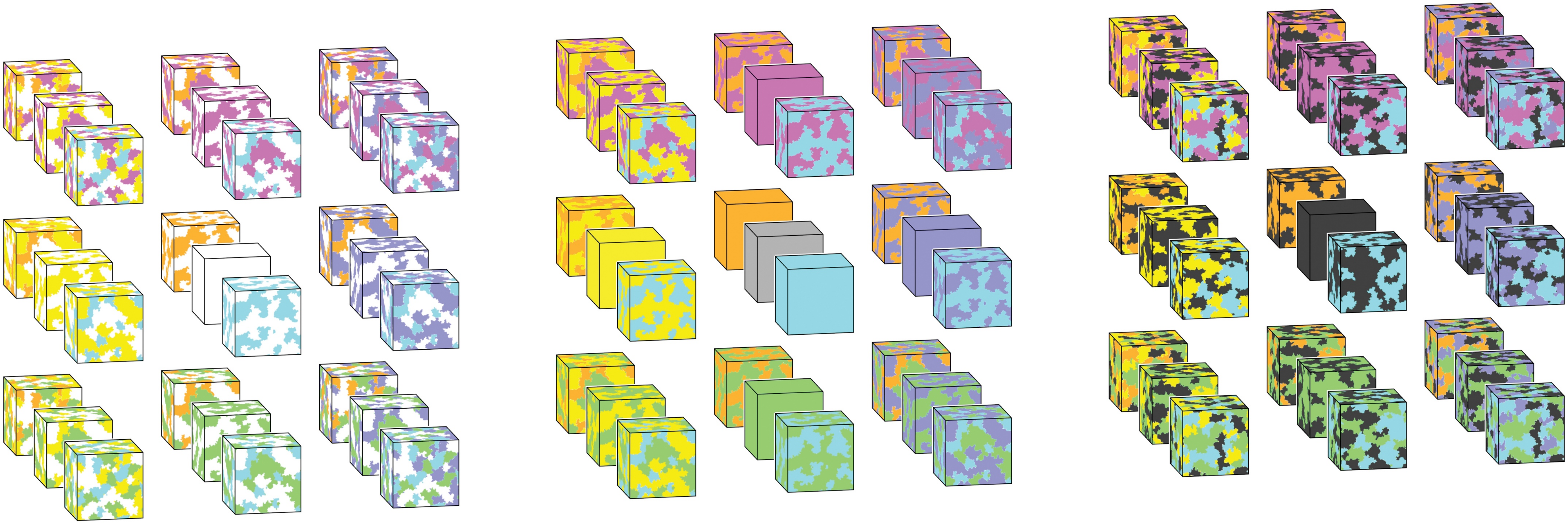
CRYSTAL FLOWERS IN HALLS OF MIRRORS:

THE CUBES OF HINTON

26.2.2019
Taneli Luotoniemi



WIKIPEDIA SOURCES:

https://en.wikipedia.org/wiki/Four-dimensional_space
https://en.wikipedia.org/wiki/Tesseract
https://en.wikipedia.org/wiki/Charles_Howard_Hinton
https://en.wikipedia.org/wiki/Alicia_Boole_Stott

Charles Howard Hinton, (1853–1907)

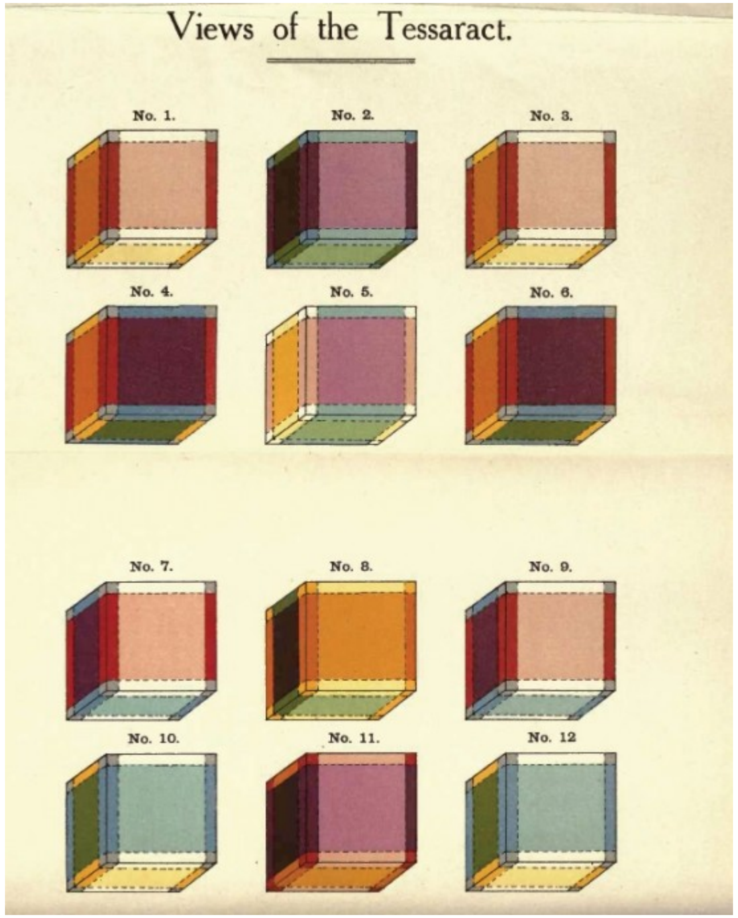
HYPERSPACE PHILOSOPHER

British mathematician Charles Howard Hinton, played a key part in the popularization of ‘hyperphilosophy’ by publishing many writings during the years 1884–1907, speculating on the physical as well as spiritual aspects of 4 space. He also anticipated the hidden dimensions of string theory by stating that the fourth dimension could perhaps be observed on the smallest details of physical matter. Hinton coined the names ana and kata, which refer to the positive and negative directions along the axis of the fourth spatial dimension.

HYPERSPACE PHILOSOPHER

Charles Howard Hinton, (1853–1907)

Hinton developed a mnemonic system of some tens of thousand cubes with individual names in Latin, serving as a 3-dimensional mental retina of a kind on which to visualize the successive cross-sections of objects in 4-space . Interested in Eastern thought, he also sought to eliminate the ‘self elements’ of his system by memorizing the different orientations and mirror reflections of the cubes. Later he developed the system into a self-help method to visualize the fourth dimension, which consisted of manipulation of coloured cubes. The cubes were available for purchase from his publisher.



Frontispiece of *The Fourth Dimension* (1901)

ALICIA BOOLE STOTT

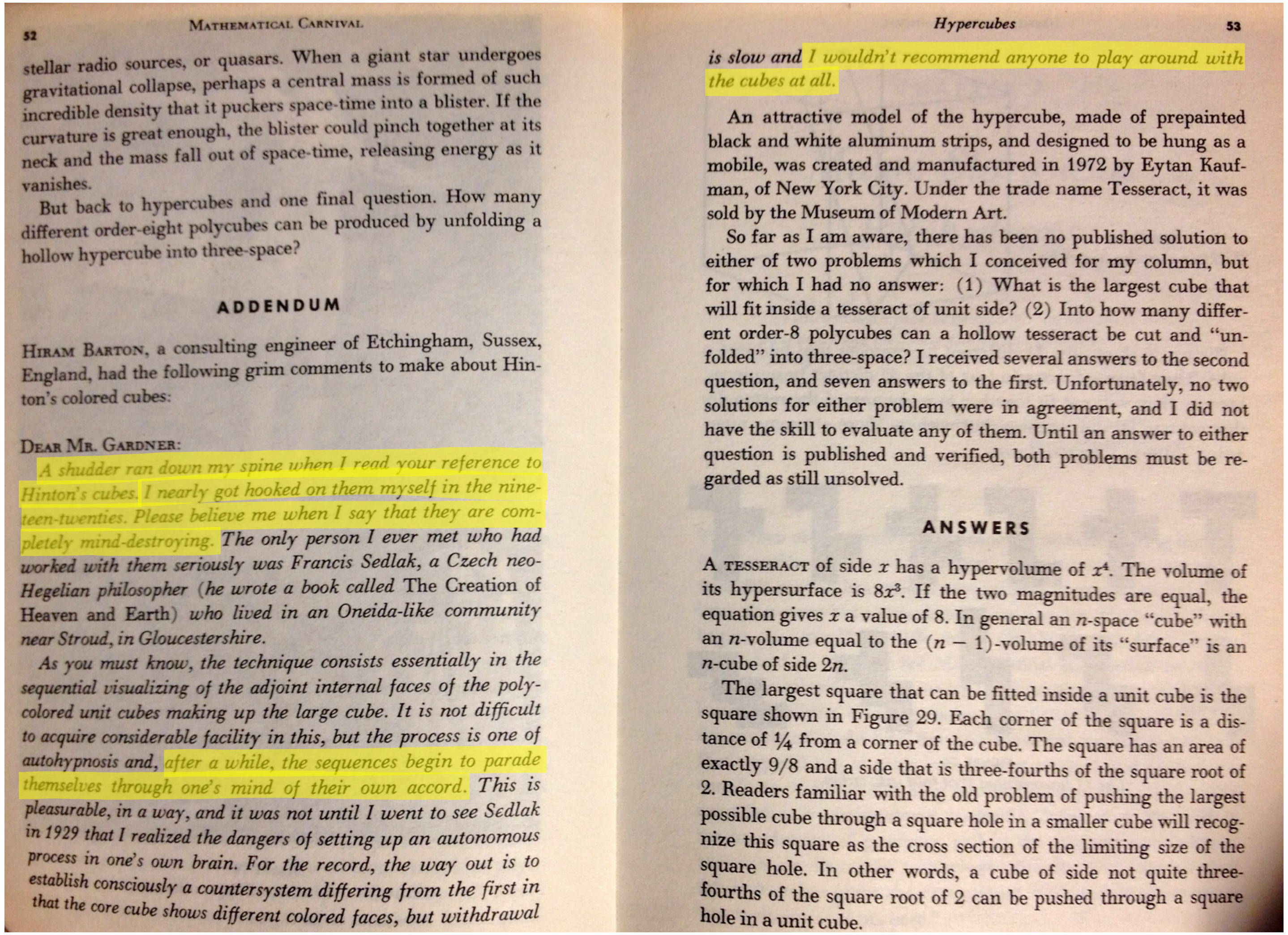
Hinton was a frequent guest at the household of Mary Everest Boole, whose husband George was famous of his Boolean algebra. During these visits he used Alicia, Boole’s young daughter as the primary guinea pig for his system of cubes, an activity encouraged by her mother who was also known for her writings on early mathematics education. Alicia showed special talent towards visualizing the fourth dimension, a skill in which she soon exceeded that of Hinton himself. Despite her restricted circumstances as a housewife without any sort of formal mathematical training, Alicia Boole Stott went on to independently prove the existence of the six regular polychora, describe their perpendicular cross-sections, and also find some of the semiregular polytopes in four dimensions.

Alicia Boole Stott, (1860–1940)

‘DANGERS’ OF THE CUBES

In Algernon Blackwood’s short science-fiction story *A Victim of Higher Space* , the ‘victim’ keeps slipping to four-dimensional space – a condition brought about by his use of Hinton’s cubes.

When Gardner featured Hinton’s cubes in his Scientific American article, he received a grave warning from an English consulting engineer who had first-hand experience with the method. The letter claimed the four-dimensional visualization exercises to take on a life of their own, and eventually the sequence of cube will “begin to parade themselves through one’s mind of their own accord”. He went on to declare the exercises “completely mind-destroying” and that he would not “recommend anyone to play with the cubes at all”.



52 MATHEMATICAL CARNIVAL

stellar radio sources, or quasars. When a giant star undergoes gravitational collapse, perhaps a central mass is formed of such incredible density that it puckers space-time into a blister. If the curvature is great enough, the blister could pinch together at its neck and the mass fall out of space-time, releasing energy as it vanishes.

But back to hypercubes and one final question. How many different order-eight polycubes can be produced by unfolding a hollow hypercube into three-space?

ADDENDUM

HIRAM BARTON, a consulting engineer of Etchingham, Sussex, England, had the following grim comments to make about Hinton's colored cubes:

DEAR MR. GARDNER:

A shudder ran down my spine when I read your reference to Hinton's cubes. I nearly got hooked on them myself in the nineteen-twenties. Please believe me when I say that they are completely mind-destroying. The only person I ever met who had worked with them seriously was Francis Sedlak, a Czech neo-Hegelian philosopher (he wrote a book called The Creation of Heaven and Earth*) who lived in an Oneida-like community near Stroud, in Gloucestershire.*

As you must know, the technique consists essentially in the sequential visualizing of the adjoint internal faces of the poly-colored unit cubes making up the large cube. It is not difficult to acquire considerable facility in this, but the process is one of autohypnosis and, after a while, the sequences begin to parade themselves through one's mind of their own accord. This is pleasurable, in a way, and it was not until I went to see Sedlak in 1929 that I realized the dangers of setting up an autonomous process in one's own brain. For the record, the way out is to establish consciously a countersystem differing from the first in that the core cube shows different colored faces, but withdrawal

Hypercubes 53

is slow and I wouldn't recommend anyone to play around with the cubes at all.

An attractive model of the hypercube, made of prepainted black and white aluminum strips, and designed to be hung as a mobile, was created and manufactured in 1972 by Eytan Kaufman, of New York City. Under the trade name Tesseract, it was sold by the Museum of Modern Art.

So far as I am aware, there has been no published solution to either of two problems which I conceived for my column, but for which I had no answer: (1) What is the largest cube that will fit inside a tesseract of unit side? (2) Into how many different order-8 polycubes can a hollow tesseract be cut and "unfolded" into three-space? I received several answers to the second question, and seven answers to the first. Unfortunately, no two solutions for either problem were in agreement, and I did not have the skill to evaluate any of them. Until an answer to either question is published and verified, both problems must be regarded as still unsolved.

ANSWERS

A TESSERACT of side x has a hypervolume of x^4. The volume of its hypersurface is $8x^3$. If the two magnitudes are equal, the equation gives x a value of 8. In general an n-space "cube" with an n-volume equal to the $(n-1)$-volume of its "surface" is an n-cube of side $2n$.

The largest square that can be fitted inside a unit cube is the square shown in Figure 29. Each corner of the square is a distance of ¼ from a corner of the cube. The square has an area of exactly 9/8 and a side that is three-fourths of the square root of 2. Readers familiar with the old problem of pushing the largest possible cube through a square hole in a smaller cube will recognize this square as the cross section of the limiting size of the square hole. In other words, a cube of side not quite three-fourths of the square root of 2 can be pushed through a square hole in a unit cube.

Martin Gardner: “Mathematical Carnival” (1965)

CROSS-SECTIONS OF THE CUBE IN A PLANE

Before discussing the tesseract itself, it is useful to observe the logic in the lower-dimensional case.

We cut the **cube** into 27 pieces, and examine their cross-sections in the plane.

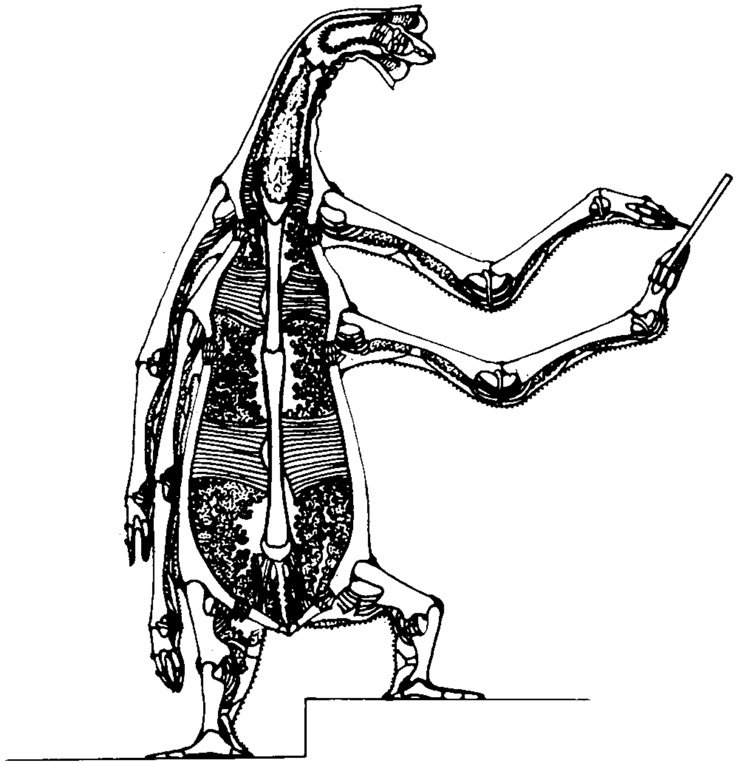
ONE STEP BACK

The first step towards understanding the concept of hyperspace has usually come in the form of analogy. Maybe surprisingly, it is instructive to first go one dimension down, and imagine a world with only two dimensions, inhabited by equally flat beings. Our challenges to understand and visualize four-dimensional space are analogous to the difficulties a two-dimensional being, confined to a plane, would have with respect to our three-dimensional space and its shapes.

In fact, there is an entire tradition of ‘flatlands’, i.e. imaginary universes of only two dimensions, whose purpose is often to depict our own challenges of with regard to non-Euclidean or higher-dimensional spaces.

The most famous of such fictitious conceptions is Edwin A. Abbott’s 1884 classic *Flatland – A Romance of Many Dimensions*. It depicts a satire of Victorian society through the life of polygons living in a plane. As noted already by Hinton (*A Plane World*, Scientific Romances, p. 129), Abbott’s main focus was not on the geometry and ‘conditions of life on the plane’. Hinton mended this issue in his own writings *A Plane World* (1884) and *An Episode on Flatland: Or How a Plain Folk Discovered the Third Dimension* (1907). Hinton’s planar world was called Astria, and it differed from Flatland with its lateral view that enabled a richer universe with heavenly bodies, gravity, etc.

ONE STEP BACK



Two-dimensional protagonist of the *Planiverse*

Of the multiple successors of Flatland trying to grasp the peculiarities of fictional worlds restricted on a plane, the most ambitious is A. K. Dewdney.

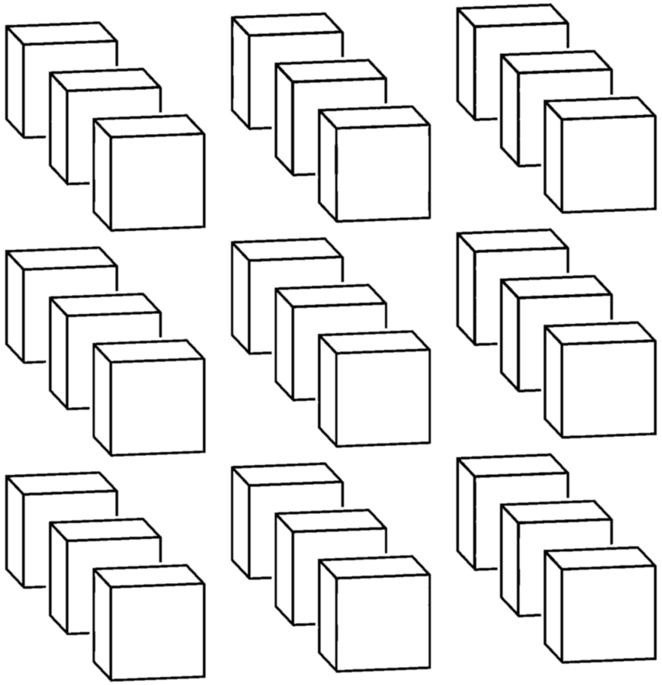
A computer scientist at the University of Western Ontario, his two dimensional world called the *Planiverse* first appeared in his 1979 article *Exploring the Planiverse*, and later in the same year in Two-dimensional Science and Technology, which he published privately.

He developed his creation into full novel in 1984 as *The Planiverse – Computer Contact with a Two-Dimensional World*, an account of a planar universe complete with its physics, chemistry, biology, politics and art.

CUTTING THE CUBE INTO PIECES

For the purpose of depicting the cube in the plane, we cut it to three slices with respect to its three primary directions.

Thus we get 3x3x3 cluster of small cubes, where each cube represents a vertex or an edge of the original cube, or the original cube itself.



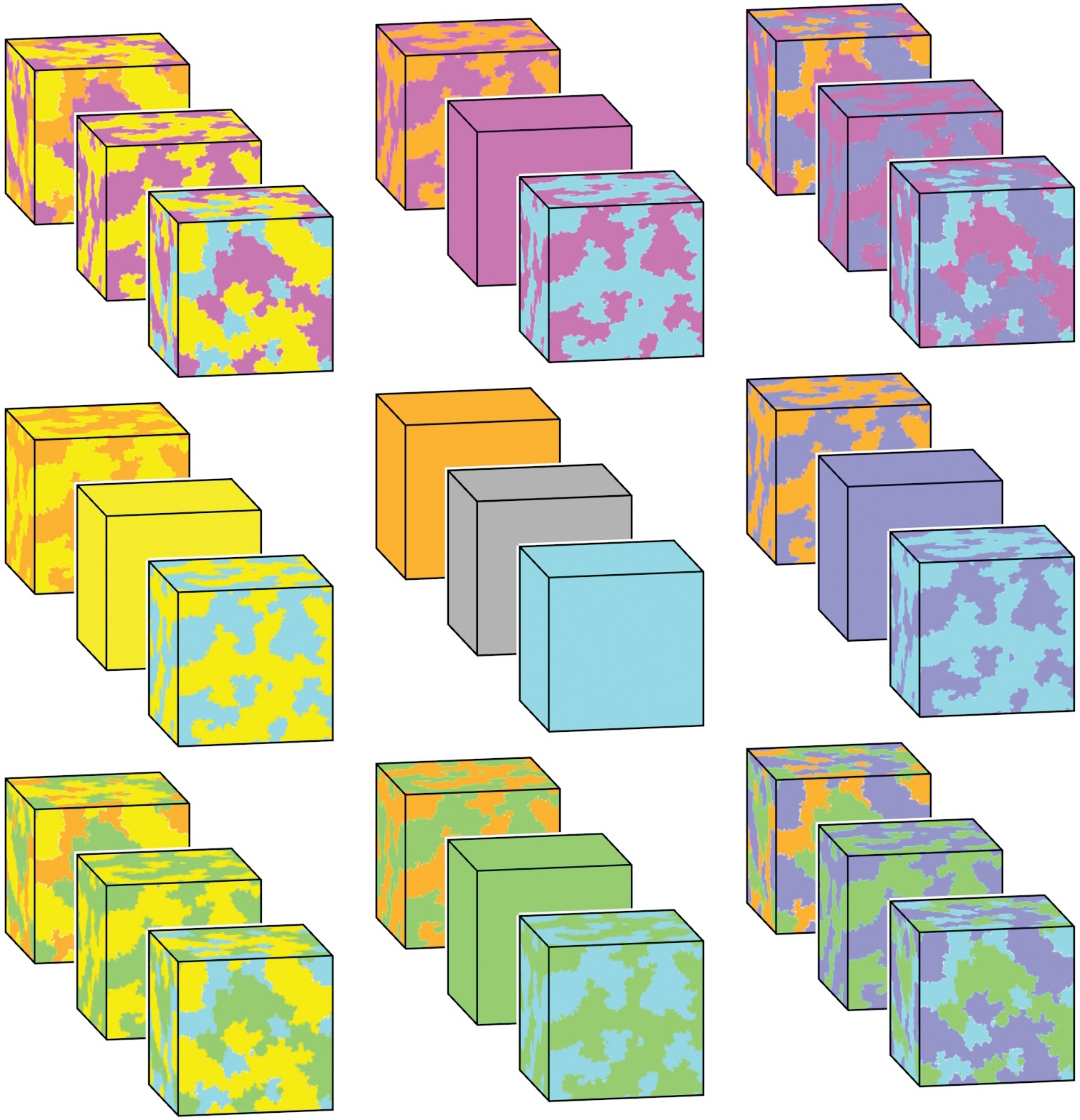
CUBE
8 vertices
12 edges
6 faces (squares)

27 CUBES
8 vertex pieces
12 edge pieces
6 face pieces
1 core piece

COLORING THE CUBE PIECES

Each cube piece gets an unique color when think the cube being exposed to colored light from three different directions (yellow, orange, pink).

On the 'shadow' side are the complementary colors (purple, blue, green).



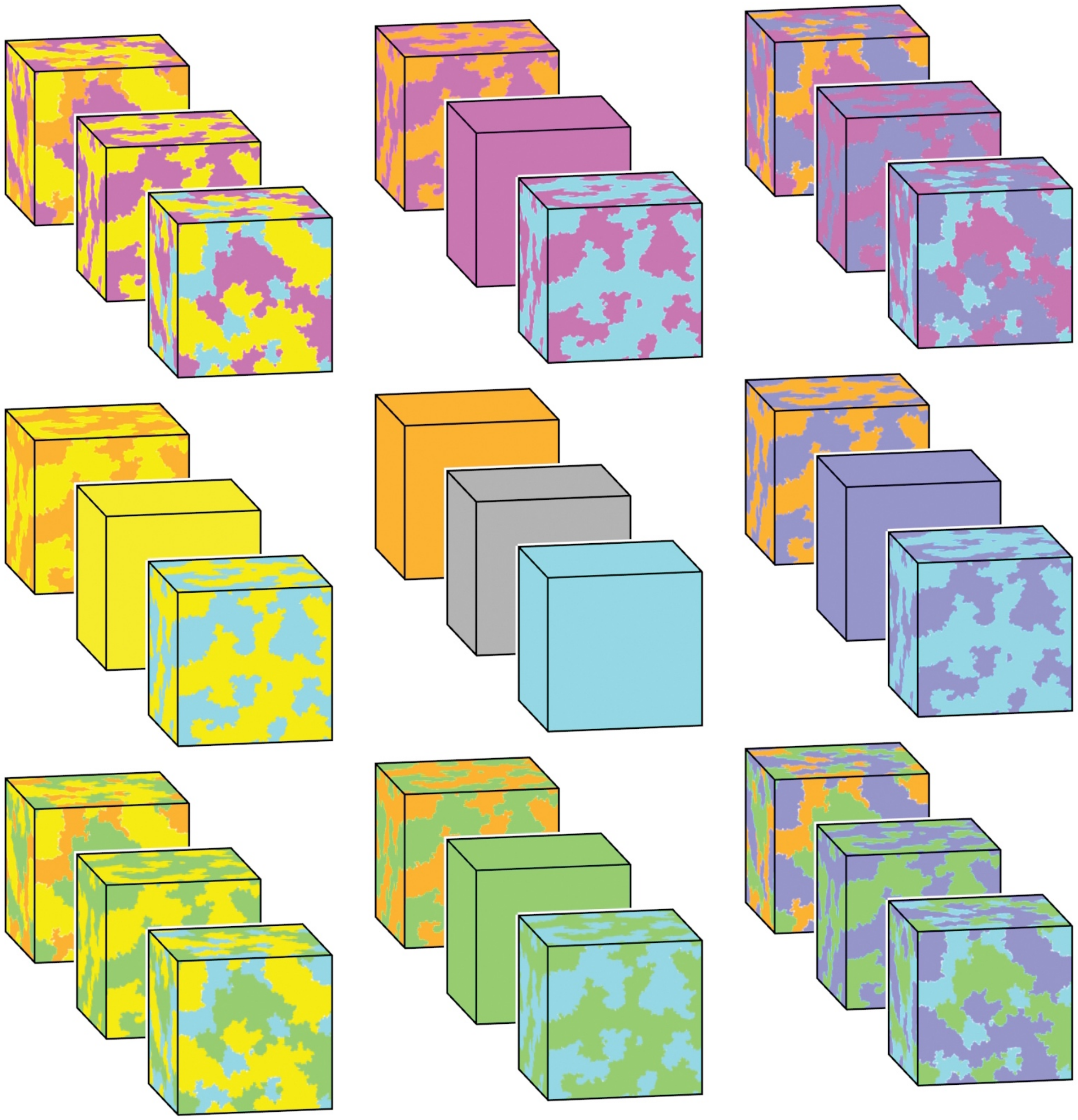
COLORING THE CUBE PIECES

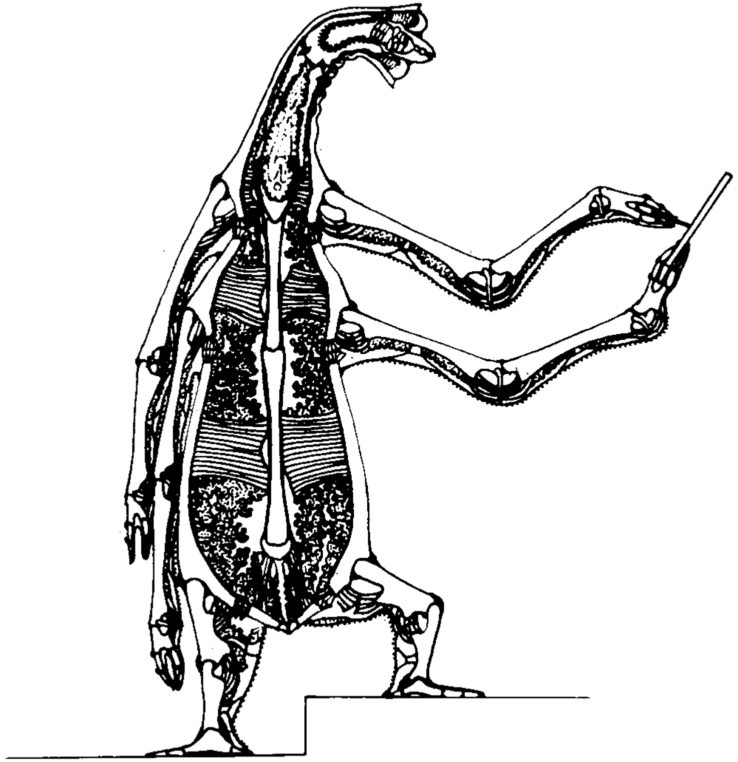
Vertex pieces are three-colored, since they are exposed to light/shadow from three directions.

Edge pieces are two-colored, since they are exposed to light/shadow from two directions.

Face pieces are one-colored, since they are exposed to light/shadow from one direction.

The core piece is colorless, since it is not exposed to any light/shadow.

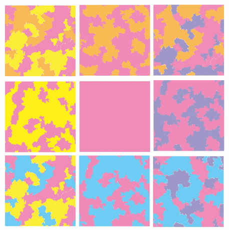


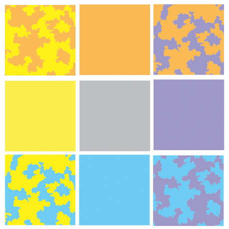


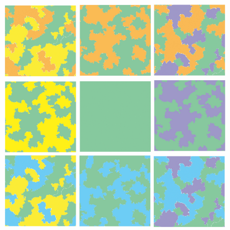
CROSS-SECTIONS OF THE CUBE IN A PLANE

Now the 27 pieces of the cube can be represented in the plane with two-dimensional slabs.

These slabs can be assembeled to show the cross-sections of the cube.

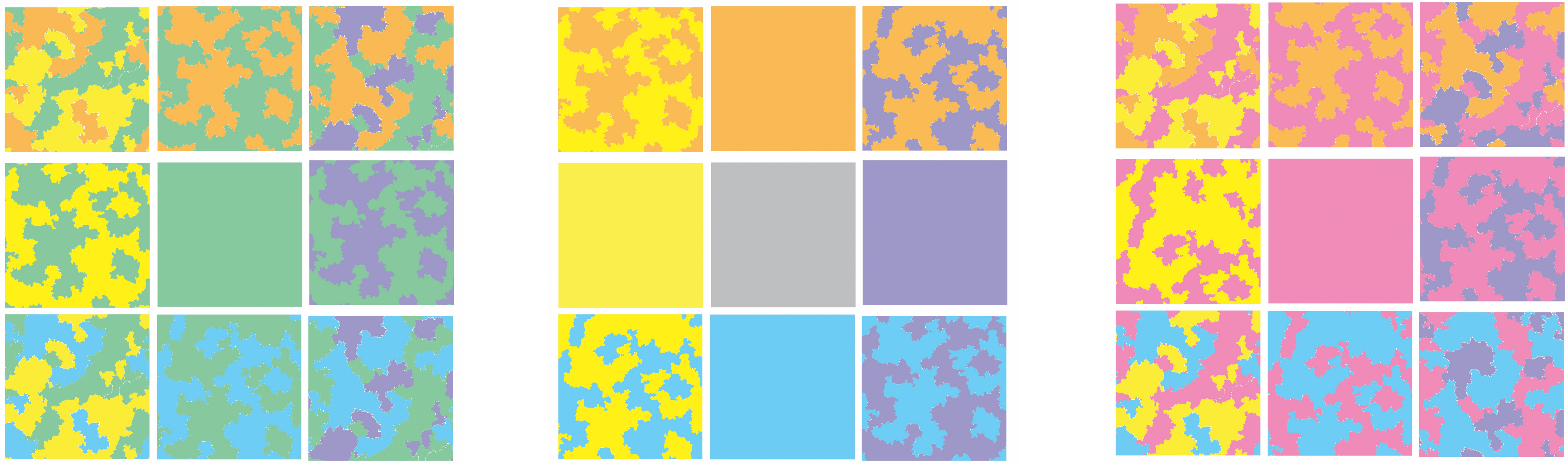






The sequence of three pictures below depicts the cross-sections perceived by the flatlander as the cube is pushed through its 2D world green face first.

First it sees the green face and the surrounding vertices and edges (a).
In the next corss-section it sees the core of the cube (grey) and the surrounding edges and faces (b).
Finally it sees the red face and the surrounding vertices and edges (c).

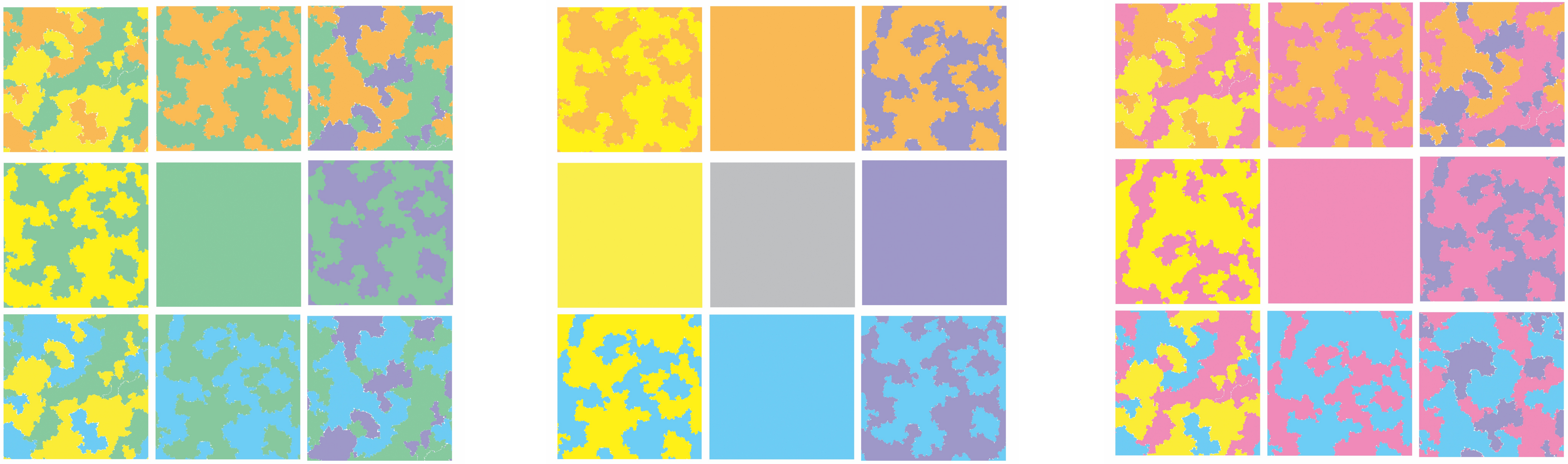


(a) (b) (c)

Exercise:

Cut out the slabs from the papers.
Assemble them again to depict a new sequence of cross-sections, from another direction.
Repeat once or twice.

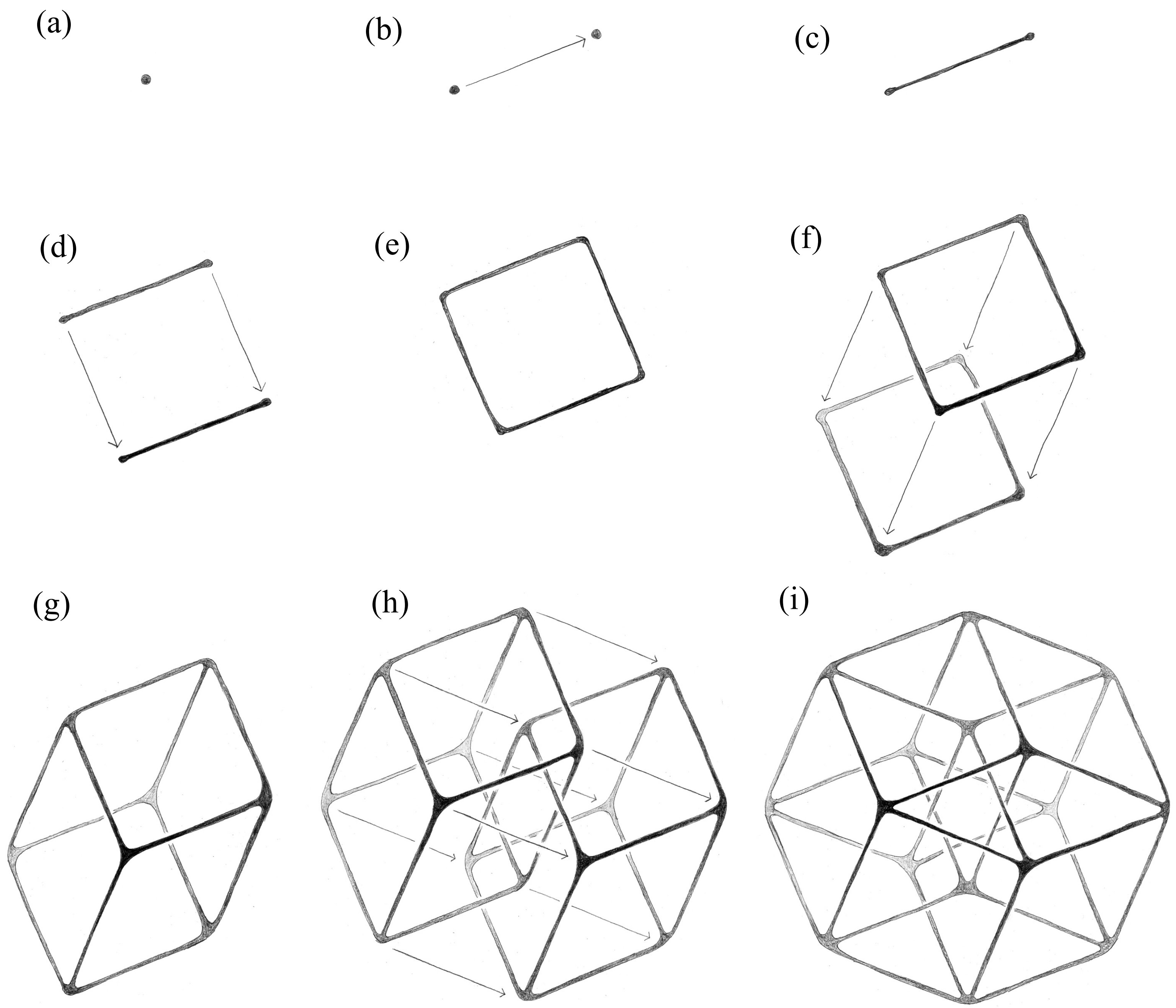
What kinds of rules you perceive see in the colorings of adjecent slabs?
When can two slabs be neighbours, when not?



CROSS-SECTIONS OF THE TESSERACT IN A 3-SPACE

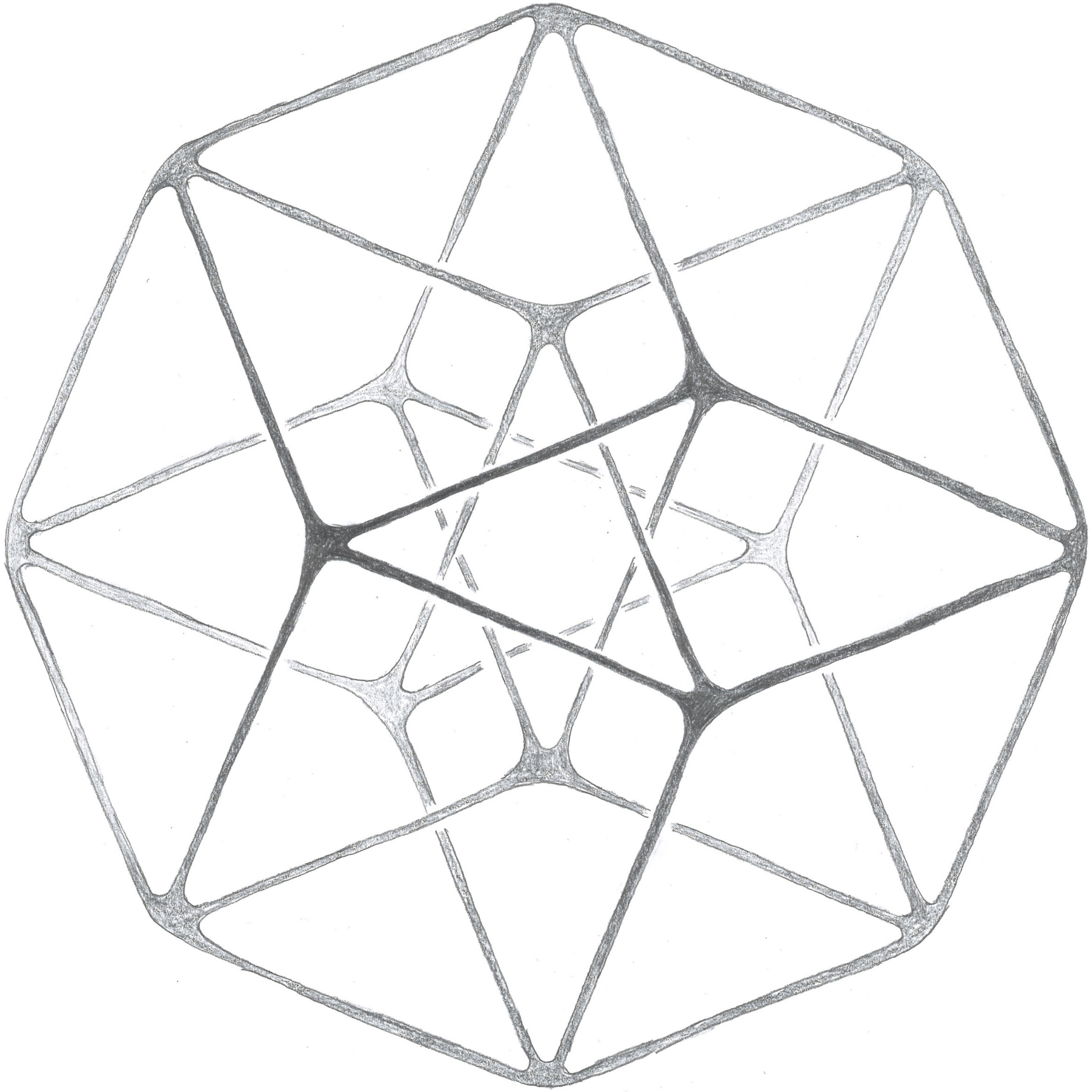
CONSRUCTING THE TESSERACT

If a point (a) is moved (b), it sweeps a line segment (c). If that line segment is then moved perpendicularly away from its line (d), over the distance of its length, it traces out a square (e). Similarly, we get the cube from the square (f,g), and finally when the cube is moved perpendicularly away from its 3 space over the distance of its edge length (h), the trace is a tesseract, also called the 8 cell (i). It has sixteen vertices, thirty-two edges, twenty-four square faces, and eight cubical cells. There are three cells meeting at each edge, and four at each vertex. This family of polytopes extends to all higher dimensions, and they are collectively called the hypercubes.



THE STRUCTURE OF THE TESSERACT

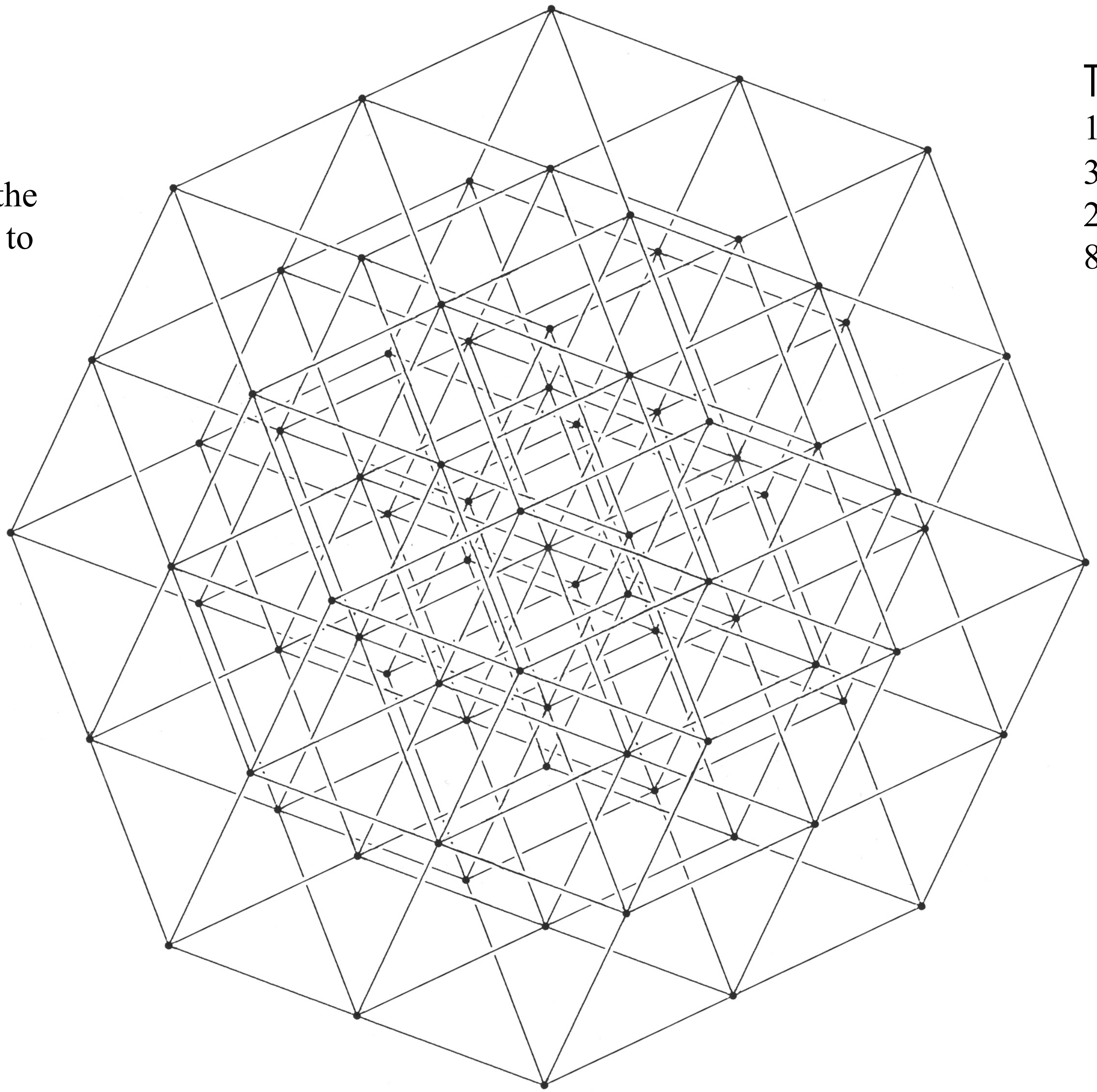
16 vertices
32 edges
24 faces (squares)
8 cells (cubes)



CUTTING THE TESSERACT INTO PIECES

For the purpose of depicting the tesseract in the 3-space, we cut it to three slices with respect to its **four** primary directions.

Thus we get 3x3x3x3 cluster of small tesseracts, where each tesseract represents a vertex, an edge, or a face of the original tesseract, or the original tesseract itself.



TESSERACT
16 vertices
32 edges
24 faces (squares)
8 cells (cubes)

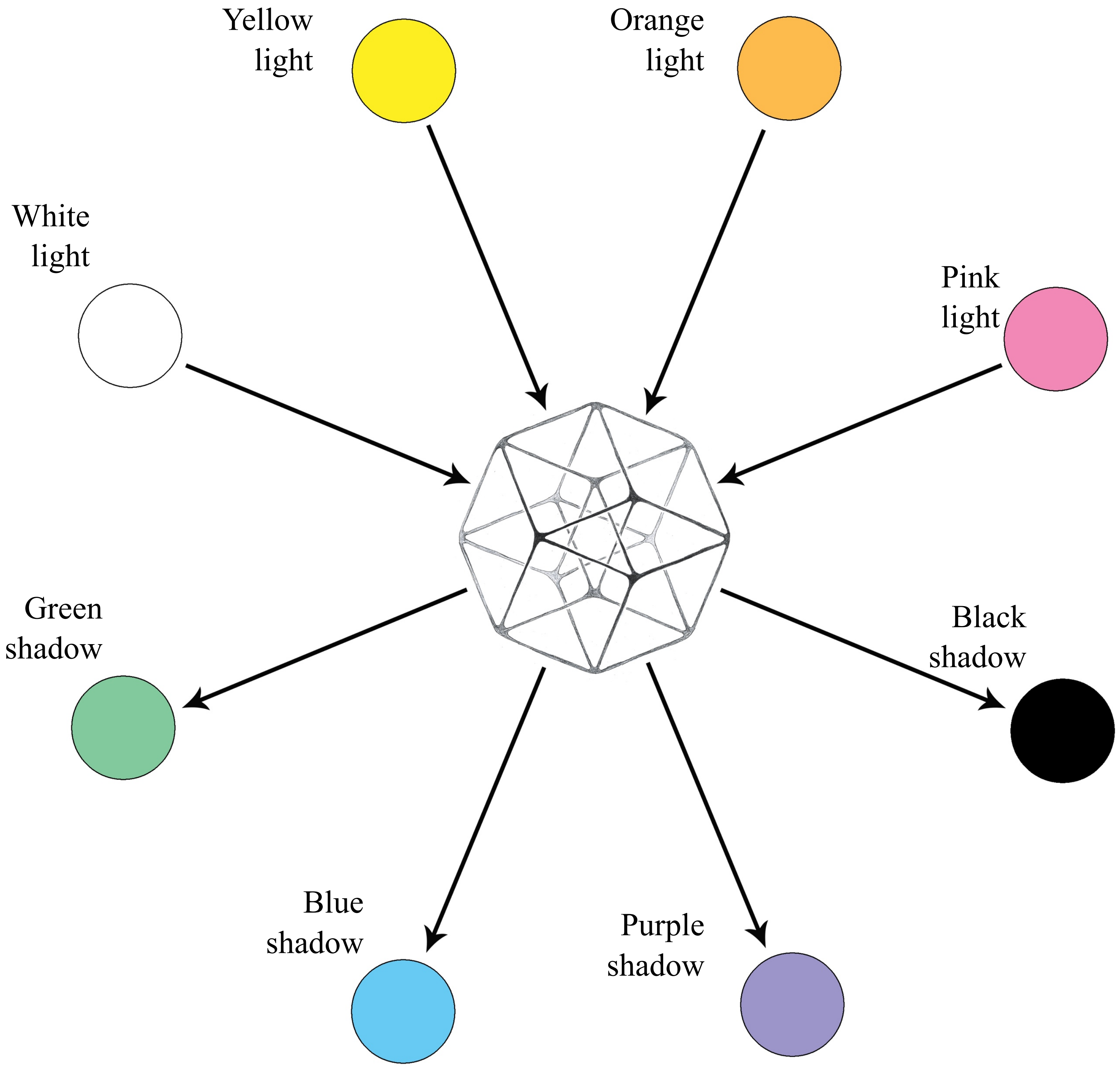
↓

81 TESSERACTS
16 vertex pieces
32 edge pieces
24 face pieces
8 cell pieces
1 core piece

COLORING THE TESSERACT PIECES

Each tesseract piece gets an unique color when think the tesseract being exposed to colored light from **four** different directions (white, yellow, orange, pink).

On the ‘shadow’ side are the complementary colors (black, purple, blue, green).



COLORING THE TESSERACT PIECES

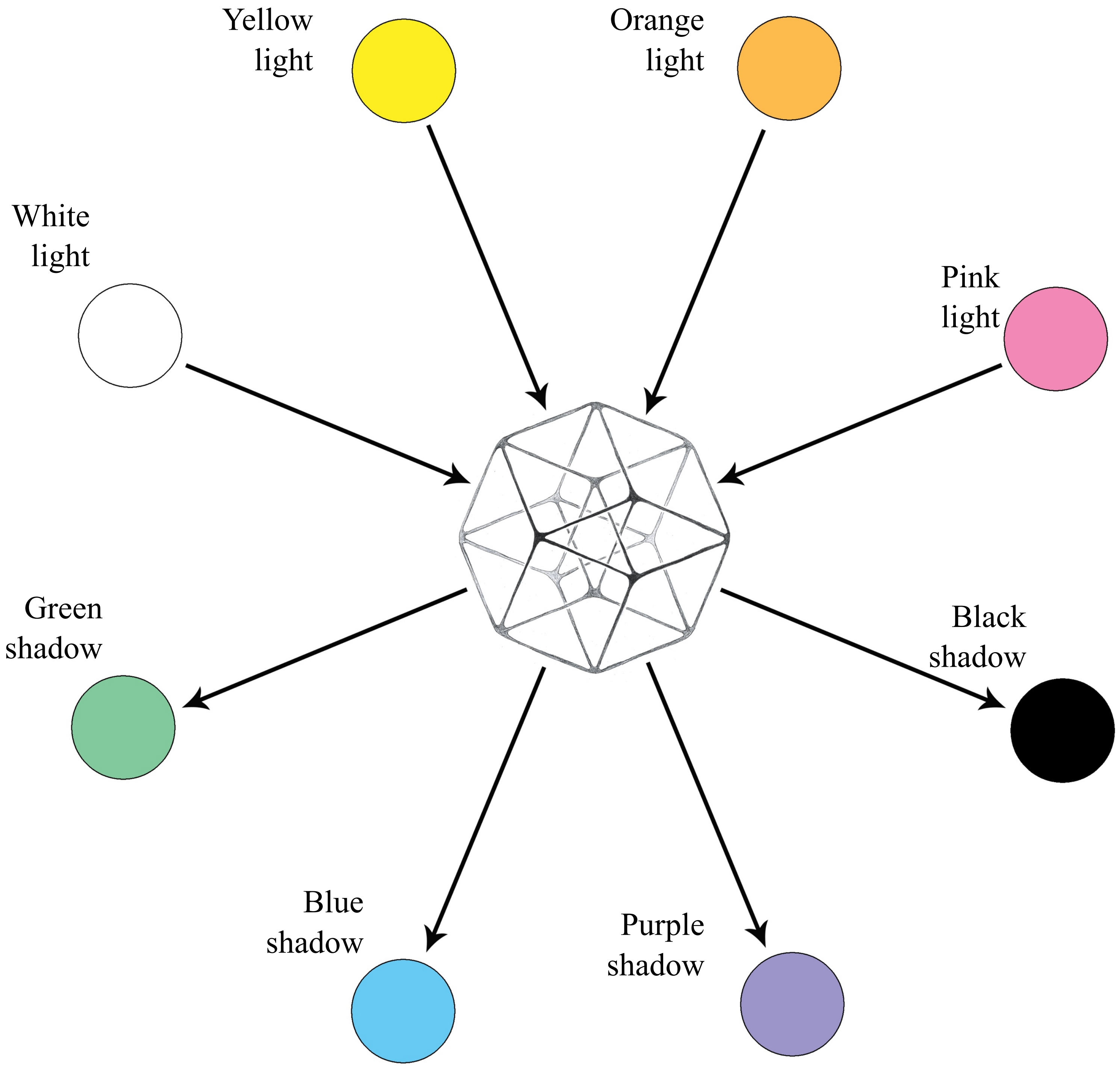
Vertex pieces are four-colored, since they are exposed to light/shadow from four directions.

Edge pieces are three-colored, since they are exposed to light/shadow from three directions.

Face pieces are two-colored, since they are exposed to light/shadow from two directions.

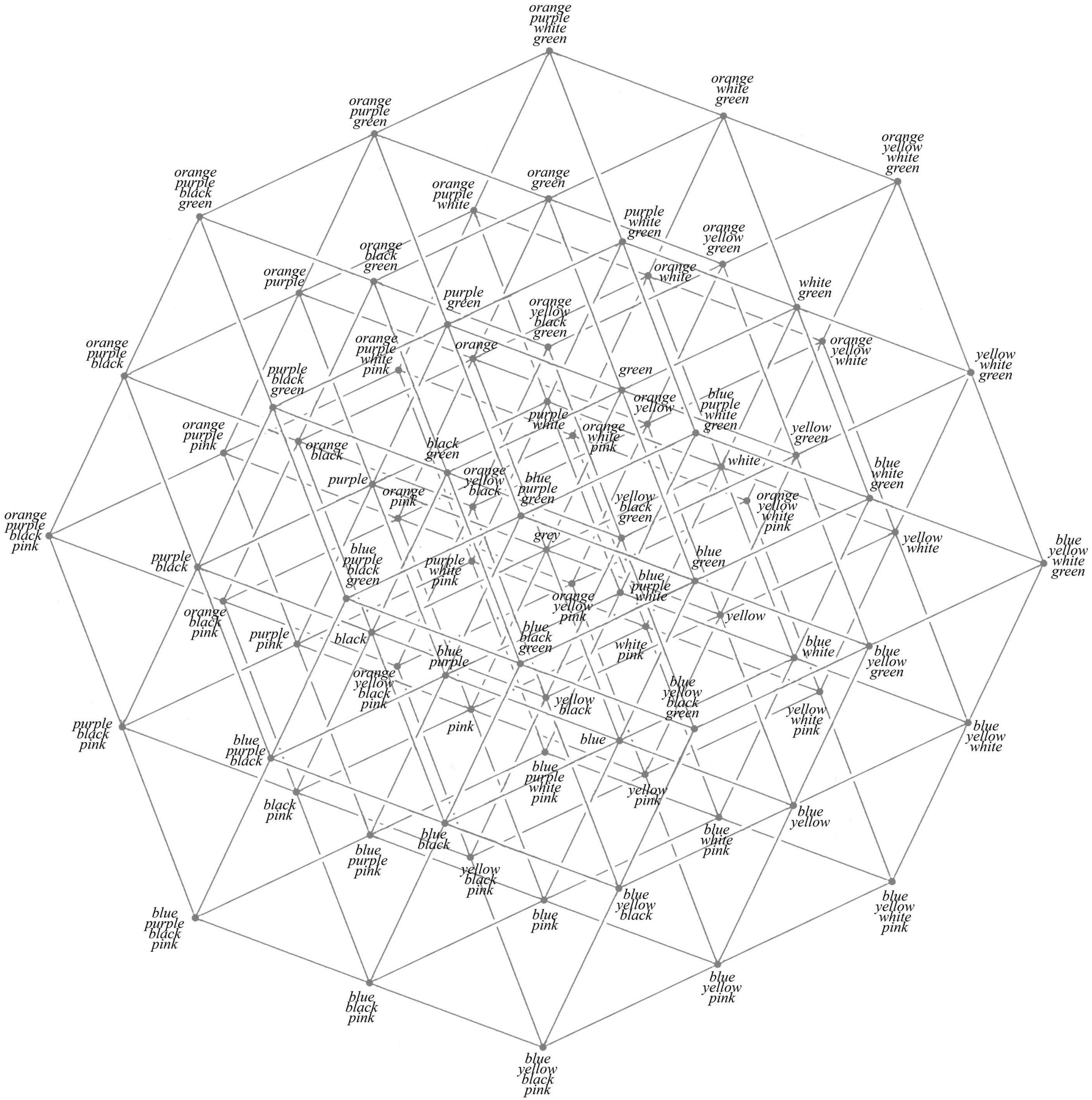
Cell pieces are one-colored, since they are exposed to light/shadow from one direction.

The core piece is colorless, since it is not exposed to any light/shadow.

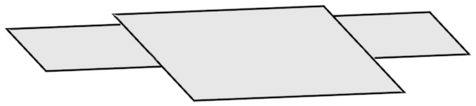


81 PIECES

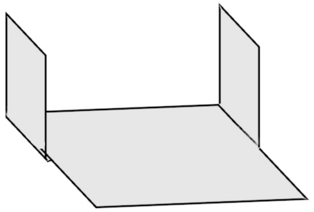
The diagram lists each piece with its coloring, and shows how they are connected to each other.



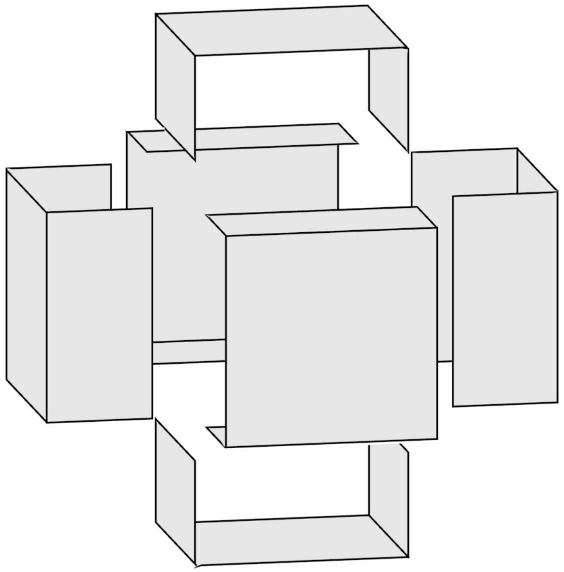
BUILDING THE CUBE SET:



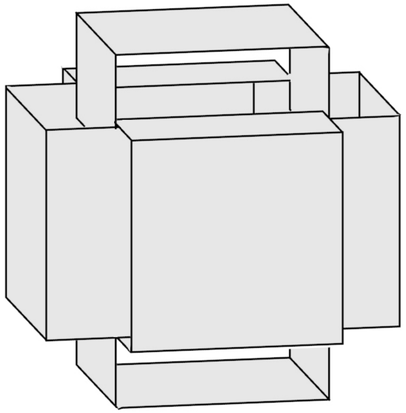
(a)



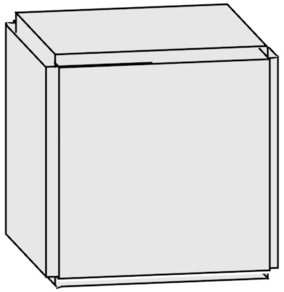
(b)



(c)



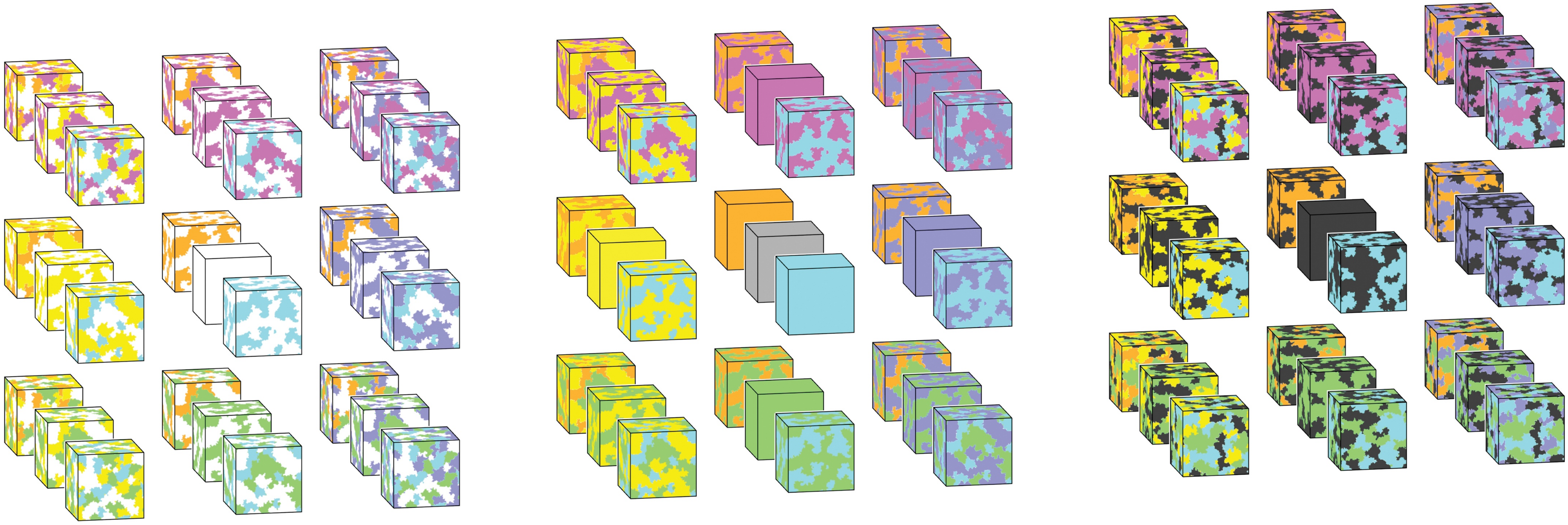
(d)



(e)

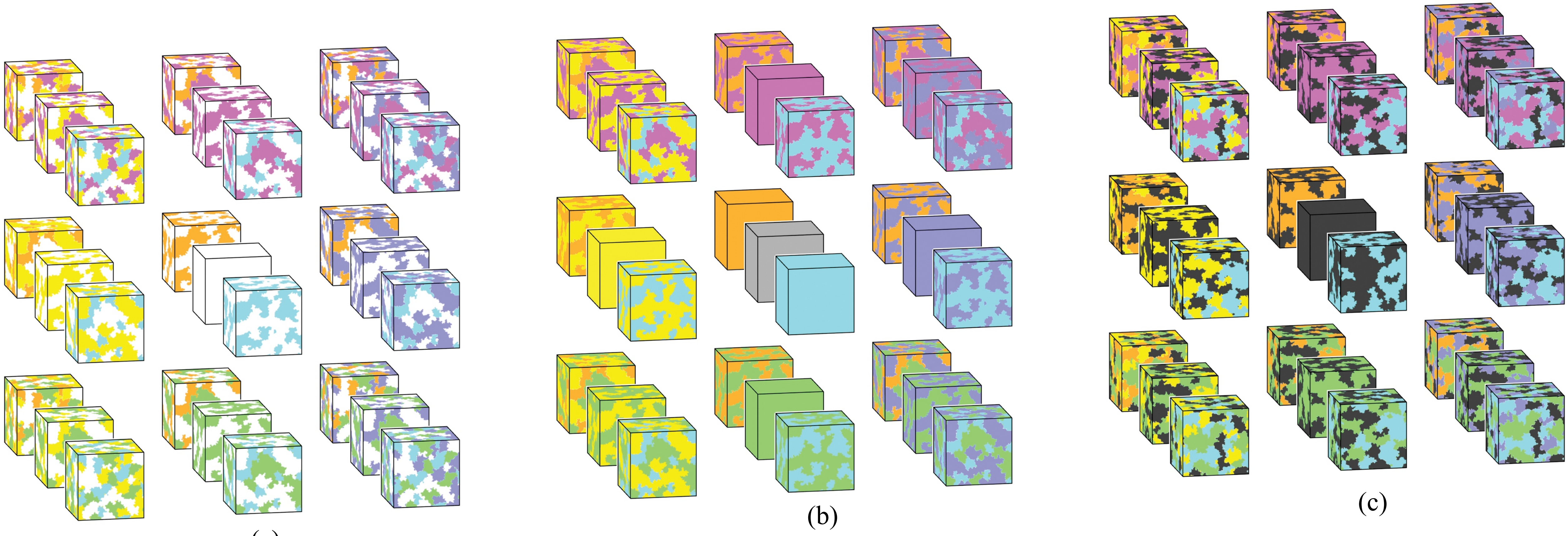
CROSS-SECTIONS OF THE CUBE IN A PLANE

Now the 81 pieces of the tesseract can be represented in 3-space with three-dimensional blocks.
These blocks can be assembeled to show the cross-sections of the tesseract.



The sequence of three models below depicts the cross-sections perceived by us as the tesseract is pushed through our 3D world white cell first.

First we see the white cell and the surrounding vertices, edges and faces (a).
In the next corss-section we seethe core of the tesseract (grey) and the surrounding edges, faces and cells (b).
Finally we see the black cell and the surrounding vertices, edges, and faces (c).



(a) (b) (c)

SOURCES:

Abbott, Edwin A. *Flatland: A Romance of Many Dimensions* [1884]. New York: Oxford University Press, 2006.

Blacklock, Mark. *The Emergence of the Fourth Dimension: Higher Spatial Thinking in the Fin De Siècle*. New York: Oxford University Press, 2018.

Blackwood, Algernon. "A Victim of Higher Space." *The Occult Review*, (William Rider and Son, Ltd) 20, no. 6 (December 1914): 318–335.

Dewdney, A. K. *The Planiverse: Computer Contact with a Two-dimensional World* [1984]. New York: Copernicus, 2000.

Gardner, Martin. "Hypercubes." In *Mathematical Carnival*, 41-54. London: Penguin Books, 1978.

Henderson, Linda Dalrymple. *The Fourth Dimension and Non-Euclidean Geometry in Modern Art* [1983]. Revised edition. Cambridge: MIT Press, 2013.

Hinton, Charles Howard. *A New Era of Thought*. London: Swann Sonnenschein & Co., 1888.

Hinton, Charles Howard. *The Fourth Dimension*. New York: Swann Sonnenschein & Co., 1901.

Rucker, Rudy. *The Fourth Dimension: Toward a Geometry of Higher Reality* [1984]. New York: Dover, 2014.

Volkert, Klaus. *In Höheren Räumen: Der Weg der Geometrie in Die Vierte Dimension*. Berlin: Springer Spektrum, 2018.

White, Cristopher G. *Other Worlds: Spirituality and the Search for Invisible Dimensions*. Cambridge: Harvard University Press, 2018.