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Brief Recap of Chapter 7 of Brown et al. (2014)

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Contents of Chapter 7 ”Signal Detection Concepts”

Introduction

- **We have already considered:**
  - Dynamics of magnetization of the measured sample
  - Rf field used to “tip” the magnetization to cause precession
- **Now we consider detection of the rotating field**
  - Based on electromotive force (emf) generated on a coil
  - We often use the same coil for transmitting and receiving

Faraday Induction

- **The Faraday's law of induction for electromotive force:**

$$emf = -\frac{d\Phi}{dt}$$

- **The flux through the coil:**

$$\Phi = \int_{\text{coil area}} \vec{B} \cdot d\vec{S}$$

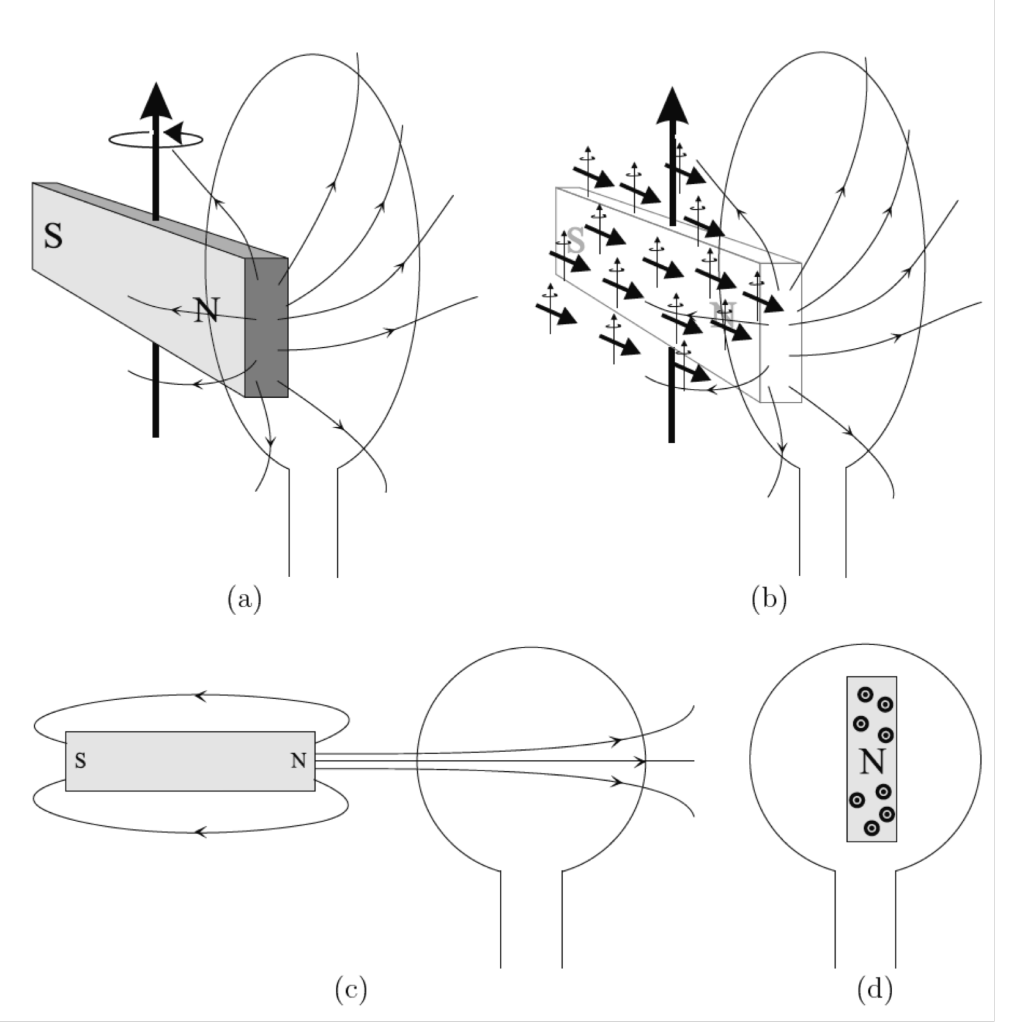
- **For example, consider**

$$\vec{B}(t) = B(\sin\theta\hat{y} + \cos\theta\hat{z})\sin\omega t$$

- **Assuming** $d\vec{S} = dx\, dy\, \hat{z}$. **we then get**

$$\begin{aligned} emf &= -\frac{d}{dt}\int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy\, \hat{z} \cdot \vec{B}(t) \\ &= -L^2 B\omega\cos\theta\cos\omega t \end{aligned}$$

Faraday Induction



(a) (b) (c) (d)

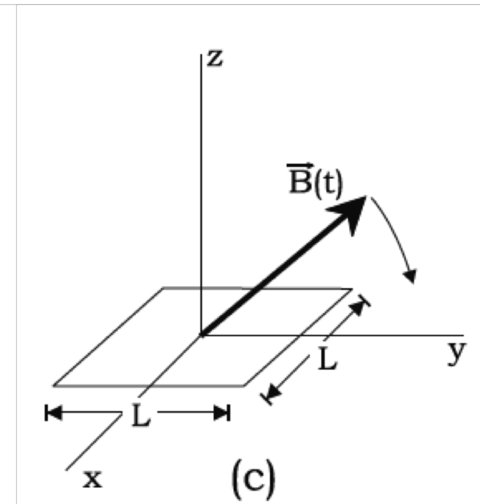
Problem

Problem 5.1

Consider the situation in Fig. 7.2c where the coil lies in the x-y plane and a spatially independent field rotates about the x-axis:

$$\vec{B}(t) = -B\sin(\omega t)\hat{z} + B\cos(\omega t)\hat{y}$$

Show that the *emf* induced in the coil is $L^2 B\omega\cos(\omega t)$.



The MRI Signal and the Principle of Reciprocity

- **The effective current density caused by magnetization is**

$$\vec{J}_M(\vec{r},t) = \vec{\nabla} \times \vec{M}(\vec{r},t)$$

- **The vector potential caused by it is**

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

- **This then gives the magnetic field**

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

- **Using the Stokes' theorem we can then write**

$$\Phi = \int_{area} \vec{B} \cdot d\vec{S} = \int_{area} (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \oint d\vec{l} \cdot \vec{A}$$

The MRI Signal and the Principle of Reciprocity

- **We now obtain**

$$\begin{aligned}\Phi_M &= \oint d\vec{l}\cdot\left[\frac{\mu_0}{4\pi}\int d^3r'\frac{\vec{\nabla}'\times\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] \\ &= \frac{\mu_0}{4\pi}\int d^3r'\oint d\vec{l}\cdot\left[\left(-\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\right)\times\vec{M}(\vec{r}')\right] \\ &= \frac{\mu_0}{4\pi}\int d^3r'\vec{M}(\vec{r}')\cdot\left[\vec{\nabla}'\times\left(\oint\frac{d\vec{l}}{|\vec{r}-\vec{r}'|}\right)\right]\end{aligned}$$

- **Which can be seen to actually have the form**

$$\Phi_M(t) = \int_{sample} d^3r\vec{\mathcal{B}}^{receive}(\vec{r})\cdot\vec{M}(\vec{r},t)$$

- **Here $B^{receive}$ is the magnetic field per unit current that would be produced by the coil at the point r**
- **Example of the principle of reciprocity**

Signal from Precessing Magnetization

- **The emf induced in the coil is expressed as**

$$\begin{aligned} emf &= -\frac{d}{dt}\Phi_M(t) \\ &= -\frac{d}{dt}\int_{sample} d^3r \vec{M}(\vec{r},t)\cdot\vec{\mathcal{B}}^{receive}(\vec{r}) \end{aligned}$$

- **Thus the detected signal is proportional to**

$$\text{signal} \propto -\frac{d}{dt}\int d^3r \left[\mathcal{B}_x^{receive}(\vec{r})M_x(\vec{r},t) + \mathcal{B}_y^{receive}(\vec{r})M_y(\vec{r},t) + \mathcal{B}_z^{receive}(\vec{r})M_z(\vec{r},t)\right]$$

- **This is nominally of the form**

$$\begin{aligned} \text{signal} &\propto \omega_0 \int d^3r e^{-t/T_2(\vec{r})}\left[\mathcal{B}_x^{receive}(\vec{r})\,\mathrm{Re}\left(iM_+(\vec{r},0)e^{-i\omega_0 t}\right) + \mathcal{B}_y^{receive}(\vec{r})\,\mathrm{Im}\left(iM_+(\vec{r},0)e^{-i\omega_0 t}\right)\right] \\ &\propto \omega_0 \int d^3r e^{-t/T_2(\vec{r})} M_\perp(\vec{r},0)[\mathcal{B}_x^{receive}(\vec{r})\sin(\omega_0 t - \phi_0(\vec{r})) \\ &\qquad + \mathcal{B}_y^{receive}(\vec{r})\cos(\omega_0 t - \phi_0(\vec{r}))] \qquad (7.18) \end{aligned}$$

- **Or**

$$\text{signal} \propto \omega_0 \int d^3r e^{-t/T_2(\vec{r})} M_\perp(\vec{r},0)\mathcal{B}_\perp(\vec{r})\sin(\omega_0 t + \theta_{\mathcal{B}}(\vec{r}) - \phi_0(\vec{r}))$$

Signal Demodulation

- **Demodulation multiplies signal with**

$$\sin(\omega_0 + \delta\omega)t$$

- **Giving**

$$\begin{aligned} \text{demodulated signal} &\propto \text{reference signal} \cdot \text{induced } emf \\ &\propto \sin(\omega_0 + \delta\omega)t \cdot \sin(\omega_0 t + \zeta) \\ &\propto \frac{1}{2}\left(\cos(\delta\omega \cdot t - \zeta) - \cos((2\omega_0 + \delta\omega)t + \zeta)\right) \end{aligned}$$

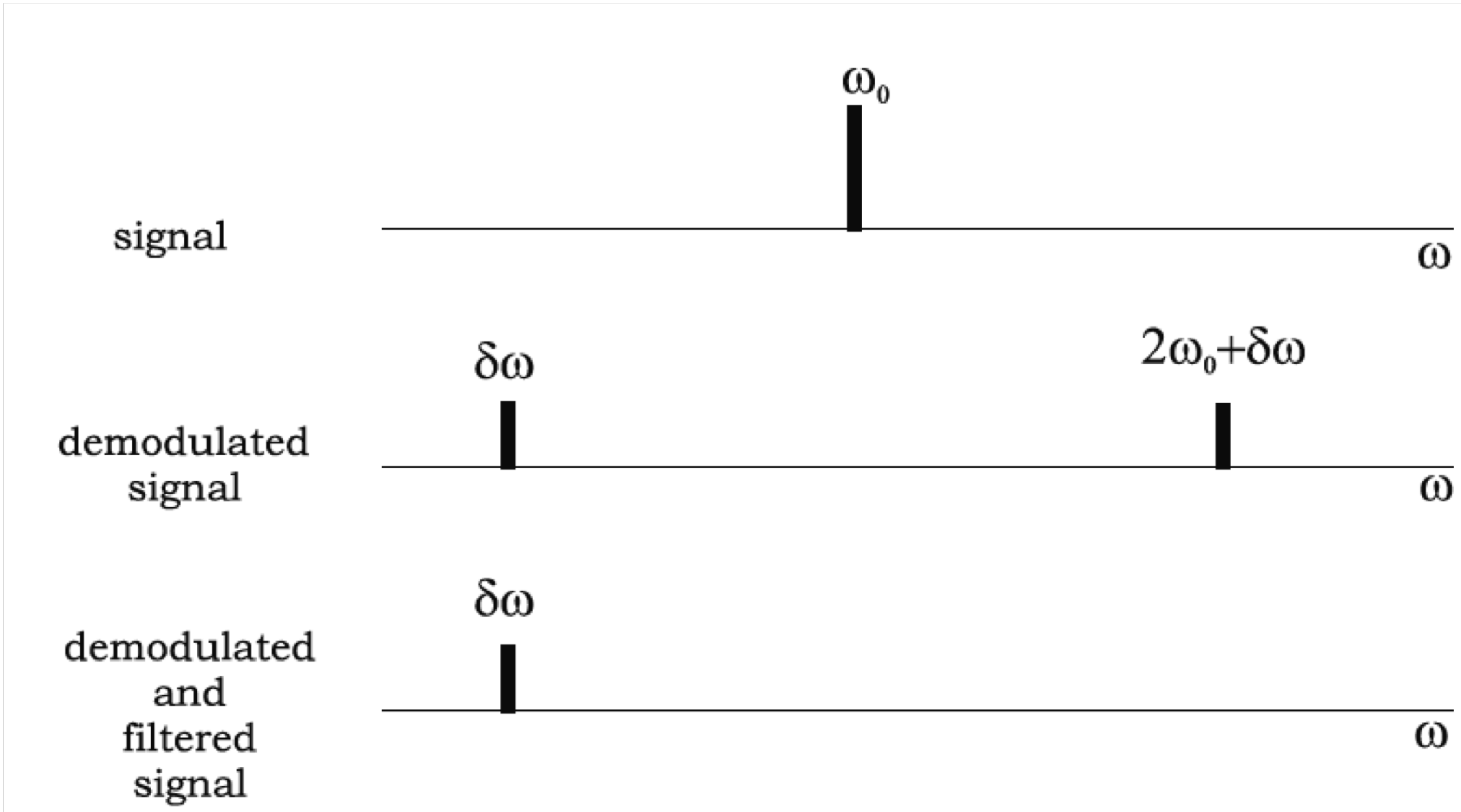
- **Then we low-pass filter away the high frequency:**

$$\text{demodulated and filtered signal} \propto \frac{1}{2}\cos(\delta\omega \cdot t - \zeta) = \frac{1}{2}\mathrm{Re}(e^{i\delta\omega \cdot t - i\zeta}) \quad \text{(real channel)} \qquad (7.24)$$

- **By multiplying with** $-\cos(\omega_0 + \delta\omega)t$ **we get**

$$\text{demodulated and filtered signal} \propto \frac{1}{2}\sin(\delta\omega \cdot t - \zeta) = \frac{1}{2}\mathrm{Im}(e^{i\delta\omega \cdot t - i\zeta}) \quad \text{(imaginary channel)} \qquad (7.25)$$

Signal Demodulation



Problem

Problem 5.2

Consider the signal resulting from two spin isochromats with identical spin densities but different frequencies of precession $\omega_a = \omega_0 + \Delta\omega$ and $\omega_b = \omega_0 - \Delta\omega$. The total signal for this experiment is just the linear addition of the signal from each isochromat. Find the demodulated signal (with zero offset, $\delta\omega = 0$) from the two-spin system and compare it to the demodulated signal (with offset) represented by (7.24) or (7.25).

Note: It will be evident in the solution that the signal from two spin isochromats with slightly different frequencies (a difference represented by a small $\Delta\omega$) exhibits beats. See the discussion on beating in Ch. 8.

$$\text{demodulated and filtered signal} \propto \frac{1}{2}\cos(\delta\omega \cdot t - \zeta) = \frac{1}{2}\text{Re}(e^{i\delta\omega \cdot t - i\zeta}) \quad \text{(real channel)} \tag{7.24}$$

$$\text{demodulated and filtered signal} \propto \frac{1}{2}\sin(\delta\omega \cdot t - \zeta) = \frac{1}{2}\text{Im}(e^{i\delta\omega \cdot t - i\zeta}) \quad \text{(imaginary channel)} \tag{7.25}$$

Complex Signal

- **Complex demodulated signal**

$$s(t) \equiv s_{re}(t) + i s_{im}(t)$$

- **Alternative forms for the signal:**

$$s(t) \propto \omega_0 V_s e^{-t/T_2} M_\perp \mathcal{B}_\perp e^{i[(\Omega-\omega_0)t+\phi_0-\theta_\mathcal{B}]} \qquad \text{(space-independent limit)}$$

$$s(t) \propto \omega_0 \int d^3r e^{-t/T_2(\vec{r})} M_\perp(\vec{r},0)\mathcal{B}_\perp(\vec{r}) e^{i((\Omega-\omega_0)t+\phi_0(\vec{r})-\theta_\mathcal{B}(\vec{r}))}$$

$$s(t) \propto \omega_0 \int d^3r M_+(\vec{r},t)\mathcal{B}_+^*(\vec{r})$$

- **where**

$$\mathcal{B}_+ \equiv \mathcal{B}_x^{receive} + i\mathcal{B}_y^{receive} = \mathcal{B}_\perp e^{i\theta_\mathcal{B}}$$

Dependence on System Parameters

- In homogeneous limit we have (ignoring amplification)

$$|s| = \omega_0 M_0 \mathcal{B}_\perp V_s$$

- As M_0 is proportional to B_0, this is proportional to B_0^2
- The noise increases proportionally to B_0 and hence signal-to-noise ratio (SNR) is proportional to B_0
- For this reason higher B_0 fields are preferred
- The relative signal strength for a given species is

$$\mathcal{R}_i \equiv |\gamma_i|^3 r_i a_i s_i (s_i + 1)$$

- A non-uniform profiles of $B^{receive}$ and $B^{transmit}$ also affect the measured image

Dependence on System Parameters

Nucleus i	γ_i (MHz/T)	r_i	a_i	s_i	$\mathcal{R}_i/\mathcal{R}(^1\text{H})$ at 1 T
^{1}H, gray matter	42.58	1.0	1.0	$\frac{1}{2}$	1.0
^{23}Na, average tissue	11.27	9.1×10^{-4}	1.0	$\frac{3}{2}$	8.4×10^{-5}
^{31}P, average tissue	17.25	8.5×10^{-4}	1.0	$\frac{1}{2}$	5.7×10^{-5}
^{17}O, gray matter	-5.77	0.5	3.8×10^{-4}	$\frac{5}{2}$	5.5×10^{-6}
^{19}F, average tissue	40.08	4.5×10^{-8}	1.0	$\frac{1}{2}$	3.8×10^{-8}