5. Functions of bounded variation

The definition of functions of bounded variation on a metric measure space is based on a relaxation method in the calculus of variations.

5.1. Definition. For $u \in L^1_{loc}(X)$ we define the total variation as

$$\|Du\|(X) = \inf \Big\{ \liminf_{i\to\infty} \int_X g_{u_i}\,d\mu : u_i \in \operatorname{Lip}_{loc}(X), \; u_i \to u \text{ in } L^1_{loc}(X) \Big\},$$

where $g_{u_i} \in L^1_{loc}(X)$ is a 1-weak upper gradient of u_i in X. A function $u \in L^1(X)$ is of bounded variation, $u \in BV(X)$, if $\|Du\|(X) < \infty$. A measurable set $E \subset X$ is said to have finite perimeter, if $\|D\chi_E\|(X) < \infty$.

By replacing X with an open set $U \subset X$ in the definition of the total variation, we can define $\|Du\|(U)$ as above. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) = \inf \{ \|Du\|(U) : U \supset A, \; U \text{ open} \}.$$

If $u \in BV(X)$, then $\|Du\|$ is a finite Borel outer measure on X, see [Miranda, Theorem 3.4], see also 5.6 below.

Next we show that there exists a minimizing sequence in the definition of the total variation.

5.2. Lemma. Assume that $u \in BV(X)$. Then there exists a sequence $(u_i)_{i\in\mathbb{N}}$ such that $u_i \in \operatorname{Lip}_{loc}(X)$ for every $i \in \mathbb{N}$ and $u_i \to u$ in $L^1_{loc}(X)$ with

$$\|Du\|(X) = \lim_{i\to\infty} \int_X g_{u_i}\,d\mu .$$

<u>Proof</u>: Since $u \in BV(X)$, for every $i \in \mathbb{N}$ there exists a sequence $(u_j^i)_{j\in\mathbb{N}}$ such that $u_j^i \to u$ in $L^1_{loc}(X)$ as $j \to \infty$ and

$$\int_X g_{u_j^i}\,d\mu < \|Du\|(X) + \frac{1}{i}, \quad j \in \mathbb{N}.$$

By a diagonalizing argument we may construct a sequence $(u_i)_{i\in\mathbb{N}}$ such that $u_i \to u$ in $L^1_{loc}(X)$ as $i \to \infty$ and

$$\int_X g_{u_i}\,d\mu < \|Du\|(X) + \frac{1}{i}$$

for every $i \in \mathbb{N}$. ~~In fact, we may take $u_i = u_i^i$, $i \in \mathbb{N}$.~~ Thus

$$\begin{aligned}
\|Du\|(X) &\le \liminf_{i\to\infty} \int_X g_{u_i}\,d\mu \\
&\le \liminf_{i\to\infty} \left(\|Du\|(X) + \frac{1}{i}\right) \\
&= \|Du\|(X)
\end{aligned}$$

and consequently

$$\|Du\|(X) = \lim_{i\to\infty} \int_X g_{u_i}\,d\mu. \qquad \square$$

5.3. <u>BV in the Euclidean case</u>. We will show that the definition above coincides with the classical definition on $\mathbb{R}^n$. We begin with recalling the definition of the functions of bounded variation in the classical case. Let $U \subset \mathbb{R}^n$ be an open set. A function $u \in L^1_{loc}(X)$ has bounded variation, $u \in BV_*(U)$, if

$$\|Du\|_*(U) = \sup\Big\{ \underbrace{\int_U \sum_{i=1}^{n} u\, D_i \varphi_i\,dx}_{=\int_U u \operatorname{div}\varphi\,dx} : \varphi \in C_0^\infty(U;\mathbb{R}^n),\ |\varphi| \le 1 \Big\} < \infty,$$

see [Evans-Gariepy]. By the Riesz representation theorem $\|Du\|_*$ is a Radon measure, that is, $\int_U u \operatorname{div}\varphi\,dx = -\int_U \varphi\cdot\sigma\,d\mu$ $\forall \varphi \in C_0^\infty(U;\mathbb{R}^n)$, $|\sigma| = 1$ μ-a.e.

It is known that BV functions can be approximated in a variational sense by smooth functions. That is, for any $u \in BV(U)$ there exists a sequence $(u_i)_{i\in\mathbb{N}}$ of functions $u_i \in C^\infty(U) \cap BV_*(U)$ such that $u_i \to u$ in $L^1(U)$ as $i \to \infty$ and

$$\|Du\|_*(U) = \lim_{i\to\infty} \|Du_i\|_*(U)$$
$$= \lim_{i\to\infty} \int_U |Du_i| \, dx.$$

<u>Remarks</u>. (1) Note that we do not claim

$$\lim_{i\to\infty} \|D(u_i - u)\|_*(U) = 0.$$

Since Sobolev spaces are Banach spaces, this would imply $u \in W^{1,1}(U)$ and then Du would be absolutely continuous with respect to the Lebesgue measure.

(2) Note that $u_i \in C^\infty(U)$ is only locally Lipschitz continuous without further assumptions on U. If $V \subset \mathbb{R}^n$ is open and bounded with ∂U Lipschitz, then we can approximate by Lipschitz functions.

5.4. <u>Theorem</u>. Let $U \subset \mathbb{R}^n$ be an open set. Then $BV(U) = BV_*(U)$.

<u>Proof</u>: $\boxed{``\subset"}$ Assume $u \in BV(U)$. Then by Lemma 5.2 there exists a sequence (u_i), with $u_i \in \mathrm{Lip}_{loc}(U)$ for every $i \in \mathbb{N}$, such that $u_i \to u$ in $L^1_{loc}(U)$ and

$$\lim_{i\to\infty} \int_U g_{u_i} \, dx = \|Du\|(U) < \infty.$$

As a locally Lipschitz function with an integrable 1-weak upper gradient we have $\|Du_i\|(U) < \infty$ for every $i \in \mathbb{N}$.
By the lower semicontinuity of the total variation, see [Evans–Gariepy, Theorem 1 on p. 272], we have

$$\|Du\|_*(U) \le \liminf_{i\to\infty} \|Du_i\|_*(U).$$

Let $\varphi \in C_0^\infty(U;\mathbb{R}^n)$, $|\varphi| \le 1$. Then

$$\int_U u_i \operatorname{div}\varphi \, dx = -\int_U Du_i \cdot \varphi \, dx$$

$$\underset{|\varphi|\le 1}{\le} \int_U |Du_i| \, dx \underset{|Du_i| \le g_{u_i} \text{ (Lemma 2.21)}}{\le} \int_U g_{u_i} \, dx.$$

Thus

$$\|Du\|_*(U) = \overset{\liminf_{i\to\infty}}{\cancel{\sup}} \left\{ \int_U u_i \operatorname{div}\varphi \, dx : \varphi \in C_0^\infty(U;\mathbb{R}^n), |\varphi| \le 1 \right\}$$

$$\le \liminf_{i\to\infty} \int_U g_{u_i} \, dx$$

$$= \|Du\|(U) < \infty.$$

This shows that $u \in BV_*(U)$.

$\boxed{\text{"}\supset\text{"}}$ Assume $u \in BV_*(U)$. Then there exists a sequence $(u_i)_{i\in\mathbb{N}}$ of functions $u_i \in C^\infty(U) \cap BV_*(U)$ such that $u_i \to u$ in $L^1(U)$ and

$$\|Du\|_*(U) = \lim_{i\to\infty} \|Du_i\|_*(U).$$

Then

consider the minimal 1-weak upper gradients, see 2.24.

$$\begin{aligned}\liminf_{i\to\infty}\int_U g_{u_i}\,dx &\le \lim_{i\to\infty}\int_U |Du_i|\,dx\\ &= \lim_{i\to\infty}\|Du_i\|_{\#}(U)\\ &= \|Du\|_{\#}(U) < \infty.\end{aligned}$$

This implies

$$\begin{aligned}\|Du\|(U) &= \inf\Big\{\liminf_{i\to\infty}\int_U g_{u_i}\,dx : u_i \in \operatorname{Lip}_{loc}(U),\ u_i \to u \text{ in } L^1_{loc}(U)\Big\}\\ &\le \|Du\|_{\#}(U) < \infty. \qquad \square\end{aligned}$$

5.5. Theorem. Let $u, v \in L^1_{loc}(X)$ and $U, V \subset X$ be open sets. Then

(i) $\|D(\alpha u)\|(U) = |\alpha|\,\|Du\|(U)$, $\alpha \in \mathbb{R}$,

(ii) $\|D(u+v)\|(U) \le \|Du\|(U) + \|Dv\|(U)$,

(iii) $\|Du\|(U\cup V) \ge \|Du\|(U) + \|Du\|(V)$ if $U \cap V = \emptyset$,

(iv) $\|Du\|(U) \le \|Du\|(V)$ if $U \subset V$

Proof: $\boxed{(i)}$ Assume that $\|Du\|(U) < \infty$. By Lemma 5.2 there exists a sequence $(u_i)_{i\in\mathbb{N}}$, with $u_i \in \operatorname{Lip}_{loc}(U)$ such that $u_i \to u$ in $L^1_{loc}(U)$ and

$$\lim_{i\to\infty}\int_U g_{u_i}\,d\mu = \|Du\|(U).$$

Then $\alpha u_i \to \alpha u$ in $L^1_{loc}(U)$ as $i \to \infty$ and $|\alpha|\, g_{u_i}$ is a 1-weak upper gradient of αu_i. Thus

$$\|D(\alpha u)\|(U) \le \liminf_{i\to\infty} \int_U |\alpha| g_{u_i} \, d\mu = |\alpha| \, \|Du\|(U).$$

From a similar argument it follows that

$$|\alpha| \, \|Du\|(U) \le \|D(\alpha u)\|(U).$$

$\boxed{\text{(ii)}}$ Assume that $\|Du\|(U) + \|Dv\|(V) < \infty$. By Lemma 5.2 there exist sequences (u_i) and (v_i) such that $u_i \to u$ and $v_i \to v$ in $L^1_{loc}(U)$ with

$$\lim_{i\to\infty} \int_U g_{u_i} \, d\mu = \|Du\|(U) \text{ and } \lim_{i\to\infty} \int_U g_{v_i} \, d\mu = \|Dv\|(U).$$

Then $u_i + v_i \to u + v$ in $L^1_{loc}(U)$ as $i \to \infty$ and $g_{u_i} + g_{v_i}$ is a 1-weak upper gradient of $u_i + v_i$ for $i \in \mathbb{N}$. Thus

$$\begin{aligned}
\|D(u+v)\|(U) &\le \liminf_{i\to\infty} \int_U (g_{u_i} + g_{v_i}) \, d\mu \\
&= \lim_{i\to\infty} \int_U g_{u_i} \, d\mu + \lim_{i\to\infty} \int_U g_{v_i} \, d\mu \\
&= \|Du\|(U) + \|Dv\|(U).
\end{aligned}$$

$\boxed{\text{(iii)}}$ Assume that $\|Du\|(U \cup V) < \infty$. By Lemma 5.2 there exists a sequence (u_i) such that $u_j \in \mathrm{Lip}_{loc}(U \cup V)$ for every $i \in \mathbb{N}$, $u_i \to u$ in $L^1_{loc}(U \cup V)$ as $i \to \infty$ and

$$\|Du\|(U \cup V) = \lim_{i\to\infty} \int_{U\cup V} g_{u_i} \, d\mu.$$

Since $U \cap V = \emptyset$, we have

$$\lim_{i\to\infty} \int_{U\cup V} g_{u_i} \, d\mu = \lim_{i\to\infty} \int_U g_{u_i} \, d\mu + \lim_{i\to\infty} \int_V g_{u_i} \, d\mu.$$

Since $u_i \to u$ in $L^1_{loc}(U \cup V)$, we have $u_i \to u$ in $L^1_{loc}(U)$ and $u_i \to u$ in $L^1_{loc}(V)$. Thus

$$\lim_{i\to\infty} \int_U g_{u_i}\, d\mu \geq \|Du\|(U) \text{ and } \lim_{i\to\infty} \int_V g_{u_i}\, d\mu \geq \|Du\|(V).$$

This implies $\|Du\|(U \cup V) \geq \|Du\|(U) + \|Du\|(V)$.

$\boxed{\text{(iv)}}$ Assume that $\|Du\|(V) < \infty$. By Lemma 5.2 there exists a sequence (u_i) such that $u_i \in \operatorname{Lip}_{loc}(V)$ for every $i \in \mathbb{N}$, $u_i \to u$ in $L^1_{loc}(V)$ and

$$\lim_{i\to\infty} \int_V g_{u_i}\, d\mu = \|Du\|(V).$$

Since $U \subset V$, we have $u_i \in \operatorname{Lip}_{loc}(U)$ and $u_i \to u$ in $L^1_{loc}(U)$. Thus

$$\begin{aligned} \|Du\|(U) &\leq \liminf_{i\to\infty} \int_U g_{u_i}\, d\mu \\ &\leq \liminf_{i\to\infty} \int_V g_{u_i}\, d\mu = \|Du\|(V). \end{aligned}$$

$\square$

5.6. <u>Total variation measure</u>. We would like to prove that $\|Du\|$ defines a measure. Observe that we cannot apply the Riesz representation theorem as in the Euclidean case. We shall use an approach that is typical in the relaxation methods. According to a lemma of De Giorgi and Letta, it is enough to prove that $\|Du\|$ has the following properties on the class of open sets in X:

(i) $\|Du\|(U) \leq \|Du\|(V)$ if $U \subset V$,

(ii) $\|Du\|(U \cup V) \leq \|Du\|(U) + \|Du\|(V)$ for every open $U, V \subset X$,

(iii) $\|Du\|(U) = \sup\{\|Du\|(V) : V \subset\subset U\}$ and

(iv) $\|Du\|(U \cup V) \geq \|Du\|(U) + \|Du\|(V)$ when $U \cap V = \emptyset$, to conclude that $\|Du\|$ is a finite Borel regular outer measure, that is, a Radon measure on X, if $u \in BV(X)$.

We shall show that (i) – (iv) imply the countable additivity on disjoint open sets. Assume that E_i is open for every $i \in \mathbb{N}$ and let $E = \bigcup_{i=1}^{\infty} E_i$. Let $\varepsilon > 0$. Property (iii) implies that there exists an open set $U \subset\subset E$ such that

$$\|Du\|(E) < \|Du\|(U) + \varepsilon.$$

Since $\overline{U}$ is compact and $\overline{U} \subset E$, we have $U \subset \overline{U} \subset \bigcup_{i=1}^{m} E_i$ for some $m \in \mathbb{N}$.

By (ii)

$$\|Du\|(U) \le \|Du\|\Big(\bigcup_{i=1}^{m} E_i\Big) \le \sum_{i=1}^{m} \|Du\|(E_i),$$

which gives

$$\begin{aligned} \|Du\|(E) &\le \sum_{i=1}^{m} \|Du\|(E_i) + \varepsilon \\ &\le \sum_{i=1}^{\infty} \|Du\|(E_i) + \varepsilon. \end{aligned}$$

By letting $\varepsilon \to 0$ we obtain

$$\|Du\|(E) \le \sum_{i=1}^{\infty} \|Du\|(E_i).$$

This is countable subadditivity. For pairwise disjoint E_i, $i \in \mathbb{N}$, by (iv)

$$\begin{aligned} \|Du\|(E) &\ge \|Du\|(E_1) + \|Du\|\Big(\bigcup_{i=2}^{\infty} E_i\Big) \\ &\ge \dots \ge \sum_{i=1}^{\infty} \|Du\|(E_i). \end{aligned}$$

Thus $\|Du\|$ is countably additive on disjoint open sets.

We refer to Miranda's paper for the proof of (ii) and (iii).