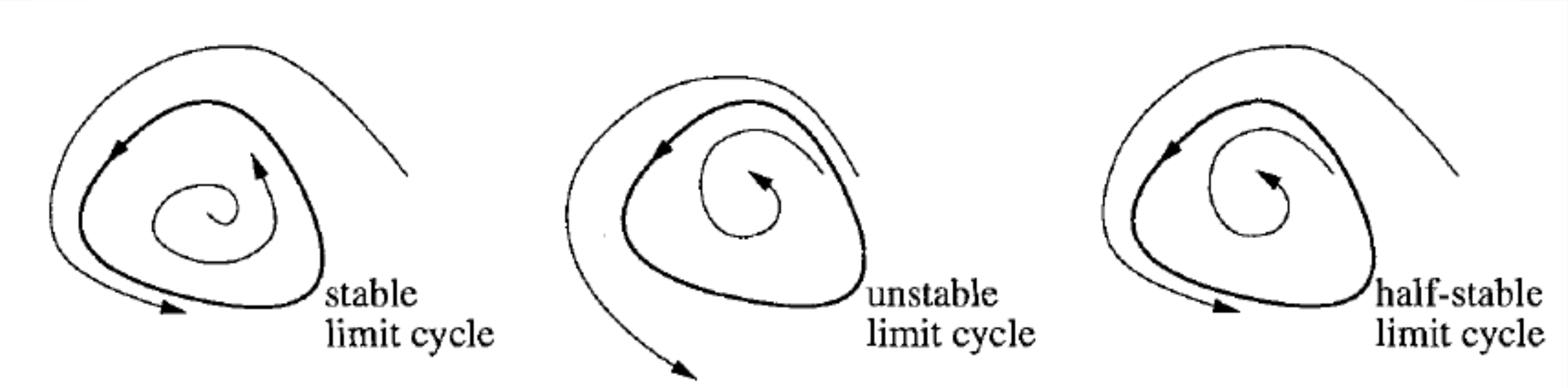
Nonlinear dynamics & chaos

Bifurcations in 2D

Lecture VII

Recap

Stable and unstable limit cycles in 2D.



Ways to rule out closed orbits.

Ways to prove existence of closed orbits.

Nonlinear oscillations.

Limit cycles are the basis for a new kind of bifurcation in 2D.

Bifurcations in 2D

Saddle-node bifurcation

Prototypical example in 2D:

$$\dot{x} = \mu - x^2$$

$$\dot{y} = -y$$

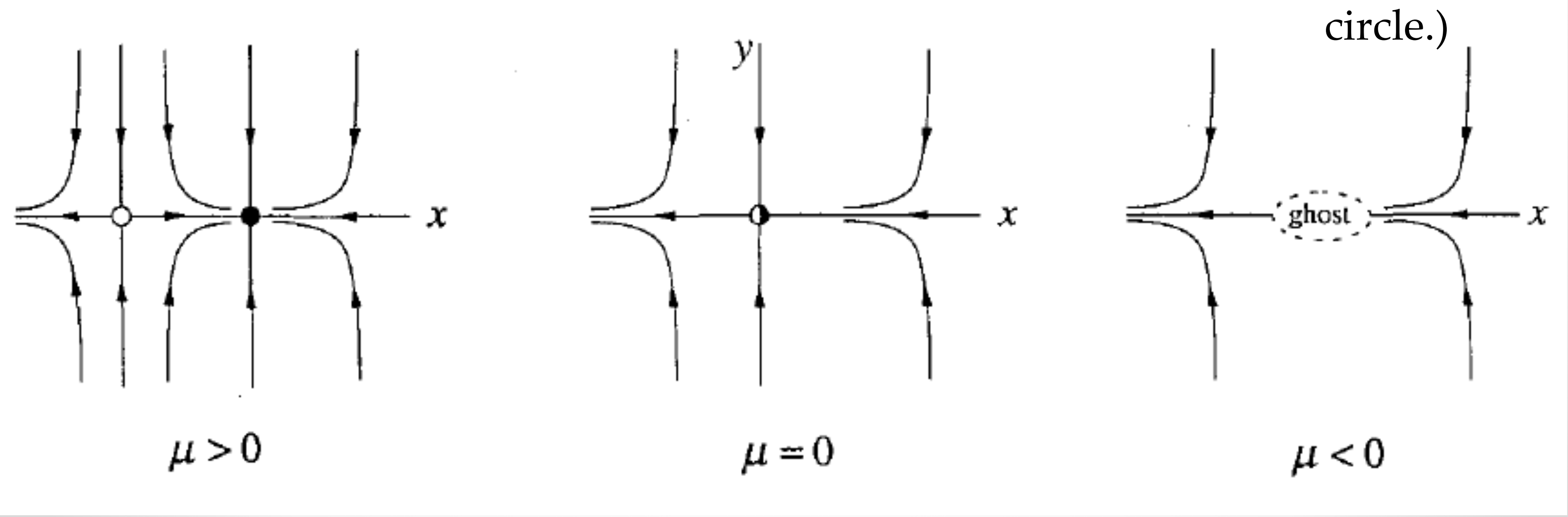
In the *x*-direction: the familiar bifurcation behaviour.

In the *y*-direction: exponential damping.

After the collision of fixed points $(\pm\sqrt{\mu},0)$, $\mu \to \mu_c=0$ a ghost, or a bottleneck region remains.

Time spent in the bottleneck: $t \propto (\mu - \mu_c)^{-1/2}$

(See nonlinear oscillators in Flows on the circle.)

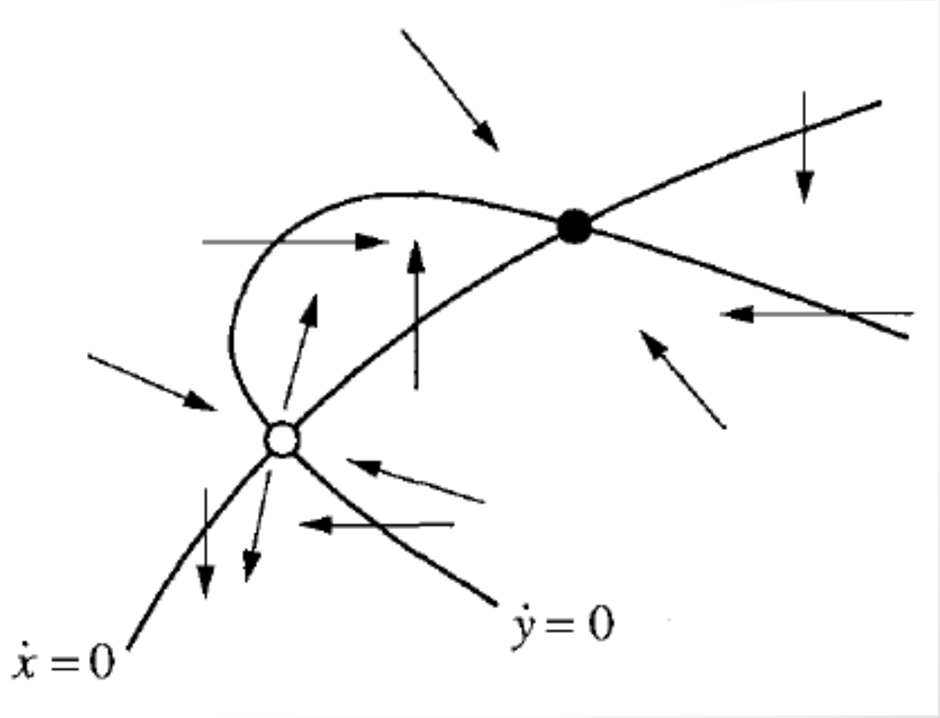


Saddle-node bifurcations

The general case

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

The fixed points at the intersections of the nullclines (ncs) get closer as the ncs pull away, collide when ncs become tangent at $\mu = \mu_c$, and disappear when ncs become detached.



Example I: Genetic control

Model for a genetic control; the activity of a gene is directly induced by two copies of the protein for which it codes. The gene is stimulated by its own product, which can lead to autocatalytic feedback process.

x and y are proportional to the concentrations of the protein and the messenger RNA from which it is translated, respectively.

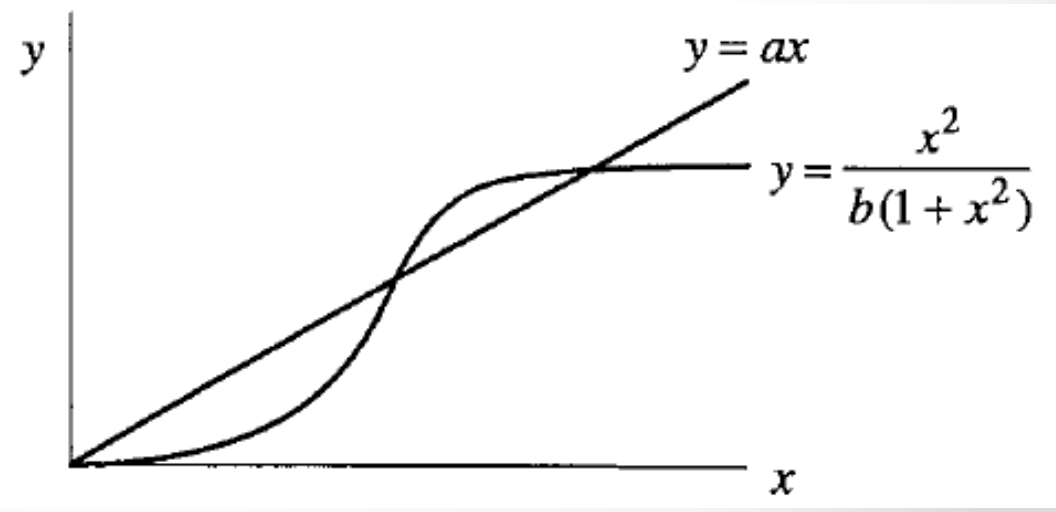
$a, b > 0$.

$$\dot{x} = -ax + y$$

$$\dot{y} = \frac{x^2}{1+x^2} - by$$

Nullclines

$$y = ax, \;\; y = \frac{x^2}{b(1+x^2)}.$$



Example I: Genetic control

$$\dot{x} = -ax + y$$

$$\dot{y} = \frac{x^2}{1+x^2} - by$$

Fixed points = intersections of nullclines:

$$ax = \frac{x^2}{b(1+x)}$$

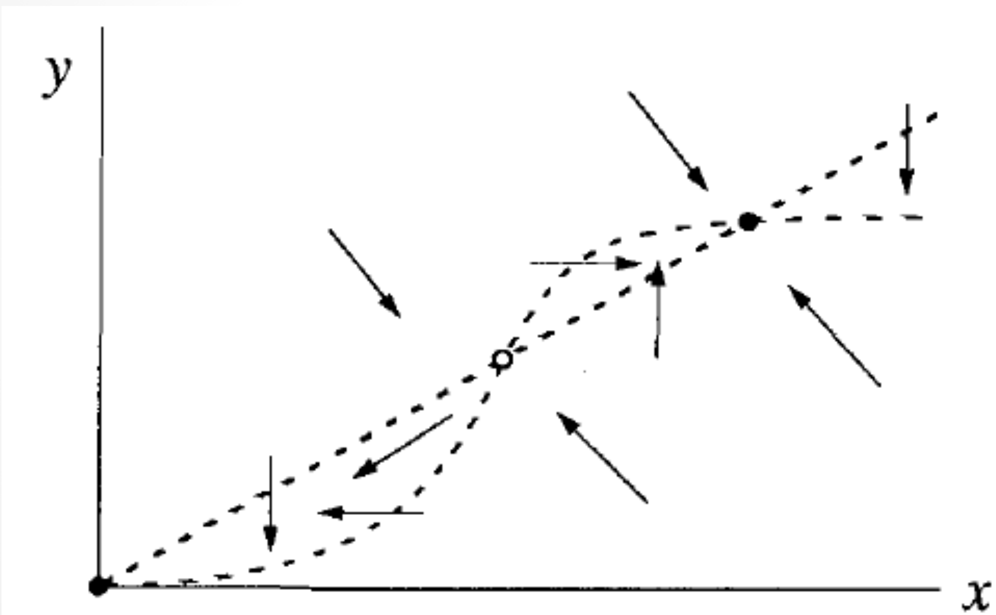
$$x* = 0, \;\; y* = 0 \text{ and } x* = \frac{1 \pm \sqrt{1 - 4a^2b^2}}{2ab}, y* = ax*,$$

$$\text{if } 1 - 4a^2b^2 > 0 \Leftrightarrow 2ab < 1.$$

The last two FP coalesce when $a_c = 1/(2b)$. At this bifurcation $x^* = 1$.

Example I: Genetic control

Determine the vector field starting from the nullclines (vertical and horizontal fields).



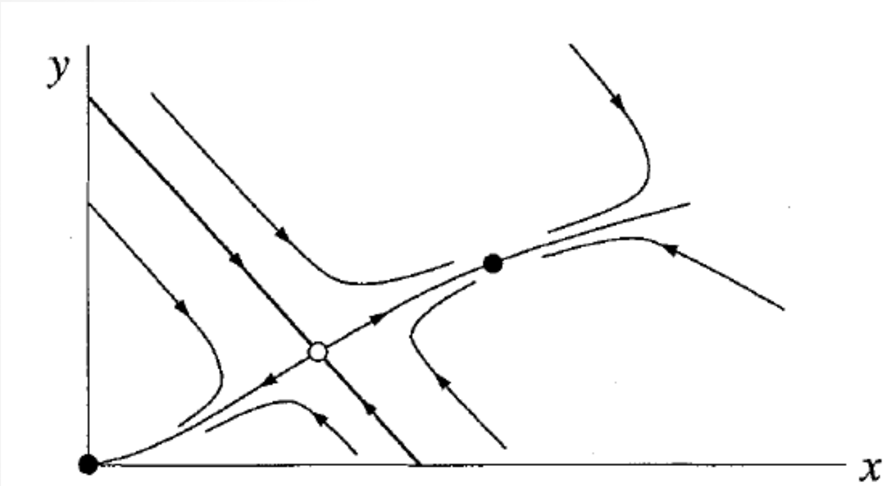
The Jacobian

$$A = \begin{pmatrix} -a & 1 \\ \frac{2x}{(1+x^2)^2} & -b \end{pmatrix}$$

$\tau < 0$, so all the fixed points are either sinks (essentially, stable) or saddles depending on Δ.

Example I: Genetic control

(0,0) is always a stable FP (we disregard the degenerate case $a = b$). For $0 < x^* < 1$ FP is saddle. For $x^* > 1$ FP is always a stable node. The phase portrait:



The unstable manifold of the saddle is trapped in between the two nullclines.
The stable manifold separates the plane into two basins of attractions of the two sinks.

Biological interpretation: The system can act as a biochemical switch, if the mRNA and protein degrade slowly enough, the decay rates satisfying $ab < ½$. Then, two steady states: 1. at the origin the gene is silent and there's no protein around to turn it on; 2. for x^*, y^* large the gene is active and sustained by the high protein level.

Transcritical and pitchfork bifurcations

In the same manner, by introducing exponential damping in the *y*-direction, we can construct prototypical example systems of other bifurcations in 2D.

$$\dot{x} = \mu x - x^2, \dot{y} = -y \text{ (transcritical)}$$

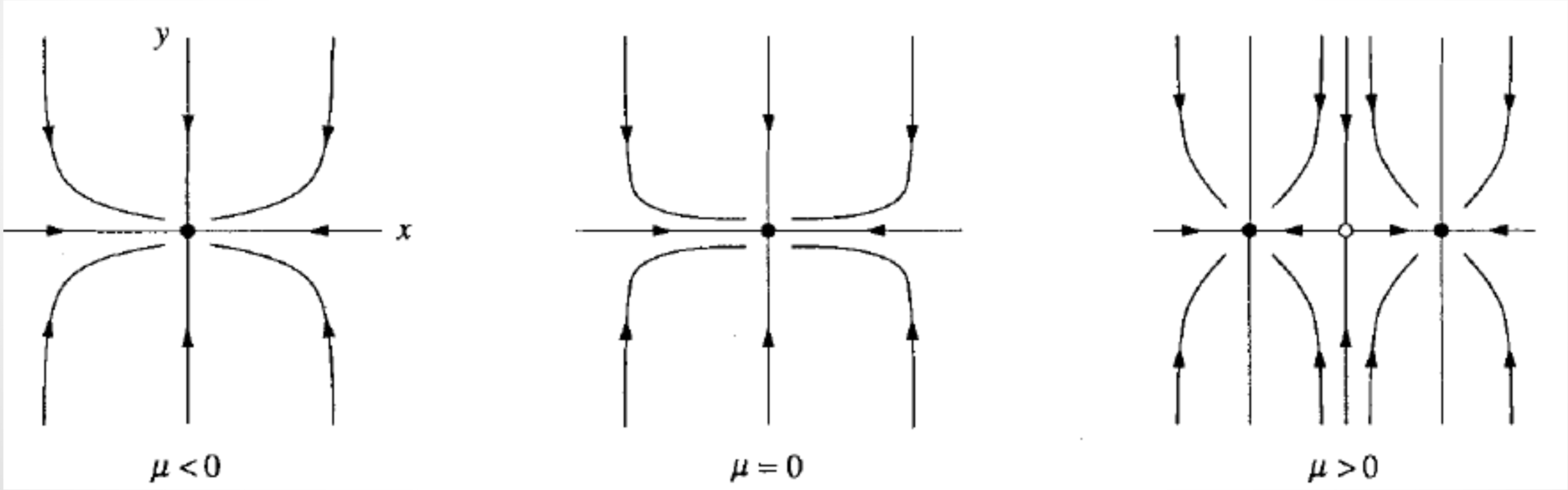
$$\dot{x} = \mu x - x^3, \dot{y} = -y \text{ (supercritical pitchfork)}$$

$$\dot{x} = \mu x + x^3, \dot{y} = -y \text{ (subcritical pitchfork)}$$

Ex.II: supercritical bifurcation

$$\dot{x} = \mu x - x^3$$

$$\dot{y} = -y$$



Example III

$$\dot{x} = \mu x + y + \sin x$$
$$\dot{y} = x - y$$

Invariance under $x \to -x$, $y \to -y$, so the phase portrait must be symmetric under reflection through the origin.

The origin is a FP for all μ. The Jacobian at (0,0):

$$A = \begin{pmatrix} \mu + 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$\tau = \mu$ and $\Delta = -(\mu + 2) \Rightarrow$

At (0,0) there's a stable FP if $\mu < -2$ and a saddle if $\mu > -2$ (and $\mu < 0$).

The symmetry and the change in the stability of (0,0) suggests a pithfork bifurcation. ➔ Look for a symmetric pair of FPs close to the origin for μ close to $\mu_c = -2$.

Example III

The fixed points satisfy $y = x \Rightarrow (\mu + 1) + \sin x = 0.$

For a small $x \neq 0$, $$(\mu + 1)x + x - \frac{x^3}{3!} + O(x^5) = 0$$

$$\Rightarrow \mu + 2 - x^2/6 \approx 0 \Rightarrow x^* \approx \pm\sqrt{6(\mu + 2)}$$

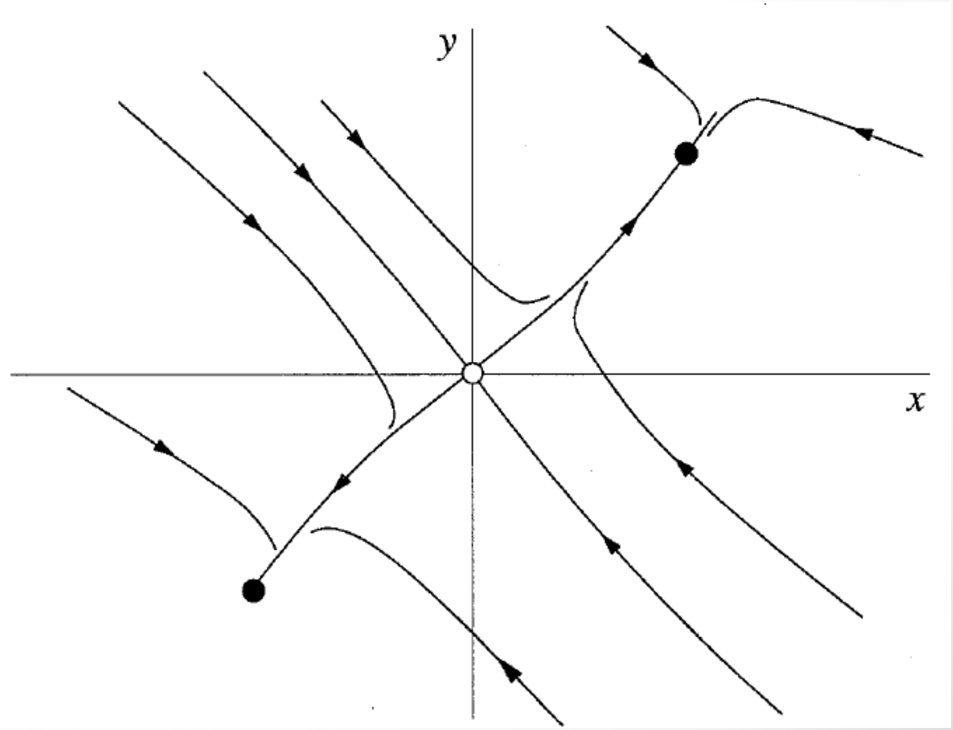
for μ slightly greater than -2.

A **supercritical** pitchfork bifurcation occurs at $\mu_c = -2$, because the pair of fixed points exist **after** the origin has become a saddle for $\mu > -2$. (In other words, the pair of FPs do not exist when the origin is still a stable FP for $\mu < -2$, which would be the case for a subcritical pitchfork bifurcation.)

Example III

Since the bifurcation is supercritical, the FPs are stable. To help drawing the phase portrait one can determine the eigenvectors at the origin at bifurcation. Using the Jacobian for $\mu = -2$, we solve for the eigenvectors as (1,1) and (1,-1).

The phase portrait for μ slightly greater than -2 and near the origin. Remember that analyses was made by linearization: It is valid only for small x and μ close to μ_c. Means: system is close to bifurcation.



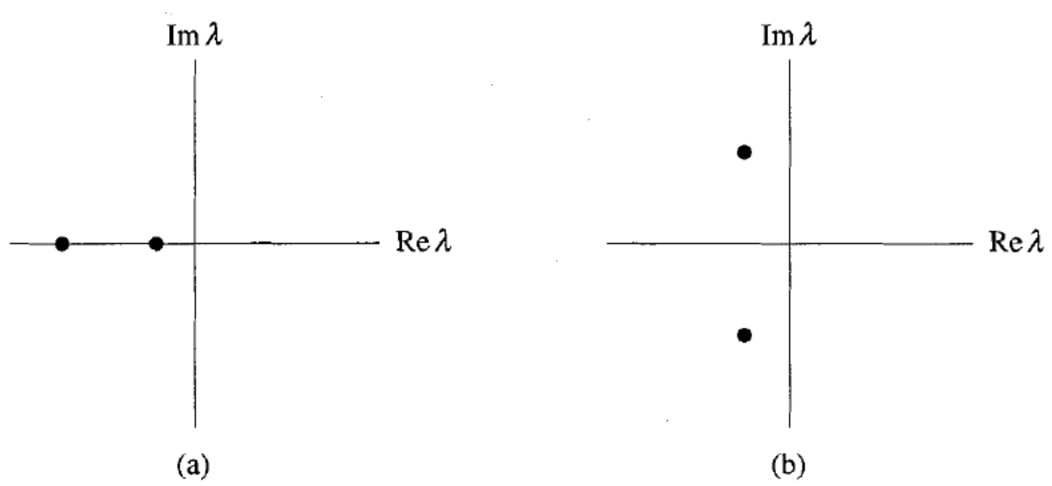
Zero-eigenvalue bifurcations

All the bifurcations up to now have occurred when $\Delta = 0$, i.e. when one eigenvalue equals zero ($\Delta = \lambda_1 \lambda_2$). These zero-eigenvalue bifurcations always involve collision of two or more fixed points. They typically have a counterpart in one-dimensional systems. This is not the case for the following Hopf bifurcations.

Hopf bifurcations

In what ways can a stable fixed point in a 2D system lose its stability? The eigenvalues of the Jacobian are the key.

A stable FP must have Re $\lambda < 0$. Destabilization: Re λ becomes positive. Two possibilities:

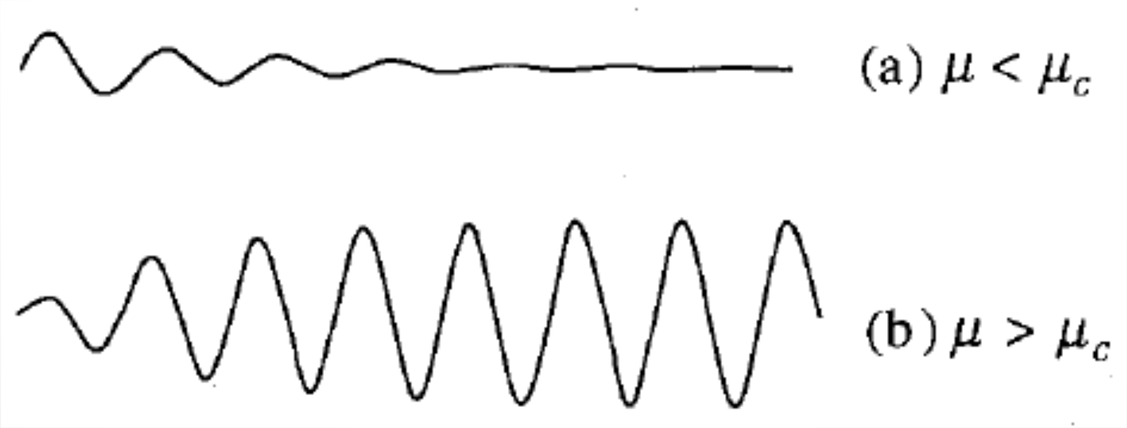


The new bifurcation: two complex-conjugate eigenvalues simultaneously cross the imaginary axis into the right half-plane. A Hopf bifurcation can occur in phase spaces of dimension $n \geq 2$.

The previous zero-eigenvalue bifurcations: a real eigenvalue passes through $\lambda = 0$.

Supercritical Hopf bifurcation

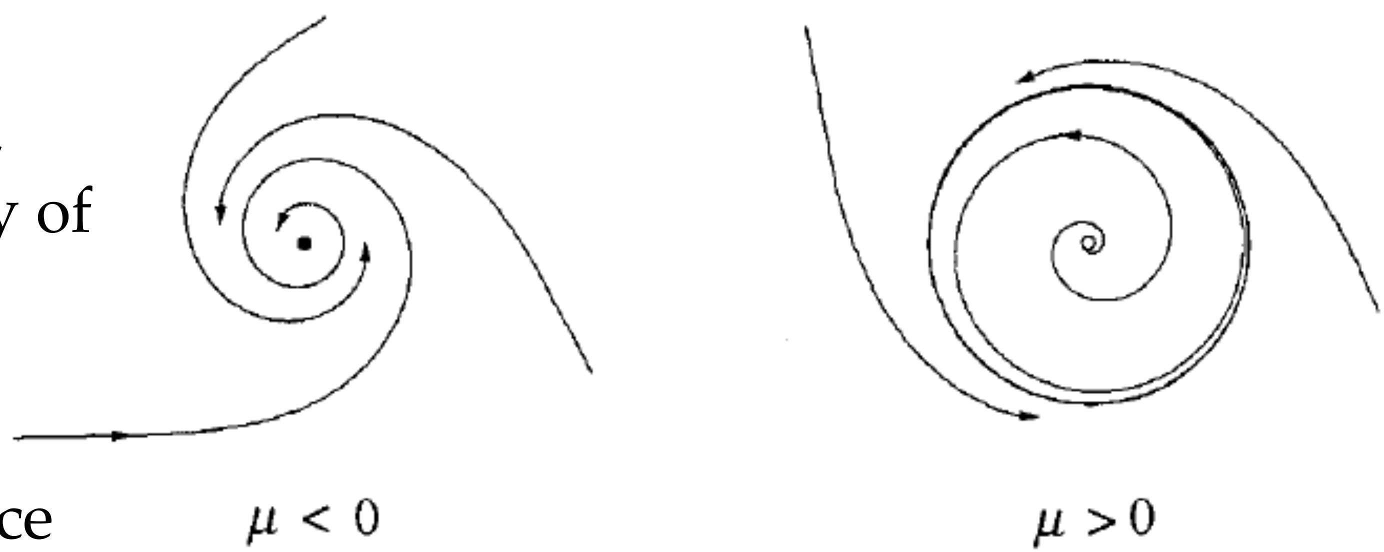
Suppose a system settles down to equilibrium through exponentially damped oscillations and the decay rate depends on a control parameter μ. If the decay becomes slower and slower and finally changes to growth at a critical value μ_c, the equilibrium state will lose stability. Often a small-amplitude sinusoidal limit-cycle oscillation about the former steady state results. These are the characteristics of a supercritical Hopf bifurcation. In the phase plane: a stable spiral ➔ an unstable spiral surrounded by small, nearly elliptical limit cycle.



Supercritical Hopf bifurcation

Example system:
$$\dot{r} = \mu r - r^3$$
$$\dot{\theta} = \omega + b r^2$$

Limit cycle at $r = \sqrt{\mu}$.



μ controls the stability of FP, ω = frequency of *infinitesimal oscillations*, b determines the dependence of frequency on amplitude for larger amplitude oscillations

Supercritical Hopf bifurcation

To see how the eigenvalues behave during the bifurcation, rewrite the system in Cartesian coordinates.

$$x = r\cos\theta,\ \ y = r\sin\theta$$

Trick to remember!

$$\begin{aligned}\dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ &= (\mu r - r^3)\cos\theta - r(\omega + br^2)\sin\theta \\ &= [\mu - (x^2+y^2)]x - [\omega + b(x^2+y^2)]y \\ &= \mu x - \omega y + \text{ cubic terms}\end{aligned}$$

Similarly,

$$\dot{y} = \omega x + \mu y + \text{ cubic terms}$$

$$\Rightarrow A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

Jacobian at the origin.

$$\Rightarrow \lambda = \mu \pm i\omega$$

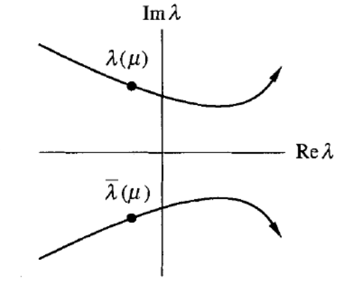
The eigenvalues cross the imaginary axis from left to right as μ increases from negative to positive values.

Supercritical Hopf bifurcation

Rules of thumb

1. The size (radius) of the limit cycle grows continuously from zero and increases proportional to $(\mu - \mu_c)^{1/2}$ for μ close to μ_c.
2. The frequency of the limit cycle $\omega = \mathrm{Im}\,\lambda$ at $\mu = \mu_c$ (=0). This is also correct within $O(\mu - \mu_c)$ for μ close to μ_c. ➔ The period $T = (2\pi/\mathrm{Im}\,\lambda) + O(\mu - \mu_c)$.

However, **in Hopf bifurcations encountered in practice**:
1. The limit cycle is elliptical, not circular.
2. The shape of the limit cycle becomes distorted as μ moves away from μ_c.
3. $\mathrm{Im}\,\lambda$ depends on μ.



Subcritical Hopf bifurcation

Example system:

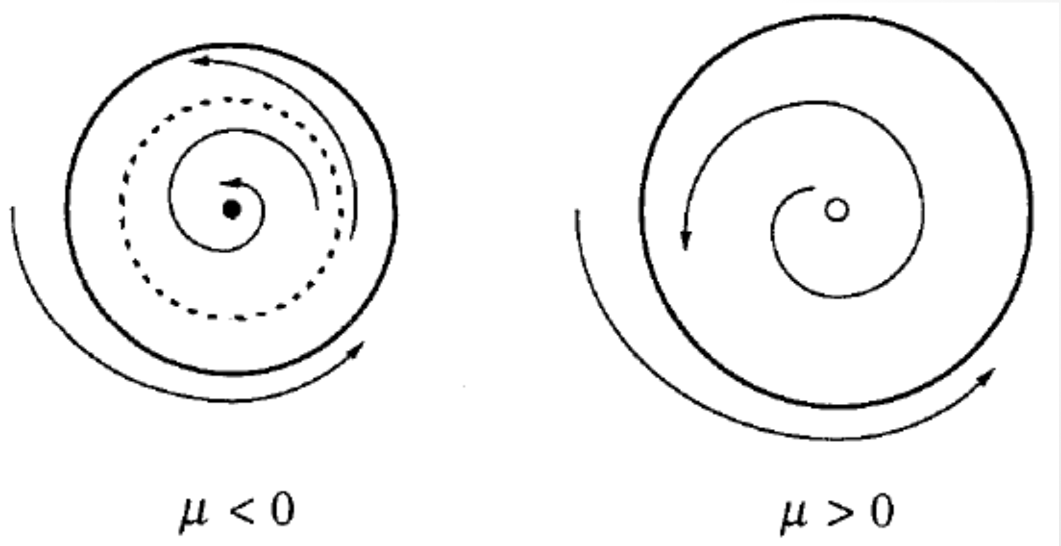
$$\dot{r} = \mu r + r^3 - r^5$$

$$\dot{\theta} = \omega + br^2$$

The crucial difference to the subcritical case: the cubic term is destabilizing.

Phase portrait:

The unstable cycle between the stable FP at the origin and the stable limit cycle tightens around the FP as μ increases.



At $\mu = \mu_c = 0$ the unstable cycle engulfs the origin, which becomes unstable. ➔ Large-amplitude oscillations.

Subcritical Hopf bifurcation

Also this **subcritical system exhibits hysteresis**: once large-amplitude oscillations have begun, they cannot be turned off by bringing μ back to zero. The large oscillations persist until $\mu = -1/4$ where the stable and unstable cycles collide and annihilate. This is a so-called saddle-node bifurcation of cycles (coming up).

Subcritical Hopf bifurcations occur for example in the dynamics of nerve cells, in aeroelastic flutter and other vibrations of airplane wings, and in instabilities of fluid flows.

Identifying Hopf bifurcations

Supercritical, if a small attracting limit cycle appears immediately after FP goes unstable, and its amplitude shrinks back to zero as the parameter is reversed (no hysteresis).

Subcritical in most other cases. If hysteresis, then for sure.

Degenerate: For example, changing the damping μ from positive to negative in the damped pendulum $\ddot{x} + \mu\dot{x} + \sin x = 0$ turns FP at the origin from a stable to an unstable spiral. However, there are no limit cycles on either side of the bifurcation, but a continuous band of closed orbits surrounding (0,0). This is not a true Hopf bifurcation. Typically happens when a nonconservative system becomes conservative **at the bifurcation point**: **FP becomes a nonlinear centre**, not a weak spiral.

Oscillating chemical reactions

The first experimental observation of Hopf bifurcations was made in the early 50’s by the Russian biochemist Boris Belusov. He was not believed and did not get his paper published.

In 1961 graduate student Zhabotinsky proved him right: BZ reactions. Nowadays oscillations in chemical reactions are the prototypical examples of Hopf bifurcations.

Expanding circular waves of oxidation.

Oscillating chemical reactions

The Belousov-Zhabotinsky Oscillating Reaction

https://youtu.be/PpyKSRo8Iec

See in the book the construction of trapping region etc. to prove the existence of oscillations.

Global bifurcations of cycles

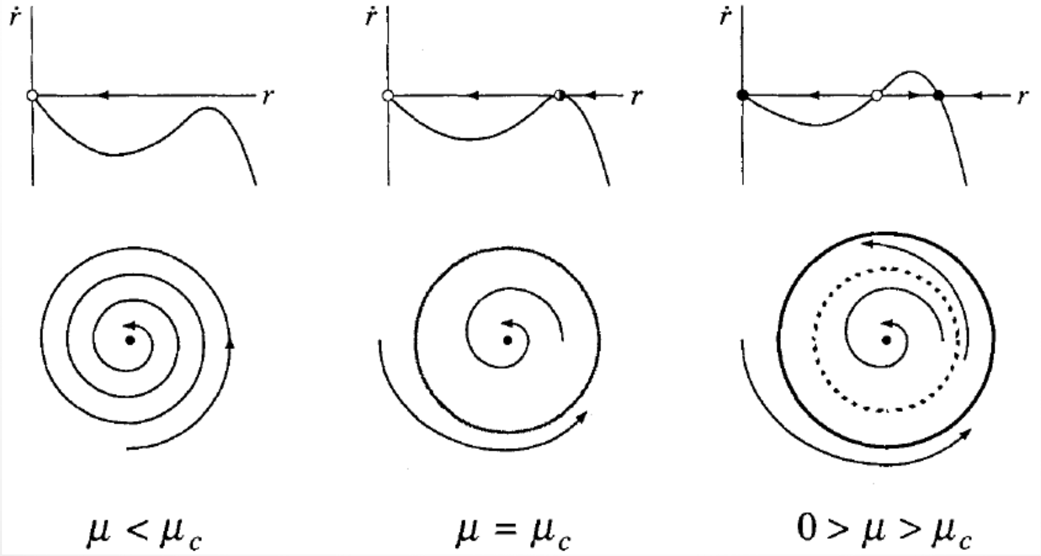
- involve large regions of the phase plane (instead of just the neighbourhood of fixed point)

Saddle-node bifurcation of cycles

$$\dot{r} = \mu r + r^3 - r^5$$
$$\dot{\theta} = \omega + br^2$$

(Exhibits the subcritical Hopf bifurcation at $\mu = 0$ as seen in the previous section.)

The radial equation exhibits a saddle-node bifurcation of fixed points at μ_c= -1/4. In 2D the FPs correspond to limit cycles.



Global bifurcations of cycles

Infinite-period bifurcation

$$\dot{r} = r(1 - r^2)$$

$$\dot{\theta} = \mu - \sin\theta, \;\; \mu \geq 0.$$

a) The radial part is of the same form as the one-dimensional supercritical bifurcation system with no control parameter (so, no bifurcation in this alone).
b) The angular part is of the same form as a nonuniform oscillator.

Radial direction: There's a FP at $r^* = 0$. All other trajectories approach the unit circle ($r^* = 1$) monotonically as $t \to \infty$.

Angular direction: $\mu_c = 1$. If $\mu > 1$, the motion is counterclockwise everywhere. If $\mu < 1$, there are two invariant rays defined by $\sin\,\theta = \mu$.

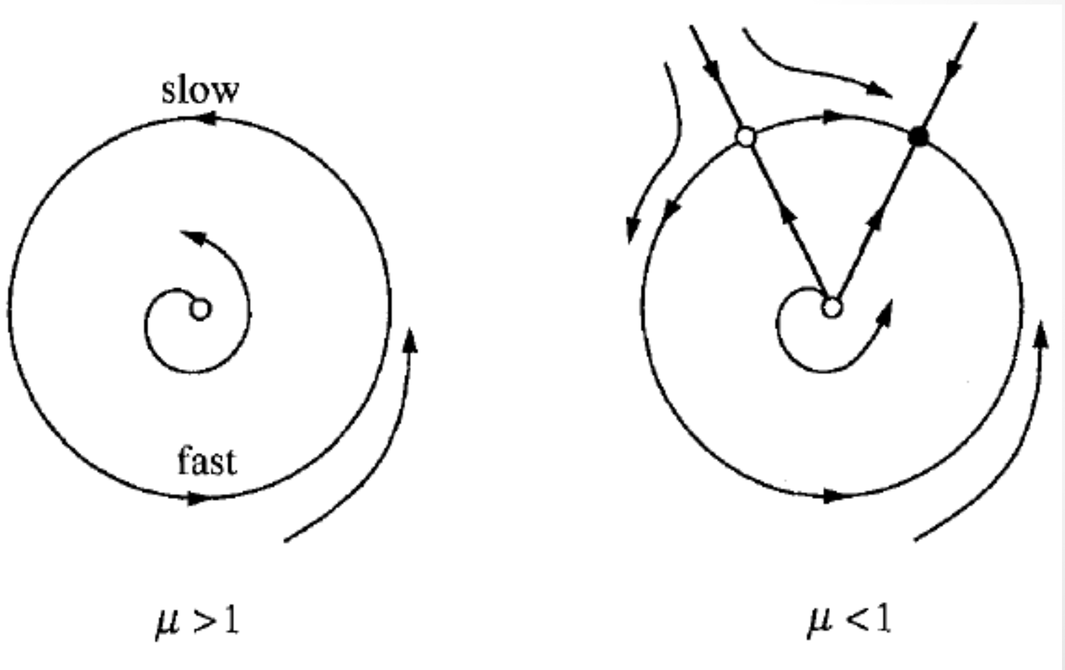
Global bifurcations of cycles

Infinite-period bifurcation

As $\mu \rightarrow 1^+$ the bottleneck in the limit cycle $r = 1$ becomes increasingly severe. **At $\mu_c = 1$** the fixed point appears on the circle and **the oscillation period becomes infinite**.

$$T \propto (\mu - \mu_c)^{-1/2}$$

For $\mu < 1$ the fixed point splits into a saddle and a node.



Global bifurcations of cycles

Homoclinic bifurcation (also called saddle-loop bifurcation)

Here, part of a limit cycle approaches a saddle point. At the bifurcation the cycle touches the saddle point and becomes a homoclinic orbit.

There is no clear analytic example. Numerical solution of the system

$$\dot{x} = y$$

$$\dot{y} = \mu y + x - x^2.$$

Global bifurcations of cycles

Homoclinic bifurcation

$$\dot{x} = y$$

$$\dot{y} = \mu y + x - x^2.$$

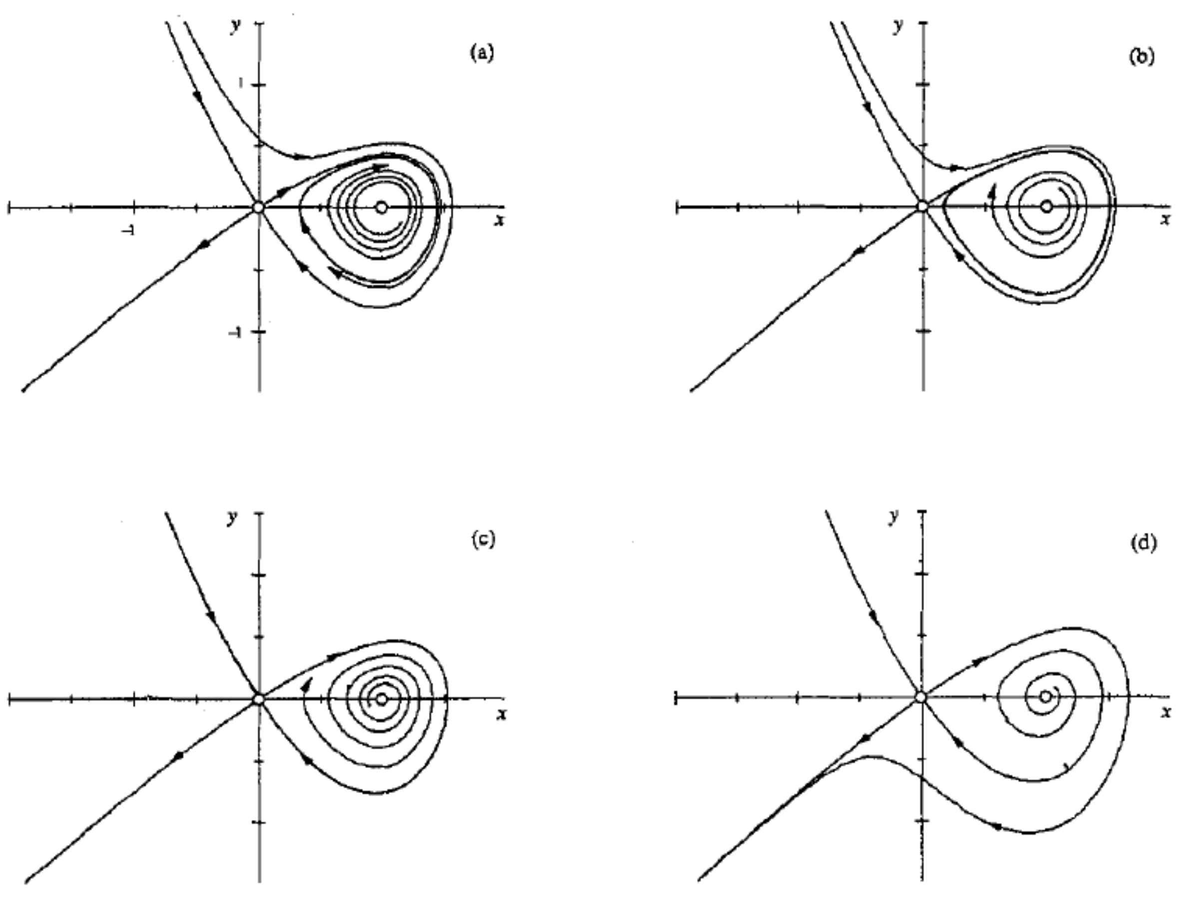
(a) $\mu < \mu_c$: a stable limit cycle passes close to a saddle point at the origin.

(b) μ increases ➔ the limit cycle swells.

(c) $\mu = \mu_c$: a homoclinic orbit is created.

(d) $\mu > \mu_c$: the saddle connection breaks, the loop breaks.

Numerically: bifurcation at $\mu_c \approx -0.8645$.



Global bifurcations of cycles

Scaling laws

Characteristic scaling laws govern the and period of the limit cycle as the bifurcation is approached. Here, $\mu << 1$ denotes a dimensionless measure of the distance from the bifurcation.

	Amplitude of stable limit cycle	Period of cycle
Supercritical Hopf	$O(\mu^{1/2})$	$O(1)$
Saddle-node bifurcation of cycles	$O(1)$	$O(1)$
Infinite-period	$O(1)$	$O(\mu^{-1/2})$
Homoclinic	$O(1)$	$O(\ln \mu)$

Global bifurcations of cycles

There are exeptions. Consider **the van der Pol oscillator**:

$$\ddot{x} + \epsilon\dot{x}(x^2 - 1) + x = 0$$

At $\varepsilon = 0$, the eigenvalues at the origin are pure imaginary ($\lambda = \pm i$) suggesting Hopf bifurcation at $\varepsilon = 0$, but we know that for $0 < \varepsilon \ll 1$ the system has a limit cycle of amplitude $r \approx 2$. The cycle is born full grown, not $O(\sqrt{\varepsilon})$.

Explanation: The bifurcation at $\varepsilon = 0$ is **degenerate**. The nonlinear term $\epsilon\dot{x}x^2$ vanishes at the same parameter value as the eigenvalues cross the imaginary axis; a **nongeneric coincidence**.

Global bifurcations of cycles

$$\ddot{x} + \epsilon \dot{x}(x^2 - 1) + x = 0$$

Rescale to remove degeneracy: Write the equation as

$\ddot{x} + x + \epsilon x^2 \dot{x} - \epsilon \dot{x} = 0$ and let $u = \epsilon x^2 \Rightarrow u = \epsilon^{1/2} x$

$$\Rightarrow \ddot{u} + u + u^2 \dot{u} - \epsilon \dot{u} = 0.$$

Now the nonlinear term is not destroyed when the eigenvalues become pure imaginary.

The limit-cycle solution is $x(t, \epsilon) \approx 2 \cos t$ for $0 < \epsilon << 1.$

In terms of u: $u(t, \epsilon) \approx 2\sqrt{\epsilon} \cos t.$

The amplitude grows like $\sqrt{\varepsilon}$ as expected for a Hopf bifurcation!

Mere reparametrisation $\rightarrow$ a change in dynamics! (?)

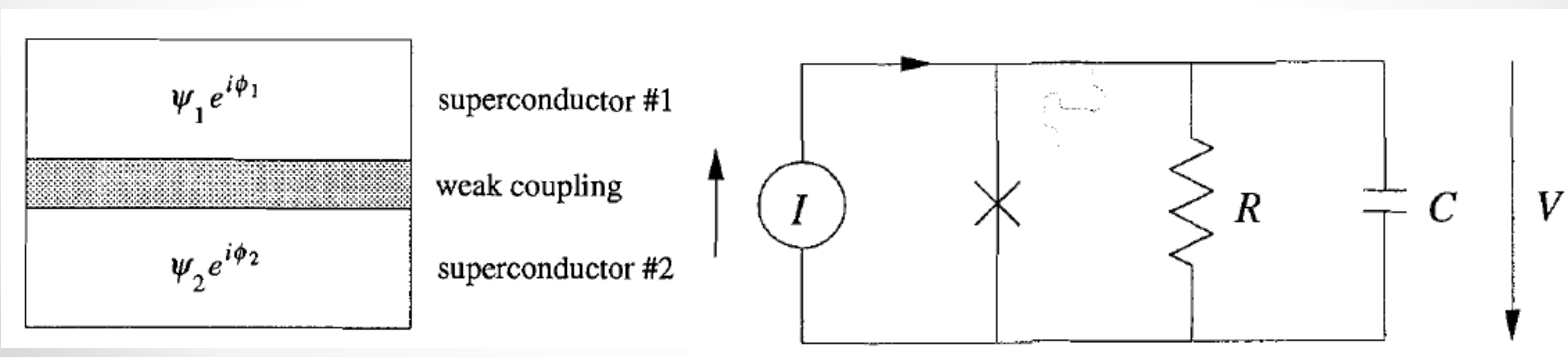
Poincaré Maps

Governing equation for the Josephson junction

$$\frac{(h/2\pi)C}{2e} + \frac{h/2\pi}{2eR} + I_c \sin\phi = I_B.$$

In a non-dimensionalised form: $\phi'' + \alpha\phi' + \sin\phi = I$

(ϕ is phase difference across junction; differentiation with respect to scaled time)



Poincaré Maps

Governing equation for the Josephson junction

$$\frac{(h/2\pi)C}{2e} + \frac{h/2\pi}{2eR} + I_c \sin\phi = I_B .$$

In a non-dimensionalised form: $\phi'' + \alpha\phi' + \sin\phi = I$

(ϕ is the phase difference across junction; differentiation with respect to scaled time)

$$\rightarrow \quad \begin{aligned} \phi' &= y \\ y' &= I - \sin\phi - \alpha y. \end{aligned}$$

The phase space is a cylinder, since ϕ is an angular variable and $y \in \mathbb{R}$.

Poincaré Maps

Josephson junction

$$\phi' = y$$
$$y' = I - \sin\phi - \alpha y.$$

Fixed points satisfy $y^* = 0$ and $\sin\phi^* = I \rightarrow$ two FPs on the cylinder if $I < 1$, and none if $I > 1$. When FPs exist, one is a saddle and other is a sink.

Jacobian $$A = \begin{pmatrix} 0 & 1 \\ -\cos\phi^* & -\alpha \end{pmatrix}$$

$$\tau = -\alpha < 0 \text{ and } \Delta = \cos\phi^* = \pm\sqrt{1 - \mathrm{I}^2}.$$

Stable node if $\tau^2 - 4\Delta = \alpha^2 - 4\sqrt{1 - I^2} > 0.$

Otherwise the sink is a stable spiral.

At $I = 1$ stable node & saddle coalesce in saddle-node bifurcation.

Poincaré Maps

Josephson junction

$$\phi' = y$$
$$y' = I - \sin\phi - \alpha y.$$

When $I > 1$ there are no more FPs available.

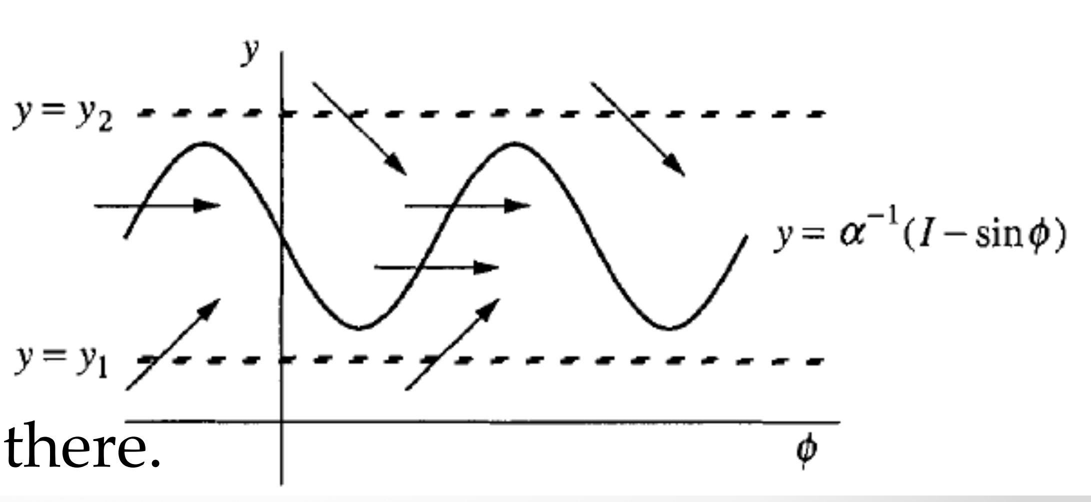
Claim: *All trajectories are attracted to a unique, stable limit cycle.*

The first step to prove this is to show that a periodic solution exists. For this we need a **Poincaré map.**

Nullcline

$$y' = 0:$$

$$y' = \alpha^{-1}(I - \sin\phi).$$

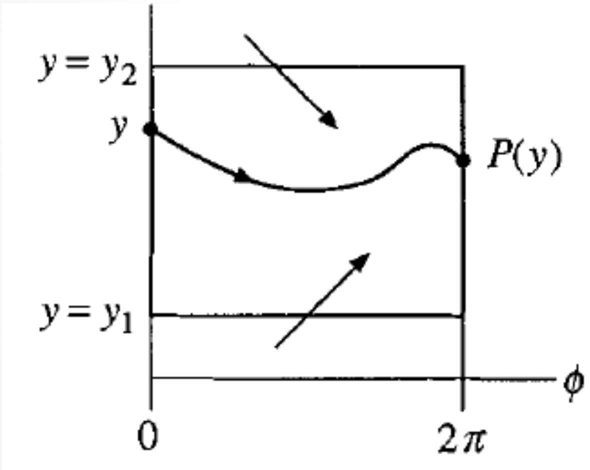


All trajectories enter the strip $y_1 \leq y \leq y_2$ and stay there.

Poincaré Maps

Josephson junction

Since $\phi = 0$ and $\phi = 2\pi$ are equivalent on the cylinder, we investigate the rectangular box $0 \leq \phi \leq 2\pi$ and $y_1 \leq y \leq y_2$.



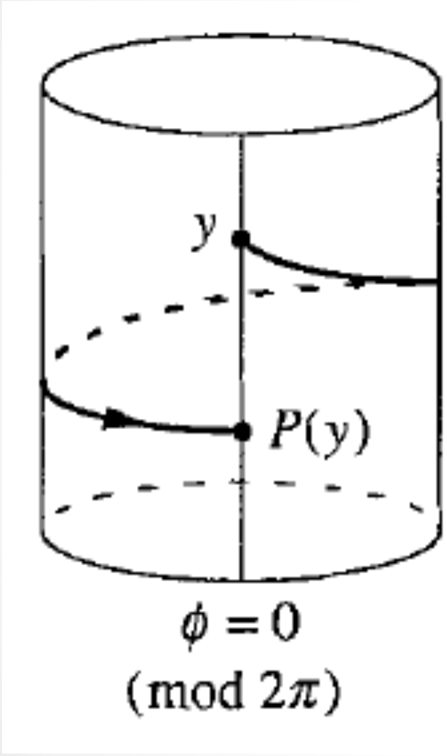
Trajectory starts at a height y on the left side and intersects the right side at height $P(y)$.

The mapping from y to $P(y)$ is called the **Poincaré map** or the **first-return map**.

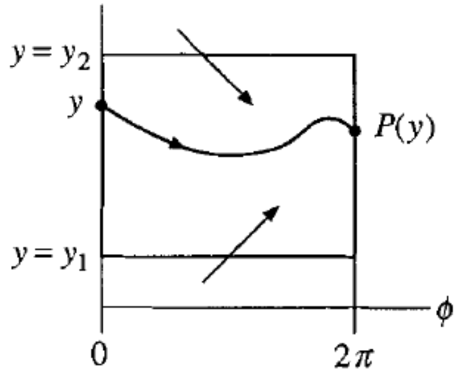
The Key: *If we can show that there's a point y^* such that $P(y^*) = y^*$, then the corresponding trajectory will be a closed orbit.*

Poincaré Maps

Josephson junction



To show that y^* exists, we need to know what the graph roughly looks like.



For a trajectory that starts at $y = y_1, \phi = 0$: $P(y_1) > y_1$.

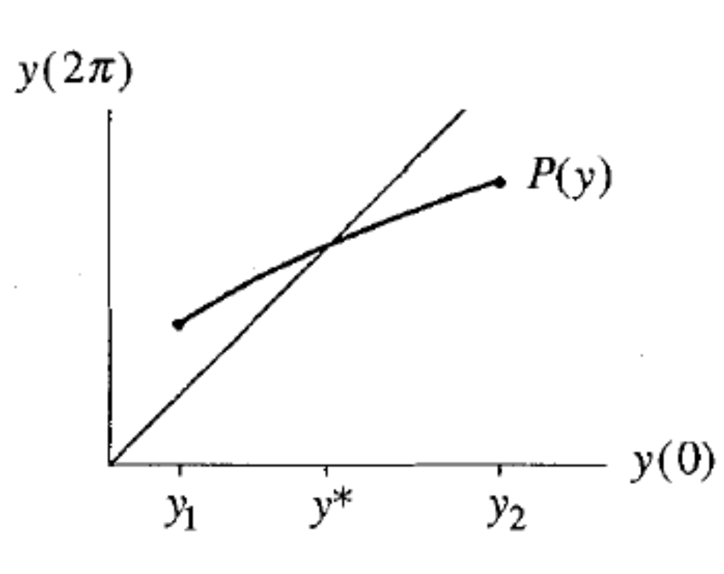
For a trajectory that starts at $y = y_2, \phi = 0$: $P(y_2) < y_2$.

Solutions of differential equations depend continuously on initial conditions, if the vector field is smooth: $P(y)$ is a *continuous* function.

Uniqueness, two trajectories cannot cross: $P(y)$ is a *monotonic* function.

Poincaré Maps

$P(y)$ is continuous and monotonic →



Intermediate value theorem or common sense: $P(y)$ must cross the 45° diagonal *somewhere*; this is the point y^*.

→ <u>There exists a closed orbit.</u>

To exclude the possibility of $P(y) \equiv y$ on some interval, *uniqueness* must be proven using $\Delta E = 0$ when $\phi: 0 \to 2\pi$.

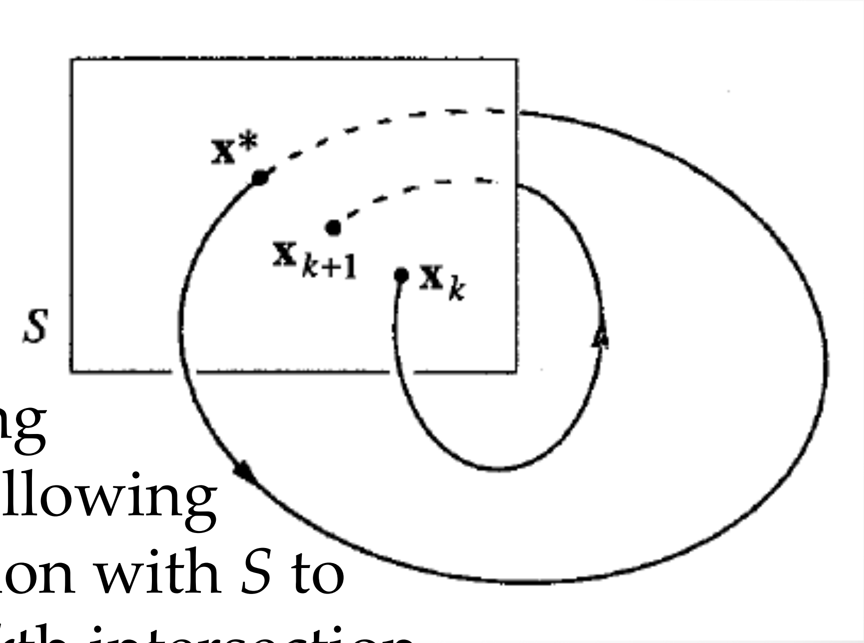
Poincaré Maps

Poincaré maps are useful studying swirling flows, e.g. flow near a periodic orbit or flow in some chaotic system.

General definition of a Poincaré map in an *n*-dimensional system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

Let S be an n-1 dimensional **surface of section** that is transverse to the flow.



The **Poincaré map** is a mapping from S to itself, obtained by following trajectories from one intersection with S to the next. If $\mathbf{x}_k \in S$ denotes the kth intersection, then then the Poincaré map is defined by $\mathbf{x}_{k+1} = P(\mathbf{x}_k)$.

Poincaré Maps

Now, suppose that $\mathbf{x}^*$ is **fixed point** of P, i.e., $P(\mathbf{x}^*) = \mathbf{x}^*$.

A trajectory starting at $\mathbf{x}^*$ returns to $\mathbf{x}^*$ after some time T, and is therefore a **closed orbit**.

The Poincaré map converts difficult problems about closed orbits into much easier problems about fixed points of a mapping. (Although finding P may be impossible.)

Poincaré Maps

Example. Vector field in polar coordinates

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$

Let S be the positive x-axis. Compute the Poincaré map and show that the system has a unique periodic orbit and classify its stability.

Solution: Let r_0 be an initial condition on S. The first return to S occurs after a ***time of flight*** $t = 2\pi$ (because $\dot{\theta} = 1$).

$\rightarrow r_1 = P(r_0)$: $$\int_{r_0}^{r_1} \frac{1}{\dot{r}} dr = \int_{r_0}^{r_1} \frac{dr}{r(1 - r^2)} = \int_0^{2\pi} dt = 2\pi.$$

$$\Rightarrow r_1 = [1 + e^{-4\pi}(r_0^{-2} - 1)]^{-1/2}$$

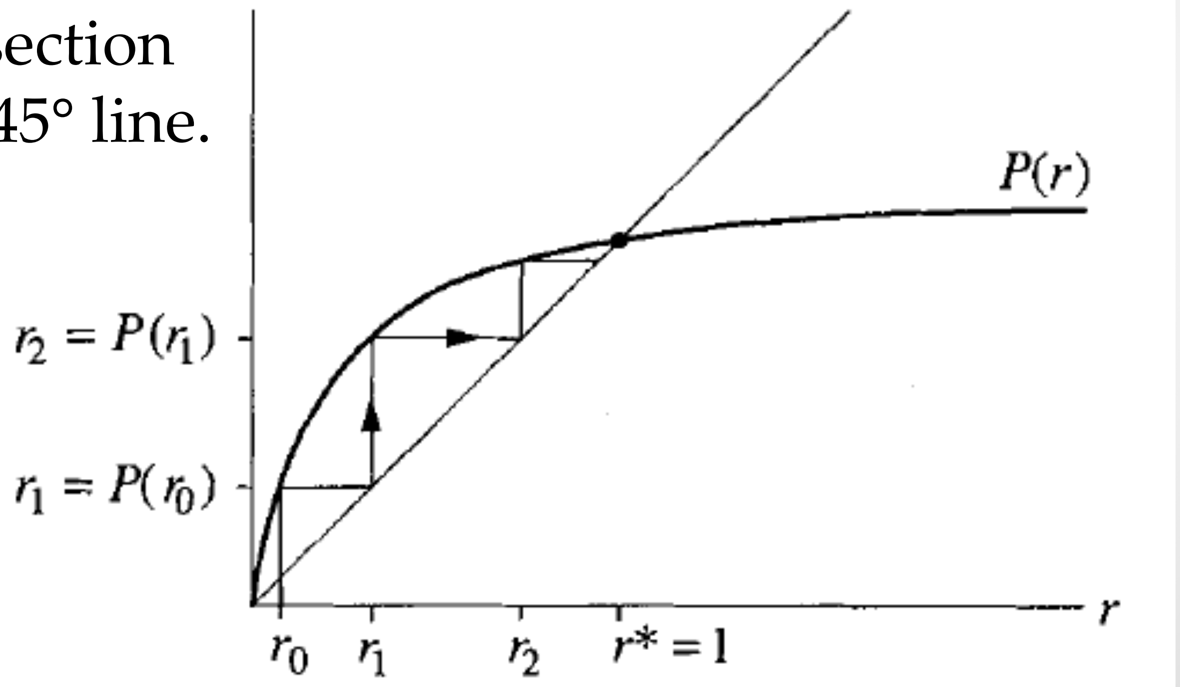
Poincaré Maps

$P(r) = [1 + e^{-4\pi}(r^{-2} - 1)]^{-1/2}$. Iteration: $r_{n+1} = r_n, n \in \mathbb{N}$.

Iteration of the map graphically by constructing a ***cobweb***.

A fixed point occurs at $r^* = 1$ that is the intersection point of $P(r)$ and the 45° line.

The **cobweb** construction is often the only doable way get an idea of **chaotic dynamics** (for example, to map out strange attractors).



Next time: Chaos – Lorenz equations.