
$$\begin{bmatrix} x_1 & y_1 & 1 \end{bmatrix} \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{12} & \omega_{22} & \omega_{23} \\ \omega_{13} & \omega_{23} & \omega_{33} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x_1 x_2 & x_1 y_2 + x_2 y_1 & y_1 y_2 & x_1 + x_2 & y_1 + y_2 & 1 \end{bmatrix} \begin{bmatrix} \omega_{11} \\ \omega_{12} \\ \omega_{22} \\ \omega_{13} \\ \omega_{23} \\ \omega_{33} \end{bmatrix} = 0$$

$$A\overline{\omega} = 0$$

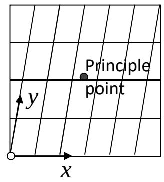
Applications of homogeneous systems: Direct solution of relative orientation





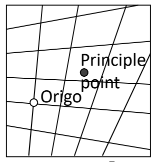
- A calibrated perspective camera (essential matrix)

$$\tilde{m}_1^T E \tilde{m}_2 = 0 \qquad \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0$$



- A non-calibrated perspective camera (fundamental matrix)

$$\tilde{m}_1^T F \tilde{m}_2 = 0 \qquad \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0$$



- The projective version of fundamental matrix

$$\tilde{m}_1^T F \tilde{m}_2 = 0$$

The solution of a homogeneous system

- In a homogeneous system $Ax = 0$, the size of a matrix *A* is $n \times u$ (n = the number of observations, u = the number of unknown parameters)
- The solution of such system has some special features:
  - The system always has a trivial solution $x = 0$ (not interesting)
  - If we find a non-trivial solution $x \neq 0$, also *kx* (k=arbitrary scalar) *is a valid solution*
  - From a homogeneous system we can only find *relative* values of unknown parameters
  - Non-trivial solutions can be found if *the dimension of the kernel N(A) > 0*

The solution of a homogeneous system

- Because the solution gives, in any case, only a relative solution of unknown parameters, the system can be solved by fixing one unknown parameter (any parameter can be selected)
- For example, if we select to fix the last parameters, the solution fulfilling conditions $Ax = 0$ and $x_u = 1$ is

$$\begin{cases} Ax = 0 \\ x_u = 1 \end{cases}$$

- If we place $x_u = 1$ to the equation $Ax = 0$, we get

$$\begin{bmatrix} a_1 & a_2 & \dots & a_{u-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{u-1} \end{bmatrix} = -a_u \quad \text{i.e.} \quad \bar{A}\bar{x} = -a_j$$

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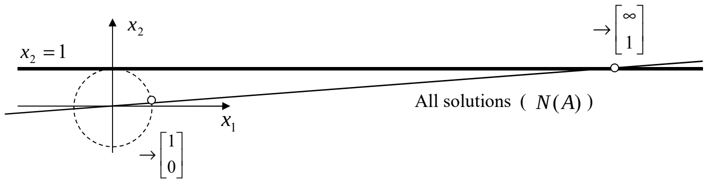
The solution of a homogeneous system

- Because we have now something at the right side of the equation (= non-zero), the system is non-homogeneous (a normal case)!
- Even if we managed to get the solution of the equation, we can have problems in special cases
- If the true value of the parameter which we fixed happens to be (close) to zero, the system cannot be solved (no solution)
- The image below illustrates how other parameters can approach infinity in such situation ($x_2 = 1 \Rightarrow x_1 \rightarrow \infty$)



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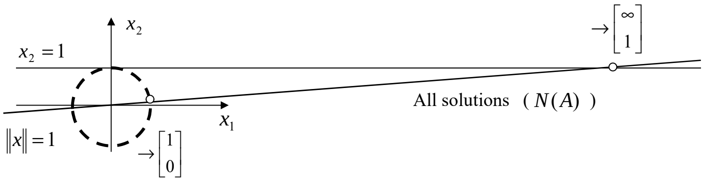
The solution of a homogeneous system



- The figure above also gives a hint that this problem can be avoided if the original condition $x_u = 1$ is replaced with $\|x\| = 1$, which fixes the length of the solution vector
- The solution (in this 2D case) can be only at the circumference of a circle (cannot go to infinity in any special cases)
- Therefore, our aim is to find a solution that fulfills this condition

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The solution of a homogeneous system

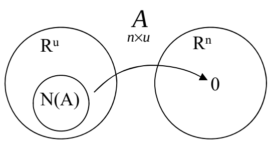
- The solution *x* of the system of equations $Ax = 0$ is the subspace $N(A)$ of the parameter space R^u i.e. the kernel of *A*, i.e.

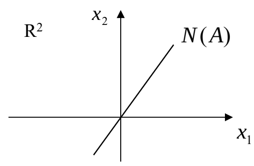
$$x = N(A)$$

- The system of equations have non-trivial solutions only if the dimension of a kernel *N(A) is > 0, i.e..*

$$d = \dim N(A) = u - s(A) > 0 \qquad (s(A) = rank\ of\ the\ matrix)$$

nxu





u = the number of columns i.e. the number of unknown parameters

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The solution of a homogeneous system

Cases A and B

The dimension of a kernel: $d = \dim N(A) = u - s(A) > 0$

n<u (Case A) nxu | n=u (Case B)

- If $n = u - 1$ and $s(A) = n \Rightarrow d = u - n = u - (u-1) = 1$ *(Case A)*
- If $n = u$ and $s(A) = u - 1 \Rightarrow d = u - (u-1) = 1$ *(Case B)*
- In both cases, the dimension of a kernel is 1, which means that the solution of a system $Ax = 0$ is *unique only up to the length of the solution vector*
- The condition $\|x\| = 1$ gives a solution that is *unique up to the sign*

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The solution of a homogeneous system

n>u (C case)

- When the elements of a matrix *A* include *measurements*, we usually have a situation of

  $n \geq u$ and $s(A) = u \Rightarrow d = u - s(A) = u - u = 0$ *(C case)*
- Unfortunately, there is no non-trivial solution to such system of equations $Ax = 0$ in which $x \neq 0$
- However, it is possible to find a least-squares solution by replacing the condition $Ax = 0$ with the least-squares condition $\|Ax\|^2 = \min$
- In following, the solutions of all three cases are illustrated

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n<u

The solution of a homogeneous system, case A

- This case is an underdetermined system
- Let's assume that $n = u - 1$, and that the rank of *A* is full i.e. $s(A) = n$ (i.e. all rows are independent from each other)
- The dimension of a kernel is one, which means that the solution is unique up to the length of the solution vector

$$d = \dim N(A) = u - s(A) = u - n = u - (u-1) = 1$$

- A unique solution is retrieved if the length of a solution vector is fixed with the additional constraint $\|x\| = 1$ (which eliminates the trivial solution $x = 0$)

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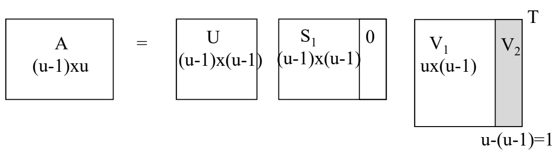
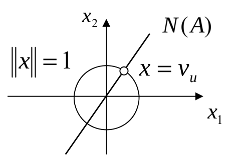
n<u

The solution of a homogeneous system, case A

- The solution that fulfills both conditions $Ax = 0$ and $\|x\| = 1$ is found by utilizing singular value decomposition

$$A = USV^T = U[S_1 \quad 0][V_1 \quad V_2]^T$$





No numerical problems!

- In this special case, the solution vector *x* is the last singular vector v_u (eigenvector)

$$\boxed{x = v_u}$$

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n=u

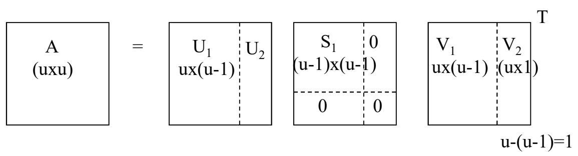
The solution of a homogeneous system, case B

- This case is a square system ($n=u$)
- If the size of a matrix A is u x u, and its rank is

$$s(A) = u - 1$$

it is possible to make a singular value decomposition

$$A = USV^T = [U_1 \quad U_2] \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} [V_1 \quad V_2]^T$$



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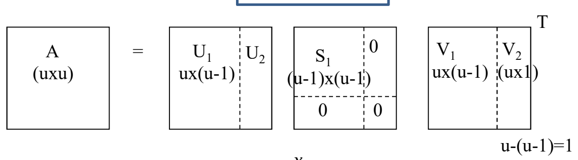
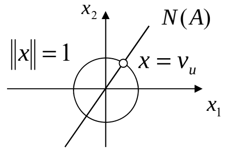
n=u

The solution of a homogeneous system, case B

- Again, it appears that the solution vector x is the last singular vector v_u of a matrix V (the last column), i.e.

$$\boxed{x = v_u}$$





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The solution of a homogeneous system, case C

n>u

- This system is overdetermined
- We try to solve a homogeneous system $Ax = 0$ when

  $n \geq u$ (overdetermined) and $s(A) = u$ (columns have a full rank)

- The latter condition (a full rank) is fulfilled in practice because the elements of *A* are results from measurements and therefore include errors. Therefore, the rank is full (columns are linearly independent) – even if in theory the rank is not full
- Because the dimension *N(A)* of the kernel is *d*=*u*-s(A)=*u*-*u*=0, the system has only a trivial solution *x*=0 (not interesting)

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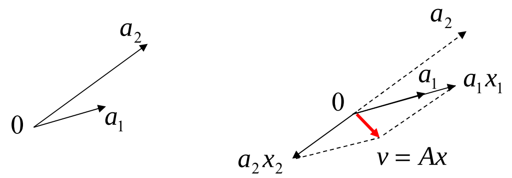
The solution of a homogeneous system, case C

n>u

- We can find the solution (e.g.) by utilizing the least-squares condition

$$S = \|Ax\|^2 = \min$$

- Geometrical interpretation of errors (2D)



- The additional condition $\|x\| = 1$ prevents the parameters to grow infinitely large in any case
- Notice that fixing one parameter is one alternative, but not recommended (no solution if the true value of the fixed parameter is close to zero)

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The solution of a homogeneous system, case C

n>u

- The task basing on conditions $S = \|Ax\|^2 = \min$ and $\|x\| = 1$ is called as a *homogeneous least-squares task*
- Two main alternatives to solve a *homogeneous least-squares task*
  - The solution based on eigenvalue decomposition (Lagrange)
  - The solution based on singular value decomposition

▸ In following, we focus on using singular value decomposition

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The solution of a homogeneous system, case C

n>u

- The solution is based on *generalized singular value decomposition* or, in special cases, on a regular singular value decomposition
- For the special case of the generalized singular value decomposition, we need a constraint matrix *B* (the number of columns must be equal for matrices *A* and *B*, but the number of rows can be different)
  - In a special case *B=I*, the solution is actually a regular singular value decomposition
- (In principle, a matrix *A* constrains rows, and a matrix *B* constrains columns)

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An example of matrix B

- An example of a 2D case, in which a line ax+by+c=0 is fitted to point observations, and the constraint matrix is

$$\|Bx\| = 1$$

- For example, following alternatives can be utilized for constraint equations (and corresponding B matrices):

$$b = 1 \qquad B = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$ (Equals to fixing one parameter)

$$\sqrt{a^2 + b^2 + c^2} = 1 \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\sqrt{a^2 + b^2} = 1 \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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The solution of a homogeneous system, case C

$n>u$

- *Generalized singular value decomposition* is (Matlab function: GSVD)

$$A = US_A F^T = U diag(\alpha_1, ..., \alpha_u) F^T \qquad \alpha_{i+1} \geq \alpha_i$$
$$B = VS_B F^T = V diag(\beta_1, ..., \beta_q) F^T \qquad \beta_{i+1} \leq \beta_i \qquad (q = \min(u, p))$$

in which *U* is an orthogonal matrix (size of *nxn*) $(U^T U = I)$, *V* is also an orthogonal matrix (size of *pxp*) ($V^T V = I$) and *F* is a regular (non-singular) matrix (size of *uxu*)

- Because $\|Ax\|^2 = \|US_A F^T x\|^2 = \|S_A F^T x\|^2$ and $\|Bx\|^2 = \|VS_B F^T x\|^2 = \|S_B F^T x\|^2$, and if we name $y = F^T x$, the original conditions $\|Ax\| = \min$ and $\|Bx\| = 1$ can be replaced with following conditions

$$\|S_A y\| = \min \quad \text{and} \quad \|S_B y\| = 1$$

Therefore, we solve *y* first.

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The solution of a homogeneous system, case C

n>u

- Because $S_A = diag(\alpha_1,\ldots,\alpha_u)$ and $\alpha_{i+1} \geq \alpha_i$ (the smallest value is in the first diagonal element), the solution fulfilling conditions $\|S_A y\| = \min$ and $\|S_B y\| = 1$ is

$$\hat{y} = [1/\beta_1 \quad 0 \quad \ldots \quad 0 \quad 0]^T$$

- The least-squares solution $\hat{x}$ of the original task is solved from the system of equations

$$F^T \hat{x} = \hat{y}$$

$$\boxed{\hat{x} = F^{-T} \hat{y}}$$

- The squared sum can be calculated (even before solving *x*) with equation

$$S = \alpha_1^2 / \beta_1^2 = s_1^2$$

- Residuals are

$$\hat{v} = A\hat{x}$$

Matlab demo…

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The solution of a homogeneous system, special case of C

n>u

- In many cases, however, we end up in a situation, in which *B=I* , and we are able to use a normal singular value decomposition
- Let say that $A = USV^T$ is a singular value decomposition of a matrix *A*, in which case $\|Ax\| = \min \Leftrightarrow \|USV^T x\| = \min$
- Because $\|USV^T x\| = \|SV^T x\|$ and $\|x\| = \|V^T x\|$, we can name $y = V^T x$
- The new constraints are then

$$\|S\hat{y}\| = \min \quad \text{and} \quad \|\hat{y}\| = 1$$

- Because *S* is a diagonal matrix that has sorted elements in the diagonal (from the largest to the smallest), the solution is

$$\hat{y} = [0 \quad 0 \quad \ldots \quad 0 \quad 1]^T$$

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The solution of a homogeneous system, special case of C

n>u

- The solution $\hat{x}$ of the original least-squares task is

$$\hat{x} = V\hat{y} = V\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = V_u$$

i.e. the last column of a matrix *V*, i.e. the singular vector that corresponds to the smallest singular values of a matrix *A*

- A residual vector is $\hat{v} = A\hat{x}$

- A squared sum: $S = \hat{y}^T S^T S\hat{y} = s_u^2$

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