Aalto University
School of Science

Decision making and problem solving – Lecture 7

- From EUT to MAUT
- Axioms for preference relations
- Assessing attribute-specific utility functions and attribute weights
- Decision recommendations
- MAVT vs. MAUT

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Motivation

- Multiattribute value theory helps generate decision recommendations, when
  - Alternatives are evaluated w.r.t. multiple attributes
  - Alternatives' attribute-specific values are certain
- What if the attribute-specific performances are *uncertain*?
  - Planning a supply chain: minimize cost, minimize supply shortage, minimize storage costs
  - Building an investment portfolio: maximize return, minimize risk

→ Multiattribute utility theory

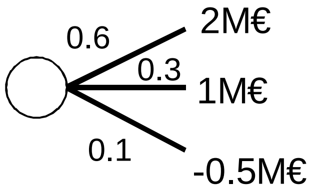
From EUT to MAUT

EUT

- Set of possible outcomes *T*:
  - E.g., revenue $T = \mathbb{R}$ euros, demand $T = \mathbb{N}$
- Set of all possible lotteries *L*:
  - A lottery $f \in L$ associates a probability $f(t) \in [0,1]$ with each possible outcome $t \in T$
- Deterministic outcomes are modeled as degenerate lotteries

Lottery

Decision tree

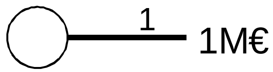


Probability mass function

$$f(t) = \begin{cases} 0.6, t = 2M€ \\ 0.3, t = 1M€ \\ 0.1, t = -0.5M€ \\ 0, elsewhere \end{cases}$$

Degenerate lottery

Decision tree



Probability distribution function

$$f(t) = \begin{cases} 1, t = 1M€ \\ 0, elsewhere \end{cases}$$

From EUT to MAUT

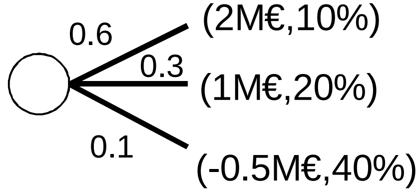
MAUT

- Multidimensional set of outcomes X:

$$X = X_1 \times \cdots \times X_n$$

  - E.g., X_1 = revenue (€), X_2 = market share
- Set of all possible lotteries L:
  - A lottery $f \in L$ associates a probability $f(t) \in [0,1]$ with each possible outcome $x = (x_1, \ldots, x_n) \in X$
- Deterministic outcomes are modelled as degenerate lotteries

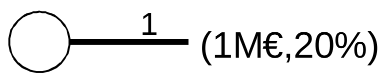
Lottery



$$f(t) = \begin{cases} 0.6, & t = (2,0.1) \\ 0.3, & t = (1,0.2) \\ 0.1, & t = (-0.5,0.4) \\ 0, & elsewhere \end{cases}$$

Degenerate lottery

Decision tree



PDF

$$f(t) = \begin{cases} 1, & t = (1,0.2) \\ 0, & elsewhere \end{cases}$$

Aggregation of utilities

- **Problem:** How to measure the overall utility of alternative $x = (x_1, x_2, \ldots x_n)$?

$$U(x_1, x_2, \ldots x_n) = ?$$

- **Question:** Could the overall utility be obtained by a weighted sum of the attribute-specific utilities?

$$U(x_1, x_2, \ldots x_n) = \sum_{i=1}^{n} w_i \, u_i(x_i)?$$

- **Answer:** Yes, if the attributes are
  - Mutually preferentially independent and
  - Additive independent (new)

Preferential independence (old)

- **Definition:** Attribute X is **preferentially independent (PI)** of the other attributes $\boldsymbol{Y}$, if the preference order of degenerate lotteries that differ only in X does not depend on the levels of attributes $\boldsymbol{Y}$

$$(x, \boldsymbol{y}) \succcurlyeq (x', \boldsymbol{y}) \Rightarrow (x, \boldsymbol{y}') \succcurlyeq (x', \boldsymbol{y}') \text{ for all } \boldsymbol{y}' \in \boldsymbol{Y}$$

- Interpretation: Preference over the <u>certain</u> level of attribute X does not depend on the <u>certain</u> levels of the other attributes, as long as they stay the same

- Same as in MAVT

Mutual preferential independence (old)

- **Definition:** Attributes A are **mutually perferentially independent (MPI),** if any subset **X** of attributes A is preferentially independent of the other attributes **Y**=A**X**. I.e., for any degenerate lotteries:

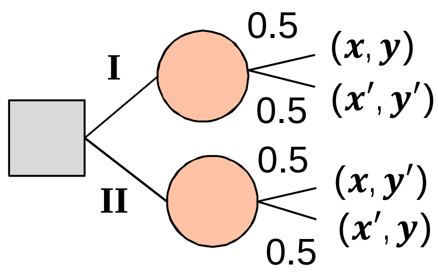
$$(\boldsymbol{x}, \boldsymbol{y}') \succcurlyeq (\boldsymbol{x}', \boldsymbol{y}') \Rightarrow (\boldsymbol{x}, \boldsymbol{y}) \succcurlyeq (\boldsymbol{x}', \boldsymbol{y}) \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}.$$

- Interpretation: Preference over the certain levels of attributes ***X*** does not depend on the certain levels of the other attributes, as long as they stay the same

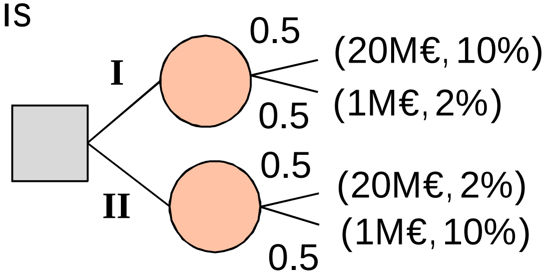
- Same as in MAVT

Additive independence (new)

- **Definition:** Subset of attributes **X**⊂A is **additive independent (AI),** if the DM is indifferent between lotteries **I** and **II** for any $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}') \in A$

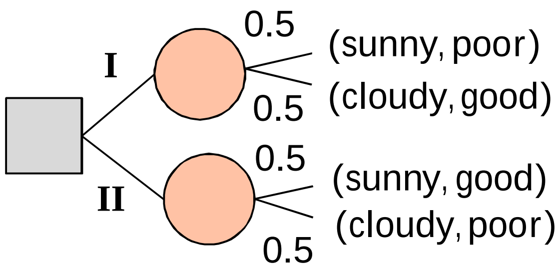


- Example:
  - Profit is AI if the DM is indifferent between I and II
  - However, she might prefer II, because it does not include an outcome where all attributes have very poor values. In this case profit is not AI.



Additive independence (new)

- Example:
  - A tourist is planning a downhill skiing weekend trip to the mountains
  - 2 attributes: sunshine ({sunny, cloudy}) and snow conditions ({good, poor})
  - Additive independence holds, if she is indifferent between I and II
    - In both, there is a 50 % probability of getting sunshine
    - In both, there is a 50 % probability of having good snow conditions
    - If the DM values sunshine and snow conditions independently of each other, then I and II can be equally preferred



Additive multiattribute utility function

- **Theorem:** The attributes are mutually preferentially independent and **single** attributes are additive independent iff preference relation $\succcurlyeq$ is represented by an **additive multi-attribute utility** function

$$U(x) = \sum_{i=1}^{n} w_i u_i^N(x_i),$$

where $u_i^N(x_i^0) = 0$, $u_i^N(x_i^*) = 1$, and $\sum_{i=1}^{n} w_i = 1$, $w_i \geq 0$.

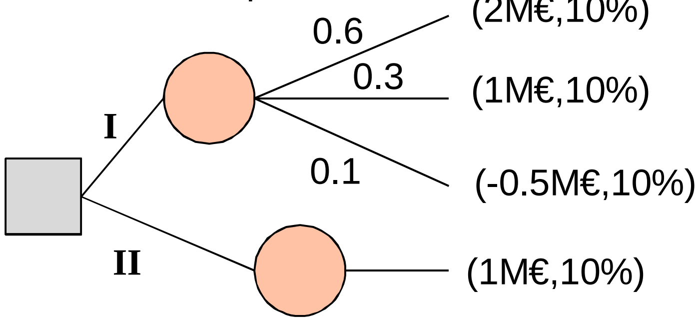
What if the MPI & AI do not hold?

- **Definition:** Attribute $X \in A$ is **utility independent (UI)** if the preference order between lotteries that have equal certain outcomes on attributes **Y**=*A**X* does not depend on the level of these outcomes, i.e.,

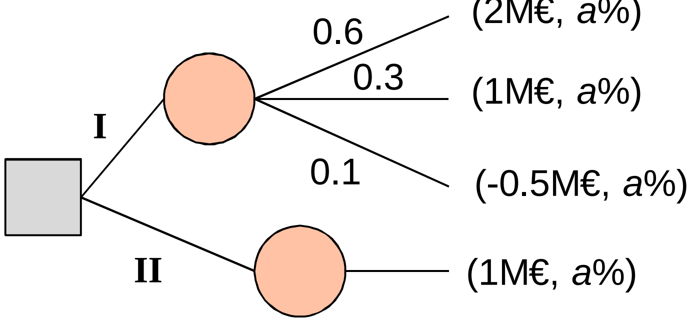
$$(\tilde{x}, \boldsymbol{y}) \succcurlyeq (\tilde{x}', \boldsymbol{y}) \Rightarrow (\tilde{x}, \boldsymbol{y}') \succcurlyeq (\tilde{x}', \boldsymbol{y}') \forall \boldsymbol{y}'$$

- Example:

Assume DM prefers **I**



If profit is UI, then the DM should prefer **I** for any *a*

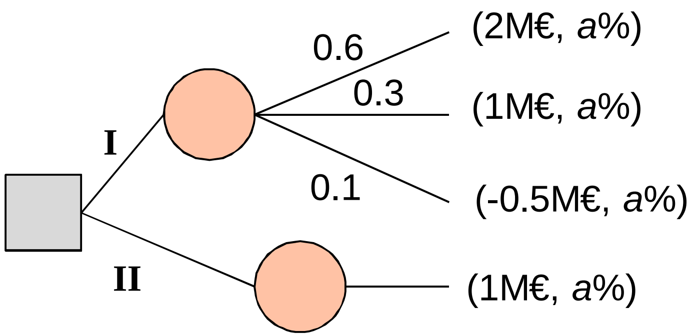


However, for a small market share (a), the DM may be more risk averse and choose **II** → profit is not UI

Mutual utility independence

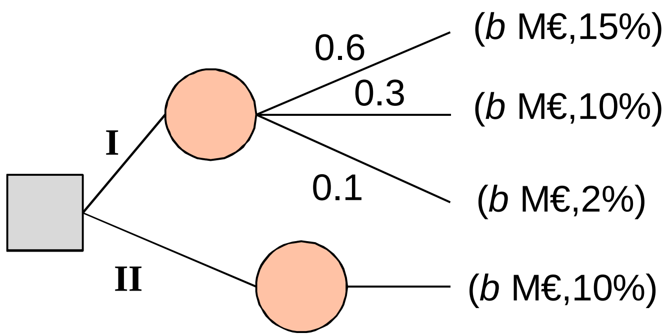
- **Definition:** Attributes A are **mutually utility independent (MPI),** if every subset $\boldsymbol{X} \subset A$ is the utility independent of the other attributes **Y**=A\X i.e.,

$$(\tilde{\boldsymbol{x}}, \boldsymbol{y}) \succcurlyeq (\tilde{\boldsymbol{x}}', \boldsymbol{y}) \Rightarrow (\tilde{\boldsymbol{x}}, \boldsymbol{y}') \succcurlyeq (\tilde{\boldsymbol{x}}', \boldsymbol{y}') \; \forall \boldsymbol{y}'$$



If DM prefers **I** for some *a*, she should prefer **I** for all *a*

AND



If DM prefers **I** for some *b*, she should prefer **I** for all *b*

Other multi-attribute utility functions

- If attributes are **mutually utility independent**, then preferences are represented by a multiplicative utility function:

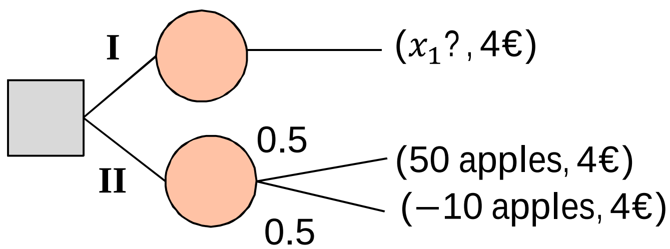
$$U(x) = \frac{\prod_{i=1}^{n}[1 + k w_i u_i(x_i)]}{k} - \frac{1}{k}$$

- If each single attribute is **utility independent**, then preferences are represented by a so-called multilinear utility function

- AI is the strongest of the three preference assumptions
  - Let X ⊂ A. Then, X is AI ⇒ X is UI ⇒ X is PI

Assessing attribute-specific utility functions

- Use the same techniques as with a unidimensional utility function
  - Certainty equivalent, probability equivalent, etc. & scale such that $u_i^N(x_i^0) = 0$, $u_i^N(x_i^*) = 1$.
  - Also direct rating often applied in practice
- What about the other attributes?
  - Fix them at the same level in every outcome
  - Do not matter! → Usually not even explicitly shown to the DM



$$U(x_1, 4) = 0.5U(50,4) + 0.5U(-10,4)$$
$$\Leftrightarrow w_1u_1(x_1) + w_2u_2(4) = 0.5w_1u_1(50) + 0.5w_2u_2(4) + 0.5w_1u_1(-10) + 0.5w_2u_2(4)$$
$$\Leftrightarrow w_1u_1(x_1) = 0.5w_1u_1(50) + 0.5w_1u_1(-10)$$
$$\Leftrightarrow u_1(x_1) = 0.5u_1(50) + 0.5u_1(-10)$$

Example: Choosing a software supplier

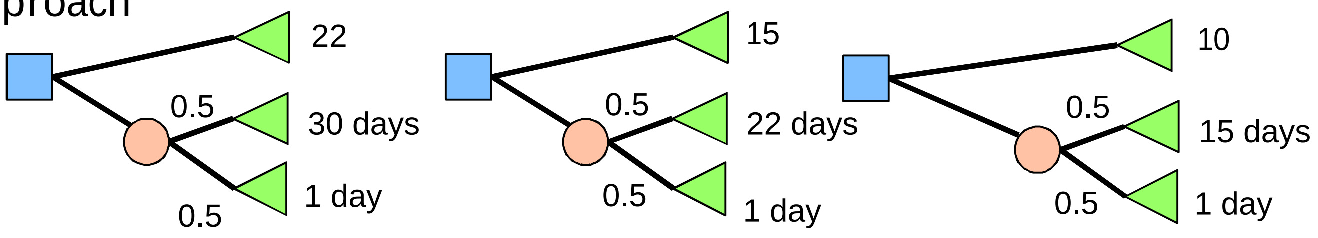
- Three attributes: cost, delay, quality

i	Name	X_i	x_i^0	x_i^*
1	Cost	[10,40] k€	40	10
2	Delay	{1,2,…,30} days	30	1
3	Quality	{fair, good, excellent}	fair	excellent

Example: Choosing a software supplier

i	Name	X_i	x_i^0	x_i^*
1	Cost	[10,40] k€	40	10
2	Delay	{1,2,…,30} days	30	1
3	Quality	{fair, good, exc.}	fair	exc.

- Assessment of the attribute-specific utility functions
  - Quality: Direct assessment
    - u_3(fair)=0, u_3(good)=0.4, u_3(excellent)=1
  - Cost: Linear decreasing utility function
    - $u_1(x_1) = \frac{40-x_1}{30}$
  - Delay: Assessment with certainty equivalent (CE) approach



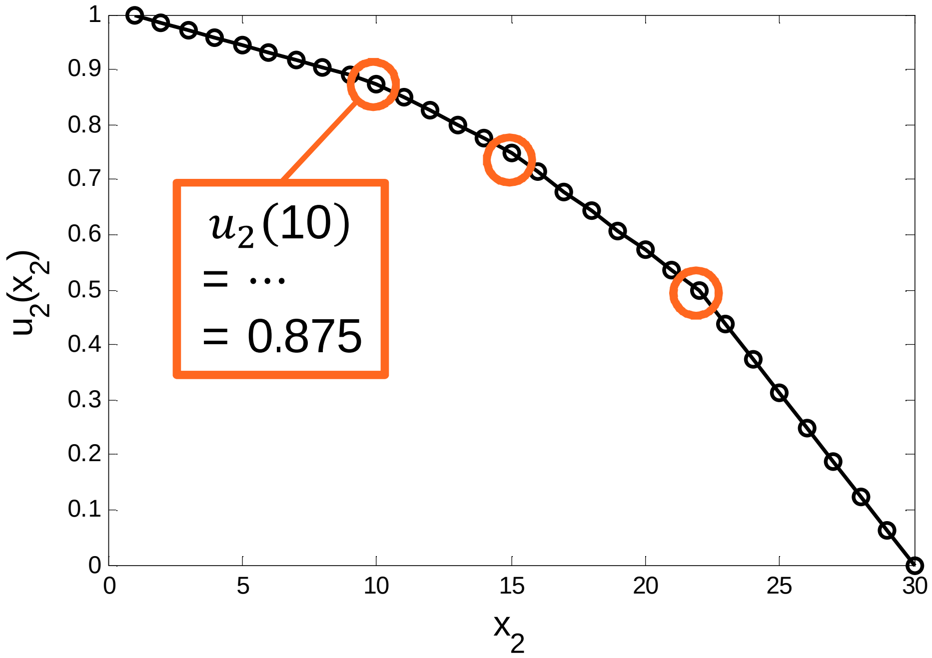
$$u_2(22) = 0.5u_2(1) + 0.5u_2(30) = 0.5 * 1 + 0.5 * 0 = 0.5$$

$$u_2(15) = 0.5u_2(1) + 0.5u_2(22) = 0.5 * 1 + 0.5 * 0.5 = 0.75$$

$$u_2(10) = 0.5u_2(1) + 0.5u_2(22) = 0.5 * 1 + 0.5 * 0.75 = 0.875$$

Example: Choosing a software supplier

For *delay*, linear interpolation between specified values



x_2	$u_2(x_2)$	x_2	$u_2(x_2)$
1	1	16	0.7143
2	0.9861	17	0.6786
3	0.9722	18	0.6429
4	0.9583	19	0.6071
5	0.9444	20	0.5714
6	0.9306	21	0.5357
7	0.9167	22	0.5
8	0.9028	23	0.4375
9	0.8889	24	0.375
10	0.875	25	0.3125
11	0.85	26	0.25
12	0.825	27	0.1875
13	0.8	28	0.125
14	0.775	29	0.0625
15	0.75	30	0

Assessing attribute weights

- Attribute weights are elicited by constructing two equally preferred degenerate lotteries
  - E.g., ask the DM to establish a preference order for n hypothetical alternatives specified so that $(x_1^0, \dots, x_i^*, \dots, x_n^0),\ i = 1, \dots, n.$
  - Assume that $(x_1^*, x_2^0, \dots, x_n^0) \succcurlyeq (x_1^0, x_2^*, \dots, x_n^0) \succcurlyeq \cdots \succcurlyeq (x_1^0, x_2^0, \dots, x_n^*)$
  - Then, for each i=1,…,n-1 ask the DM to define $x_i \in X_i$ such that

$$\left(\dots x_i, x_{i+1}^0, \dots\right) \sim \left(\dots x_i^0, x_{i+1}^*, \dots\right)$$
$$\Rightarrow U\left(\dots x_i, x_{i+1}^0, \dots\right) = U\left(\dots x_i^0, x_{i+1}^*, \dots\right)$$
$$\Rightarrow w_i u_i(x_i) = w_{i+1}$$

  - n-1 such comparisons + 1 normalization constraint ⇒ unique set of weights

Example: Choosing a software supplier

- Assessment of the attribute weights
  - Assume preferences (40k€, 1 day, fair)≽(10k€, 30 days, fair)≽(40k€, 30 days, exc.)
  - Choose delay $x_2 \in \{1, \ldots, 30\}$ such that $(40, x_2, x_3) \sim (10{,}30, x_3)$
  - Answer $x_2 = 8$ gives

$$w_1u_1(40) + w_2u_2(8) + w_3u_3(x_3) = w_1u_1(10) + w_2u_2(30) + w_3u_3(x_3)$$
$$w_2u_2(8) = w_1$$
$$\Leftrightarrow w_2 \cdot 0.9028 = w_1$$

  - Choose cost $x_1 \in [10{,}40]$ such that $(x_1, x_2, \text{fair}) \sim (40, x_2, \text{excellent})$
  - Answer $x_1 = 20$ gives

$$w_1u_1(20) + w_2u_2(x_2) + w_3u_3(\text{fair}) = w_1u_1(40) + w_2u_2(x_2) + w_3u_3(\text{excellent})$$
$$w_1u_1(20) = w_3$$
$$\Leftrightarrow w_1 \cdot \frac{2}{3} = w_3$$

  - Attribute weights: $w \approx \left(\frac{9}{25}, \frac{10}{25}, \frac{6}{25}\right)$

MAUT: Decision recommendations

- Consider *m* decision alternatives $x^j = (x_1^j, \dots, x_n^j)$, $j = 1, \dots, m$, where x^j is a random variable with PDF $f_{x^j}(x)$
- Alternatives are ranked by their expected (multiattribute) utilities

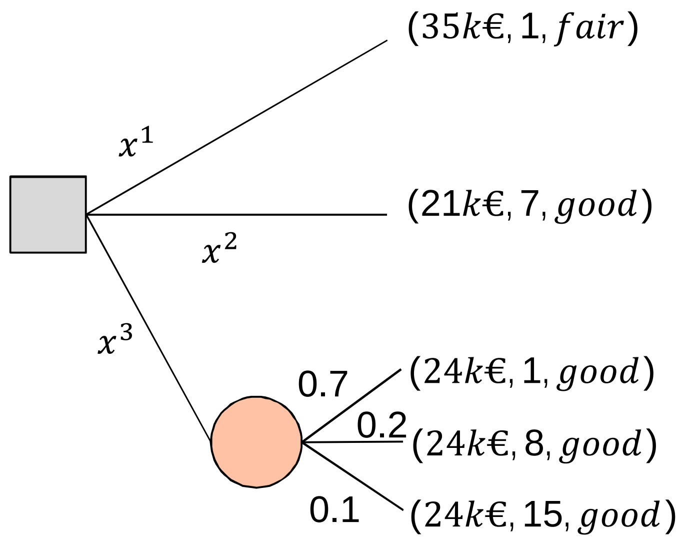
$$E[U(x^j)] = \sum_{x \in A} f_{x^j}(x)\, U(x) = \sum_{x \in A} f_{x^j}(x) \sum_i w_i u_i(x)$$

  – Integral for continuous random variables

- In a decision tree, MAU is used just like unidimensional utility

Example: Choosing a software supplier

- Consider three suppliers:
  - Supplier 1: Expensive, fair quality, can deliver without delay
    $$x^1 = (35k€, 1 \text{ day}, fair)$$
  - Supplier 2: Cheap, good quality, can deliver in 1 week
    $$x^2 = (21k€, 7 \text{ days}, good)$$
  - Supplier 3: Moderate price, good quality, 20% chance of 1-week delay and 10% chance of 2-week delay
    $$x^3 = (24k€, \tilde{x}_2^3, good),$$
    $$f_{\tilde{x}_2^3}(x) = \begin{cases} 0.7, x = (24k€, 1 \text{ day}, good) \\ 0.2, x = (24k€, 8 \text{ days}, good) \\ 0.1, x = (24k€, 15 \text{ days}, good) \end{cases}$$



Example: Choosing a software supplier

	u_1^N	u_2^N	u_3^N	U	$f_{x_k^j}$	$\mathrm{E}[U]$
x^1	0.17	1.00	0.00	0.46	1	0.46
$\boldsymbol{x^2}$	**0.63**	**0.92**	**0.40**	**0.69**	**1**	**0.69**
$x^3\ (s_1)$	0.53	1.00	0.40	0.69	0.7	0.67
$x^3\ (s_2)$	0.53	0.90	0.40	0.65	0.2	
$x^3\ (s_3)$	0.53	0.75	0.40	0.59	0.1	
w	0.36	0.40	0.24			

MAVT vs. MAUT

- **MAVT:** Preference between <u>alternatives with certain outcomes</u> can be represented by an additive multiattribute value function, iff the attributes are
  - Mutually preferentially independent
  - Difference independent

- **MAUT**: Preference between <u>lotteries with uncertain outcomes</u> can be represented by additive multiattribute utility function, iff the attributes are
  - Mutually preferentially independent
  - Additive independent

MAVT vs. MAUT

- **Attribute-specific <u>value</u> functions** are elicited by asking the DM to specify equally preferred differences in attribute levels
  - E.g., ”Specify salary x such that you would be indifferent between change 1500€ → x€ and x€ → 2000€”
- **Attribute-specific <u>utility</u> functions** are elicited by asking the DM to specify equally preferred lotteries
  - E.g., ”Specify salary x such that you would be indifferent between getting x€ for certain and a 50-50 gamble between getting 1500€ or 2000€”
- **Attribute weights** are elicited similarly in MAVT and MAUT

MAVT vs. MAUT

- In principal, the natural / measurement scale is first mapped to value scale and then (if needed) to utility scale

- Yet, in practice the value function is ”hidden” in the utility function
  - E.g, if certainty equivalent of 50-50 gamble between 3k€ and 5k€ salary is 3.9k€, is this a sign of risk aversion or decreasing marginal value of salary?

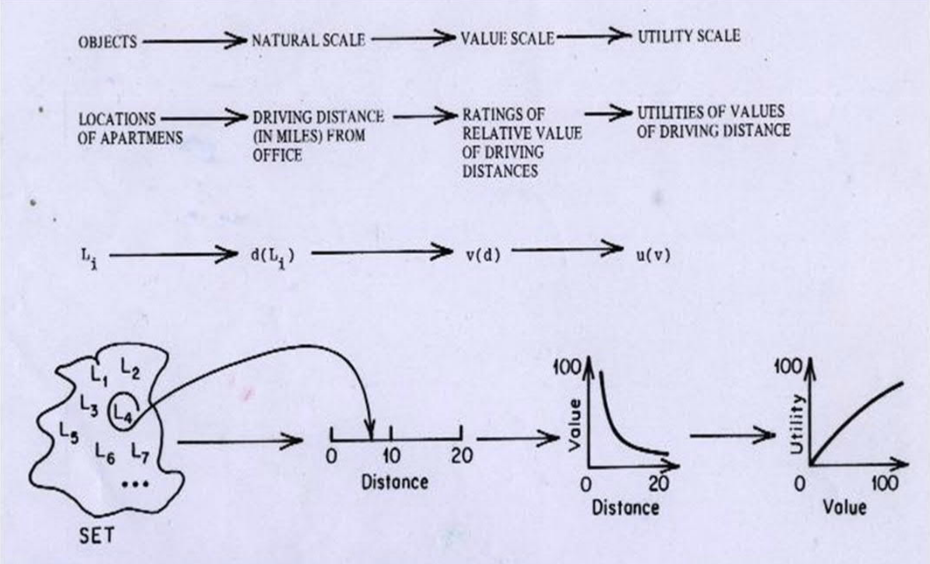


Figure 7.2. The four steps needed to construct value and utility functions.

Summary

- Multiattribute utility theory helps establish a preference relation between alternatives with uncertain outcomes on multiple attributes
- Preference relation is represented by an additive utility function, iff the attributes are mutually preferentially independent and additive independent
- Attribute-specific utility functions are elicited as in the unidimensional case
- Attribute weights are elicited as in MAVT
- Decision recommendation: the alternative with highest expected utility
- Robust methods can also be used with MAUT