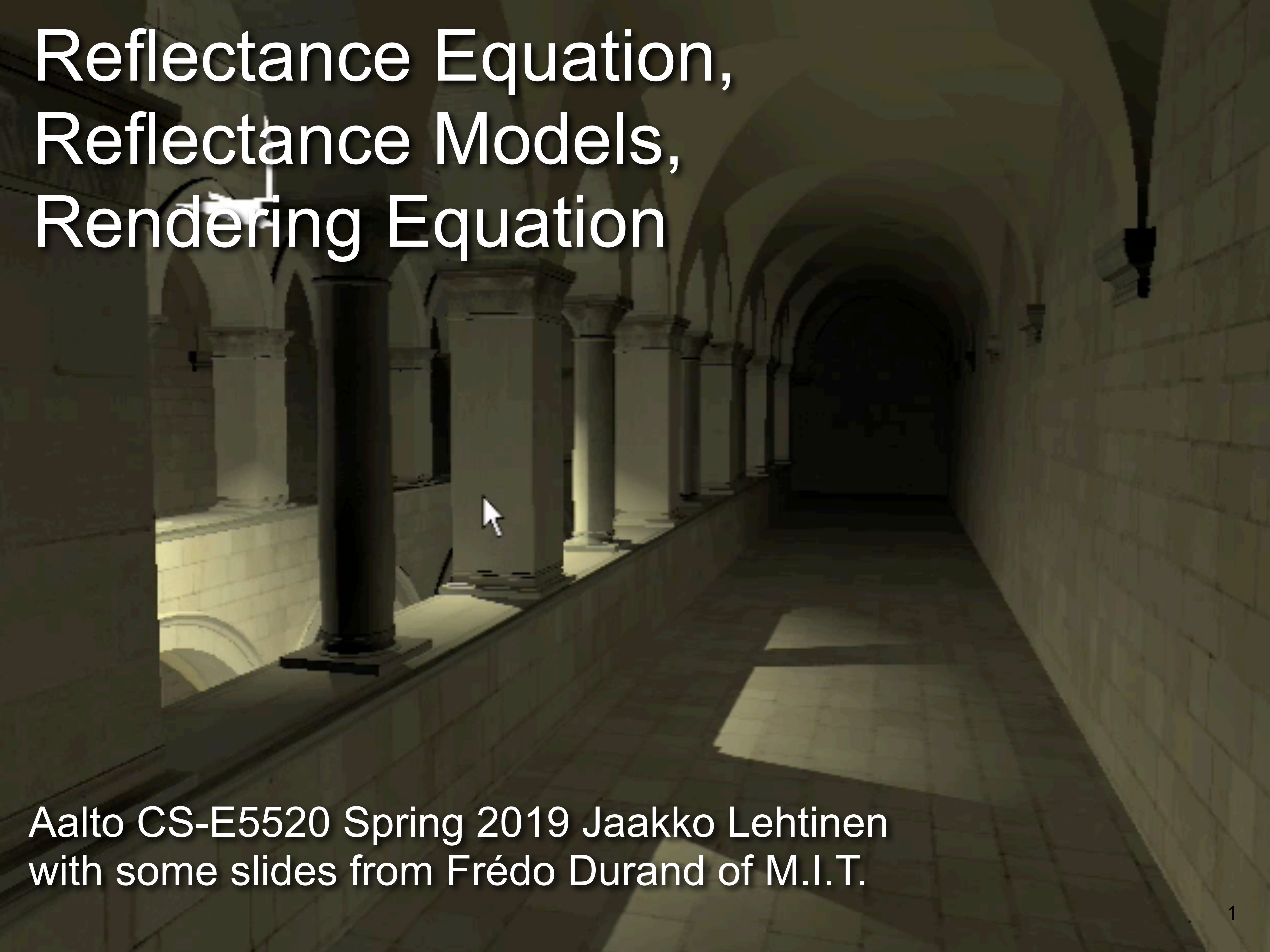
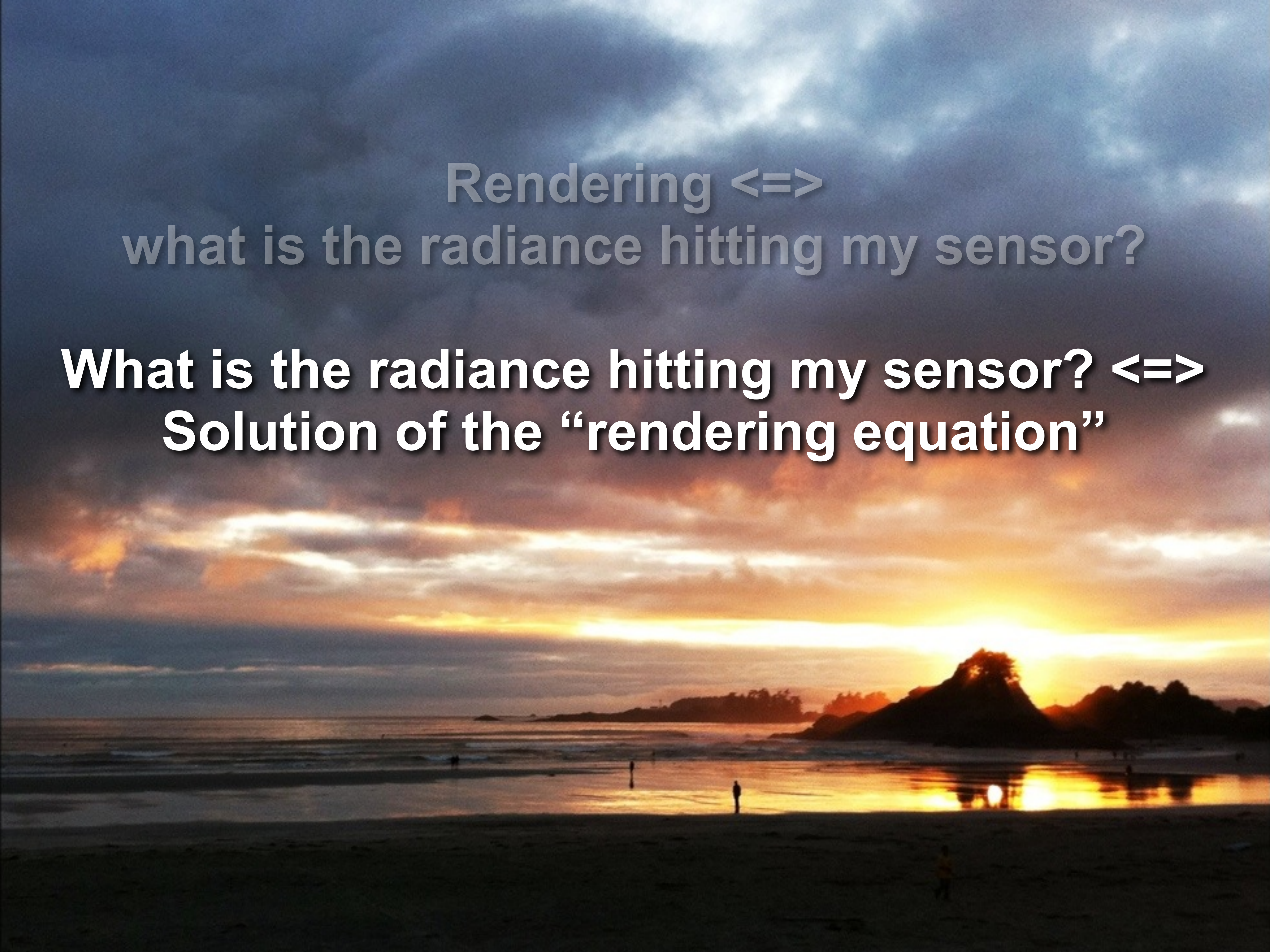


Reflectance Equation, Reflectance Models, Rendering Equation

Aalto CS-E5520 Spring 2019 Jaakko Lehtinen
with some slides from Frédo Durand of M.I.T.


Rendering <=>
what is the radiance hitting my sensor?



Rendering <=>
what is the radiance hitting my sensor?

What is the radiance hitting my sensor? <=>
Solution of the "rendering equation"

Today

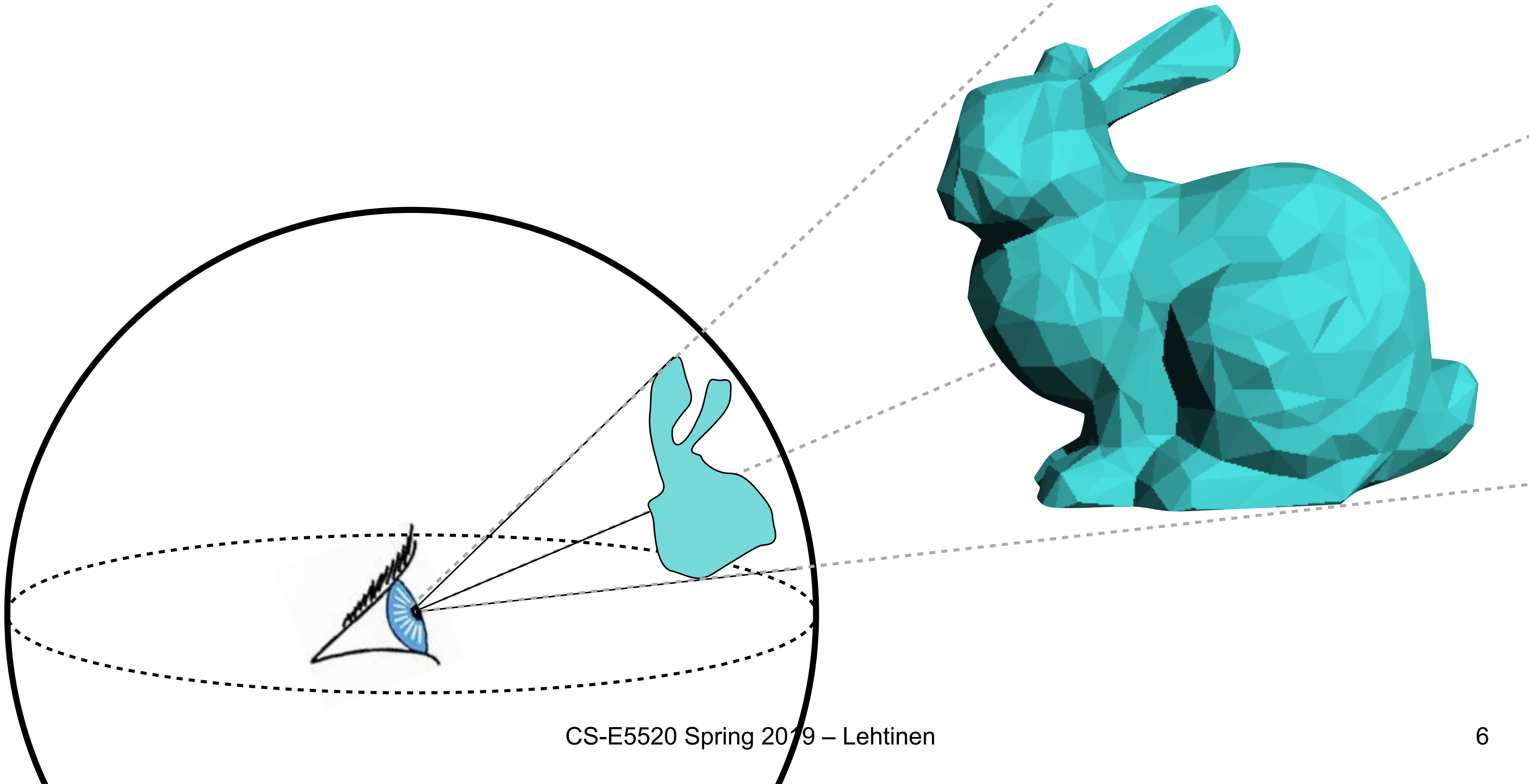
- Reflectance Equation
  - Recap of the BRDF, plus details
- Global Illumination
  - Rendering Equation
  - Gets us indirect lighting
- Next time
  - Monte Carlo integration
  - Better sampling
    - importance
    - stratification

Recap, Last Week

- How “*bright*” something is doesn’t directly tell you how brightly it *illuminates* something
  - The lamp appears just as bright from across the room and when you stick your nose to it (“intensity does not attenuate”)
  - Also, the lamp’s apparent brightness does not change much with the angle of exitance
- **However**
  - if you take the receiving surface further away, it will reflect less light and appear darker
  - If you tilt the receiving surface, it will reflect less light and appear darker

Remember: "How Big Something Looks"

- Solid angle <=> projected area on unit sphere



Recap: Flux

- Flux Φ measures luminous energy per unit time, i.e., power, $[\Phi] = [J/s] = [W]$

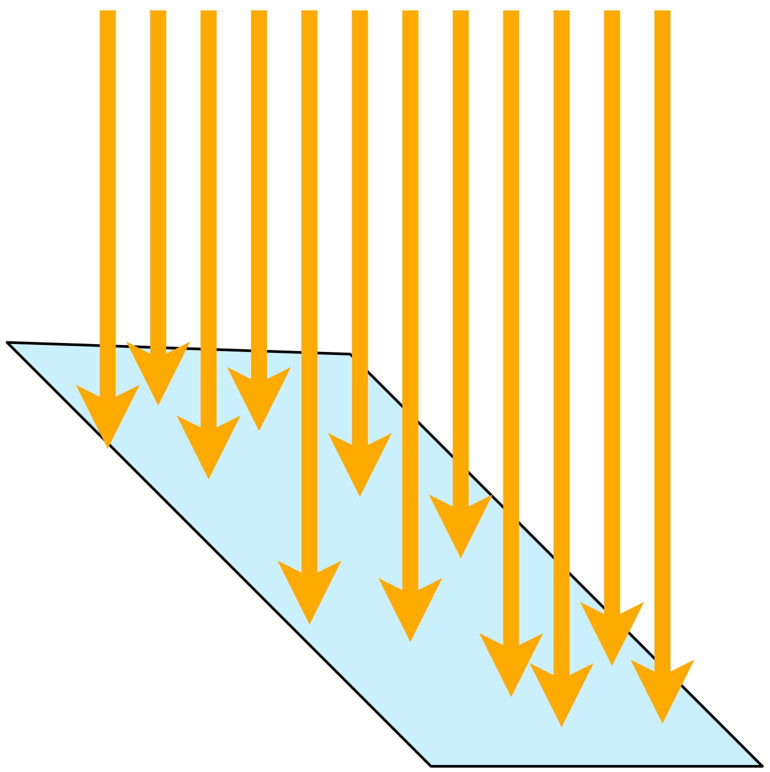
- You can think of photons/second, with the limit of infinitely many infinitely low-energy photons
  - (In reality, every photon carries some non-infinitesimal flux)

Recap: Irradiance

- **Irradiance** E is the flux Φ [W] per unit area [1/m²] landing on a surface

$$E = \frac{\mathrm{d}\Phi}{\mathrm{d}A} \qquad \left[\frac{W}{m^2}\right]$$

  – You can really think of counting photons



- (Brightness of diffuse surface determined directly by irradiance)
  – (We'll come to this in a bit)

Recap: Radiance

- **Sensors are sensitive to radiance**
  - It's what you assign to pixels
  - The fundamental quantity in image synthesis

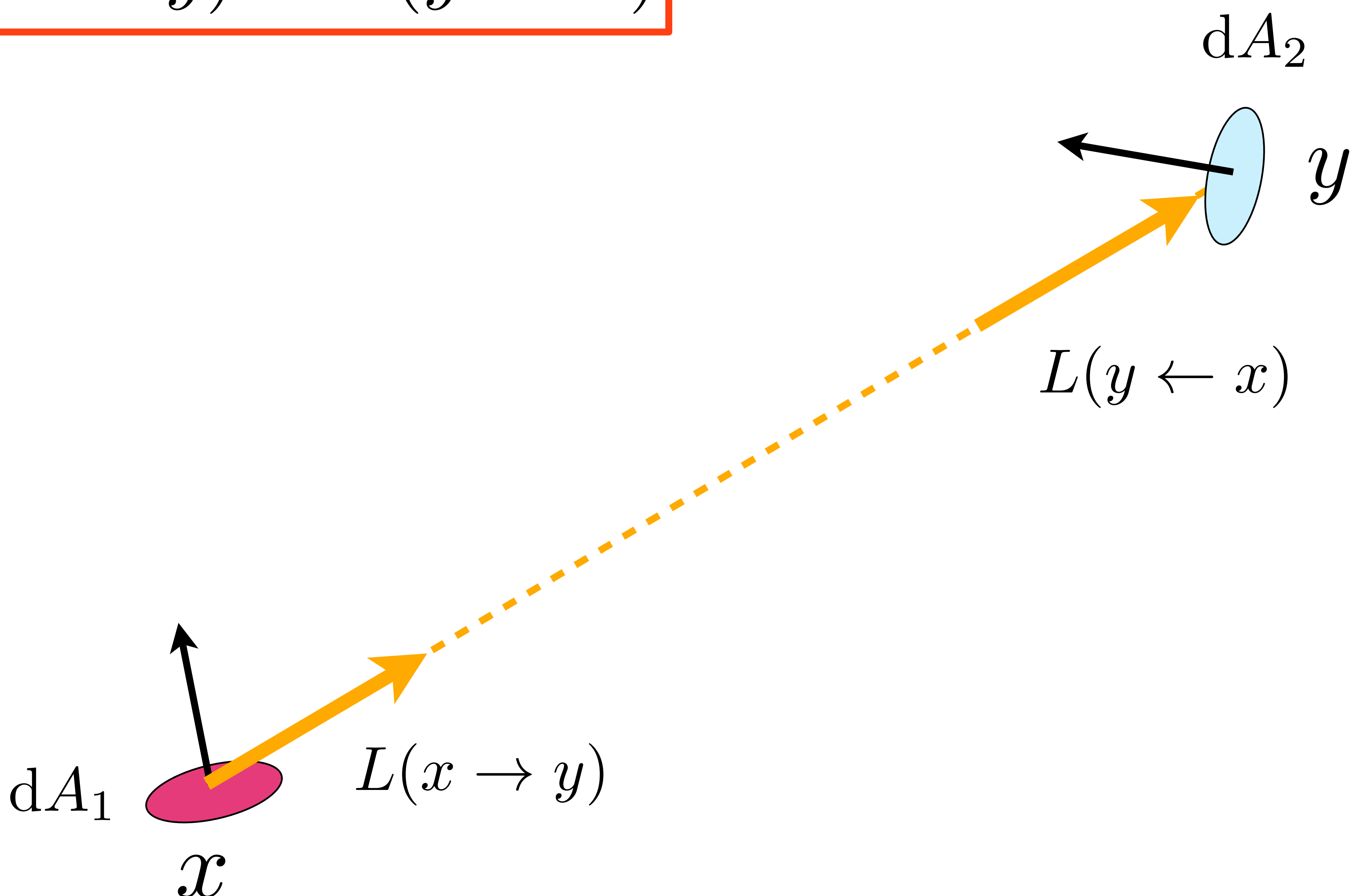
- "Intensity does not attenuate with distance"
  <=> **radiance stays constant along straight lines****

- **All relevant quantities (irradiance, etc.) can be derived from radiance**

**unless the medium is participating, e.g., smoke, fog

Constancy Along Straight Lines

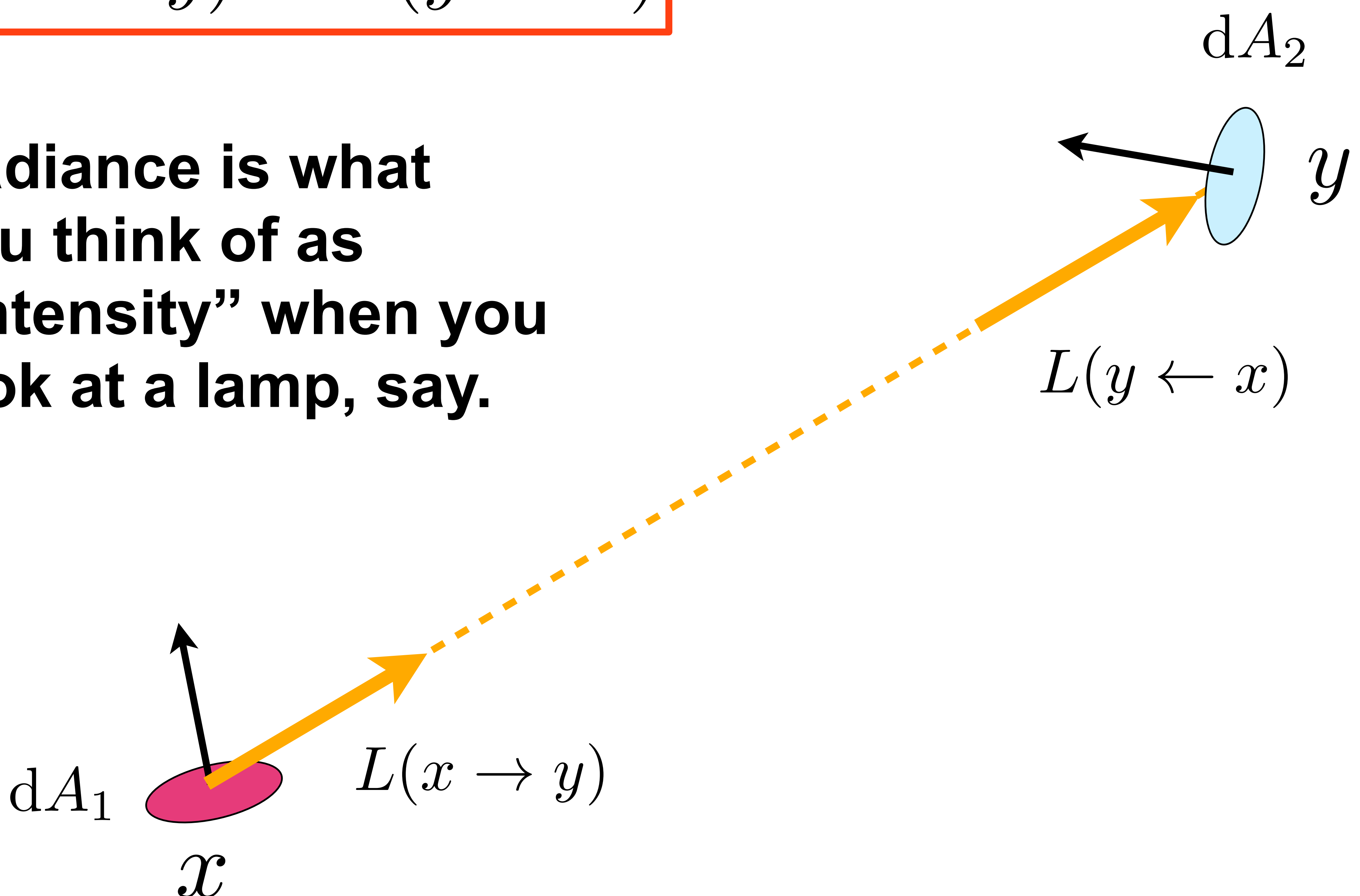
$$\boxed{L(x \to y) = L(y \leftarrow x)}$$



Constancy Along Straight Lines

$$\boxed{L(x \to y) = L(y \leftarrow x)}$$

Radiance is what you think of as "intensity" when you look at a lamp, say.



Recap: Radiance

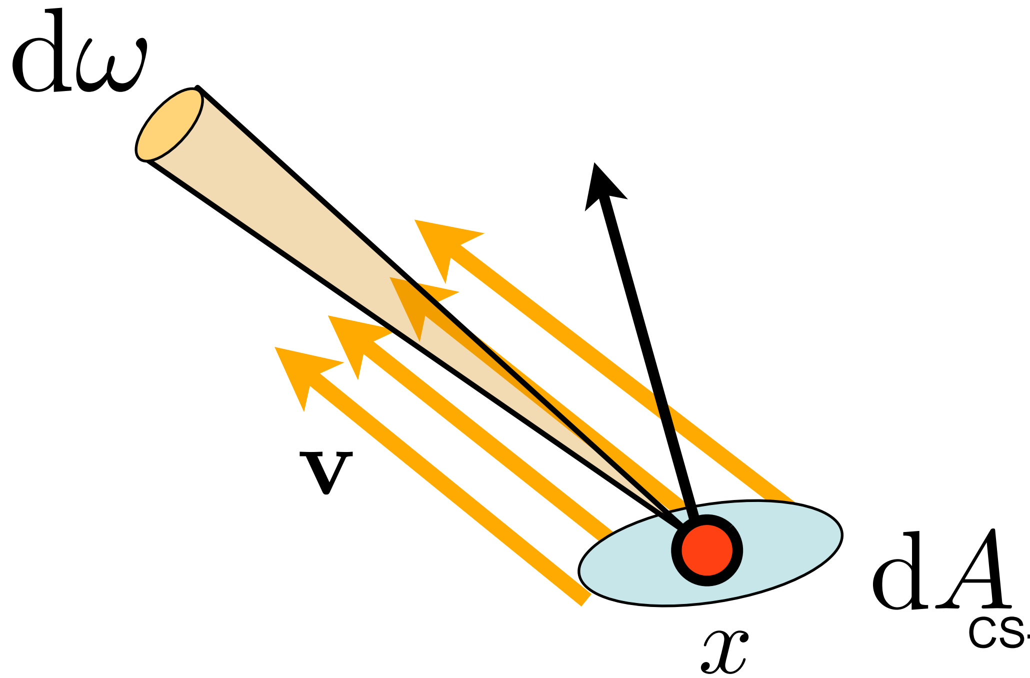
- Radiance L =
  flux per unit projected area per unit solid angle

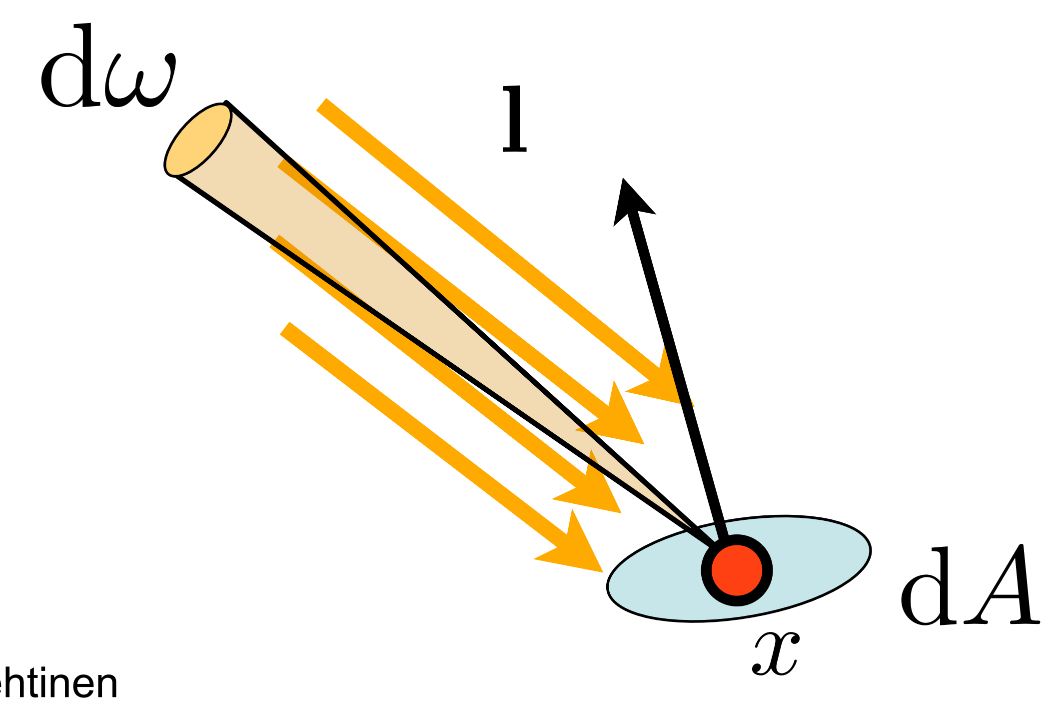
$$L = \frac{\mathrm{d}\Phi}{\mathrm{d}A^{\perp}\,\mathrm{d}\omega}$$

$$[L] = \left[\frac{W}{m^2\;sr}\right]$$

Recap: Radiance Notation

- $L(x \to \mathbf{v})$ denotes radiance leaving dA located at point x towards direction $\mathbf{v}$
  - Alternative notation: $L_{\text{out}}(x, \mathbf{v})$
- $L(x \leftarrow \mathbf{l})$ denotes radiance impinging on dA located at point x from direction $\mathbf{l}$
  - Alternative notation: $L_{\text{in}}(x, \mathbf{l})$





Recap: Irradiance

- Integrate incident radiance times cosine over the hemisphere Ω

$$\boxed{E = \int_{\Omega} L(\omega)\ \cos\theta\ \mathrm{d}\omega}$$

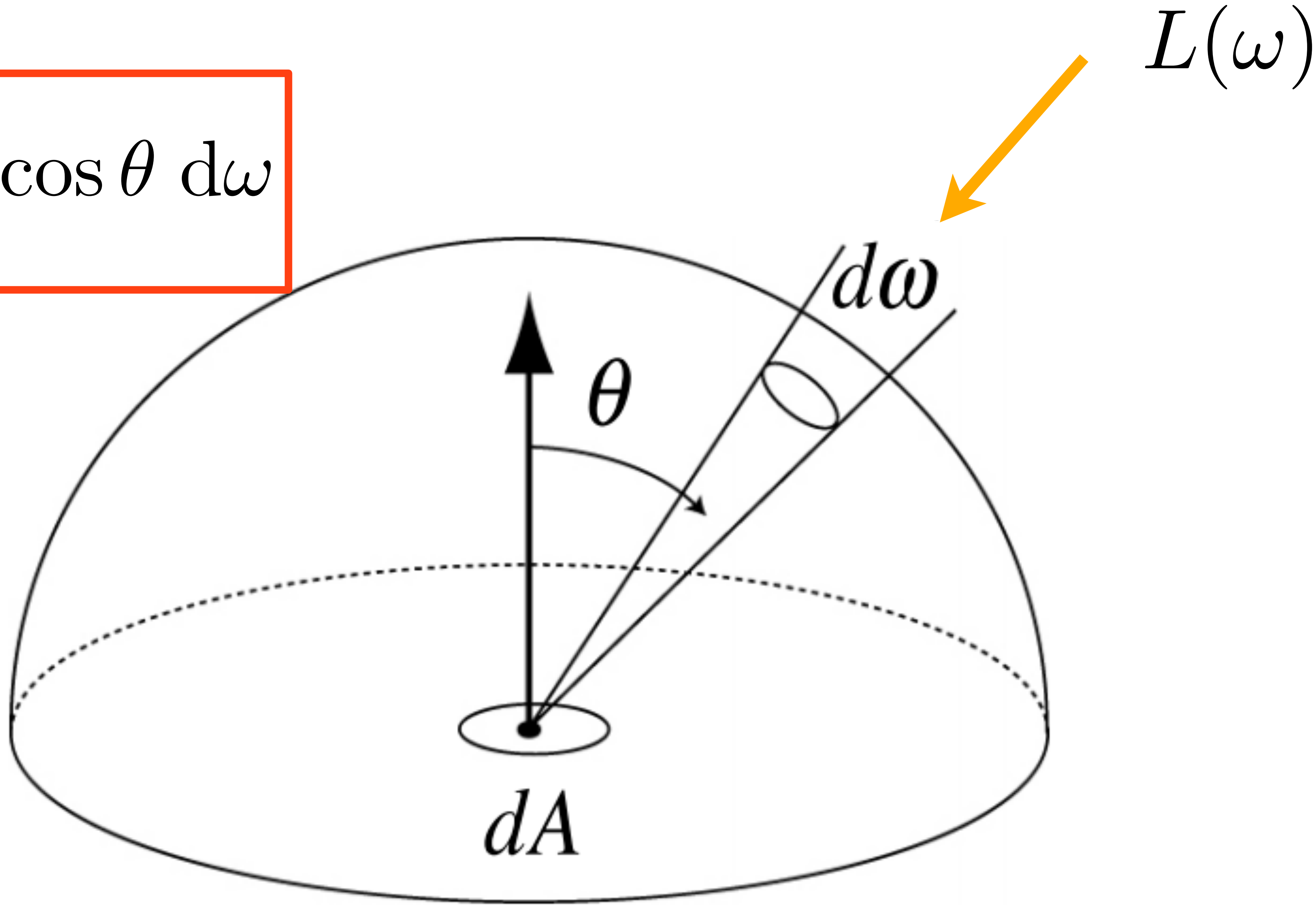
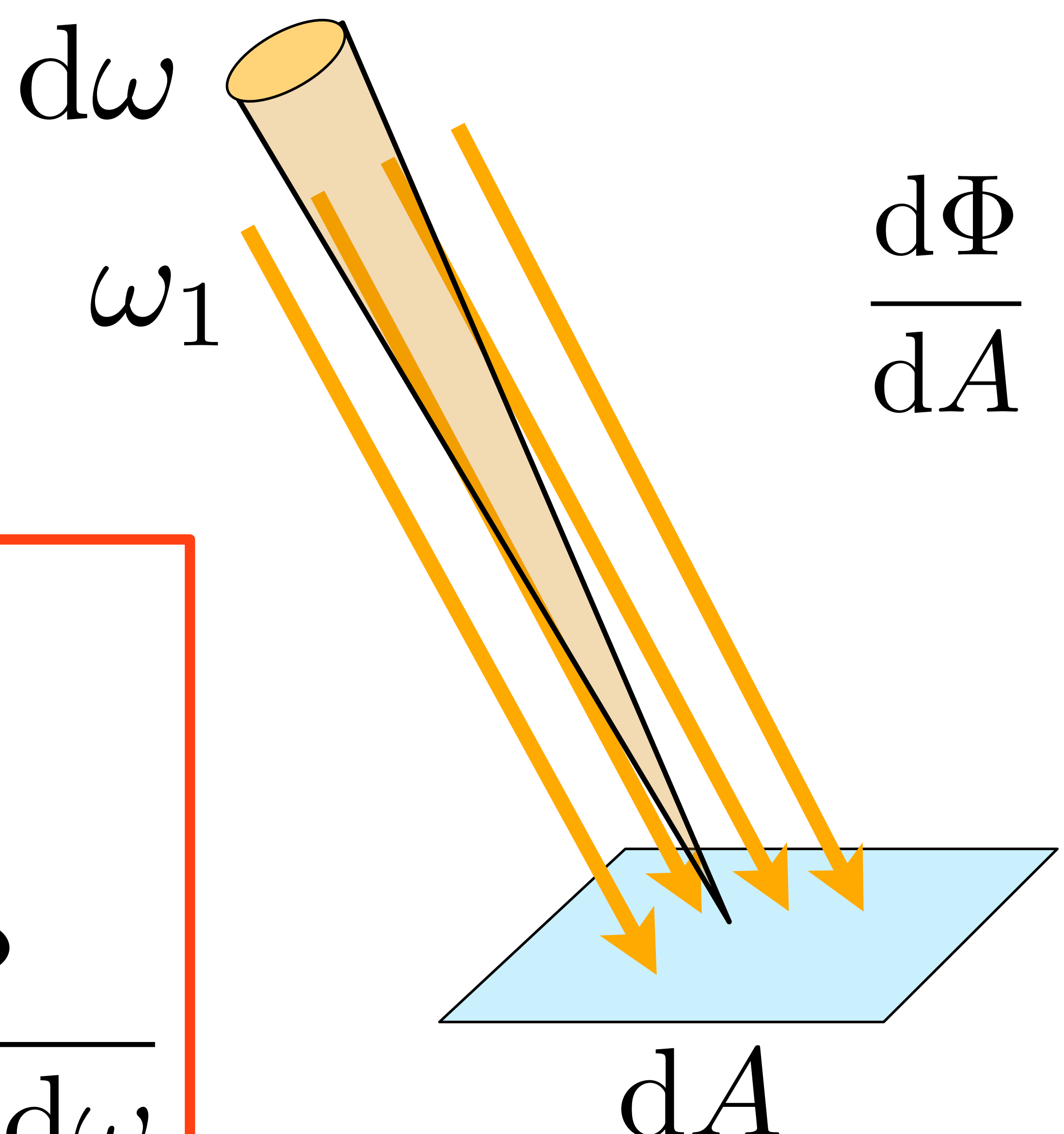


Figure adapted from Pat Hanrahan

Recap: Differential Irradiance

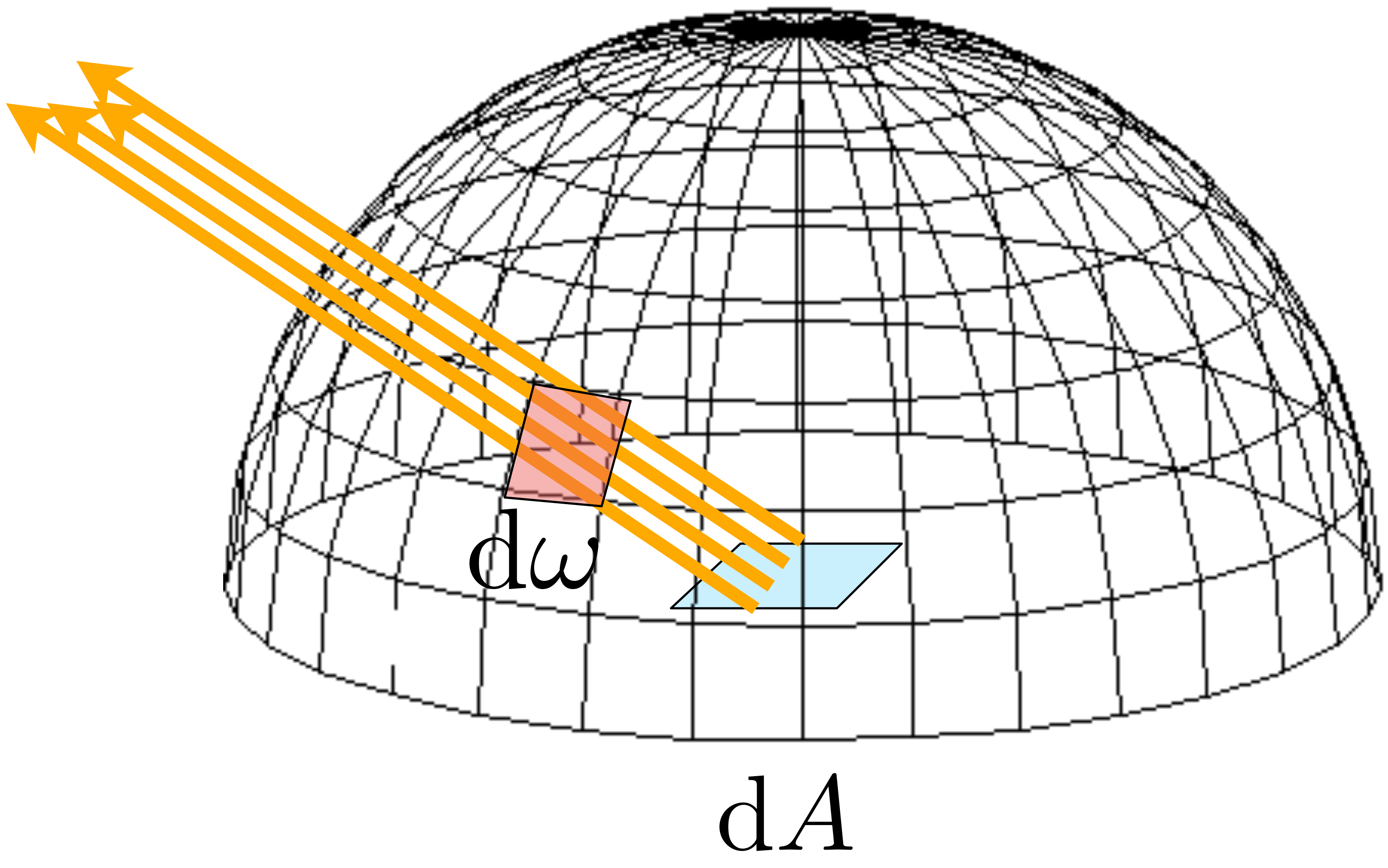
- To measure irradiance, add up the radiance from all the differential beams from all directions

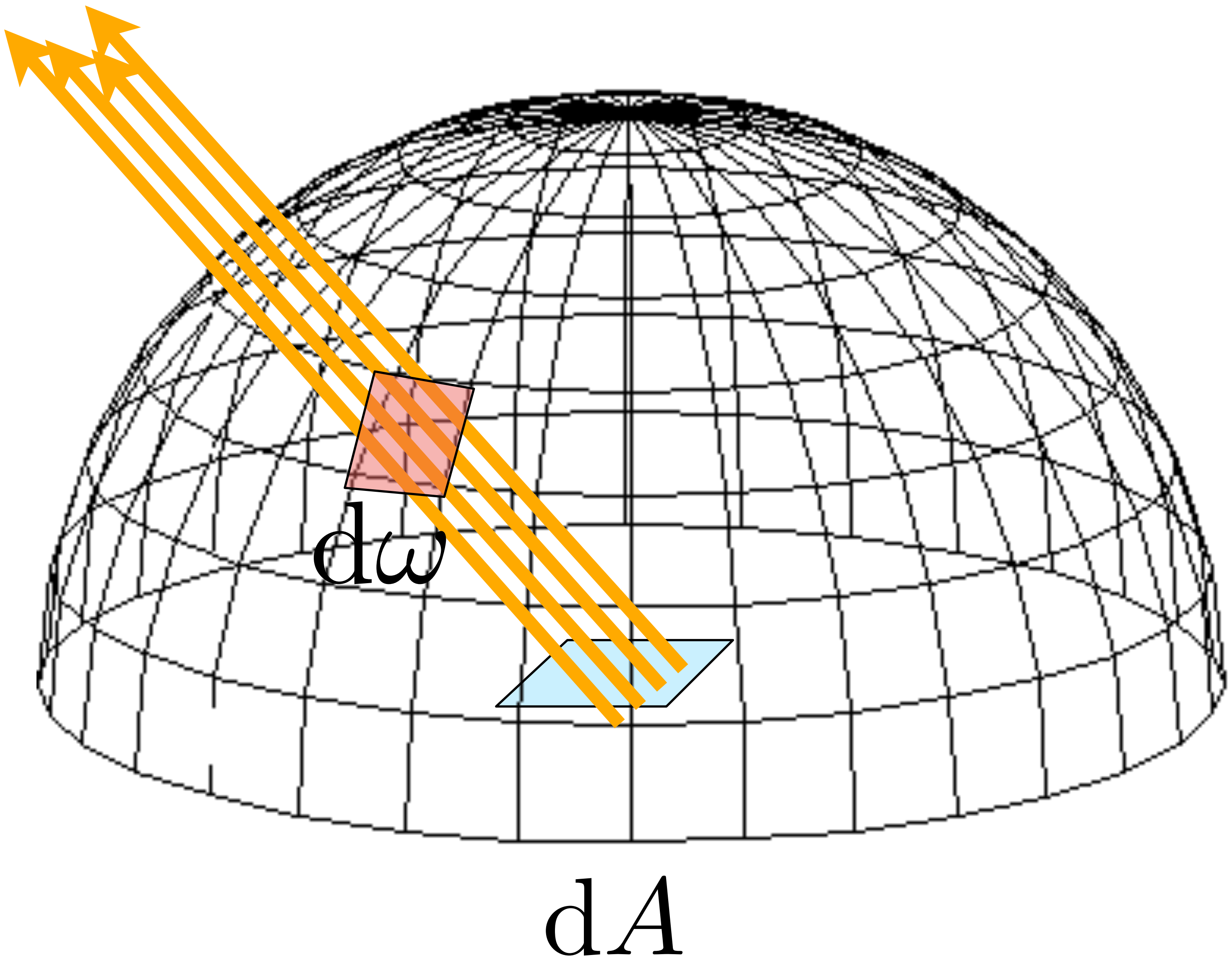


$$\frac{\mathrm{d}\Phi}{\mathrm{d}A} = \underbrace{L(\omega_1)\cos\theta}_{\text{Differential irradiance}}\,\mathrm{d}\omega$$

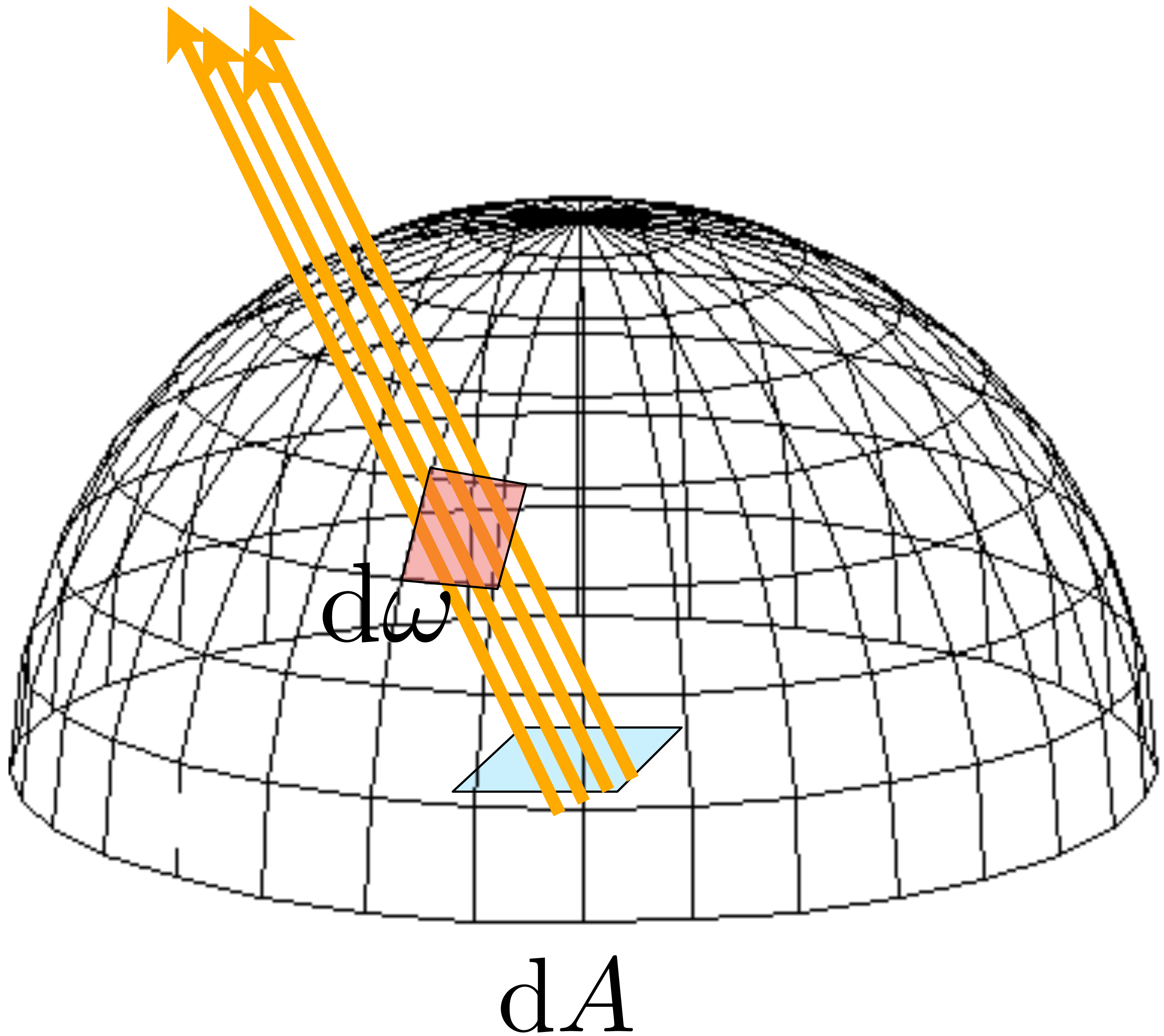
$$E = \frac{\mathrm{d}\Phi}{\mathrm{d}A}$$

$$L = \frac{\mathrm{d}\Phi}{\mathrm{d}A^{\perp}\,\mathrm{d}\omega}$$


$\mathrm{d}\omega$
$\mathrm{d}A$


$\mathrm{d}\omega$
$\mathrm{d}A$

- …



$\mathrm{d}\omega$
$\mathrm{d}A$

Recap: Irradiance to Radiosity

- The reflectivity of a diffuse surface is determined by its *albedo* $\rho \in [0, 1)$
  - This is the "diffuse color k_d" from your ray tracer in 4310
- The flux emitted by a diffuse surface per unit area is called *radiosity* B
  - Same units as irradiance, [B] = [W/m^2]
  - Hence

$$\boxed{B = \frac{\rho E}{\pi}}$$

Recap: Lambertian Soft Shadows

differential solid angle

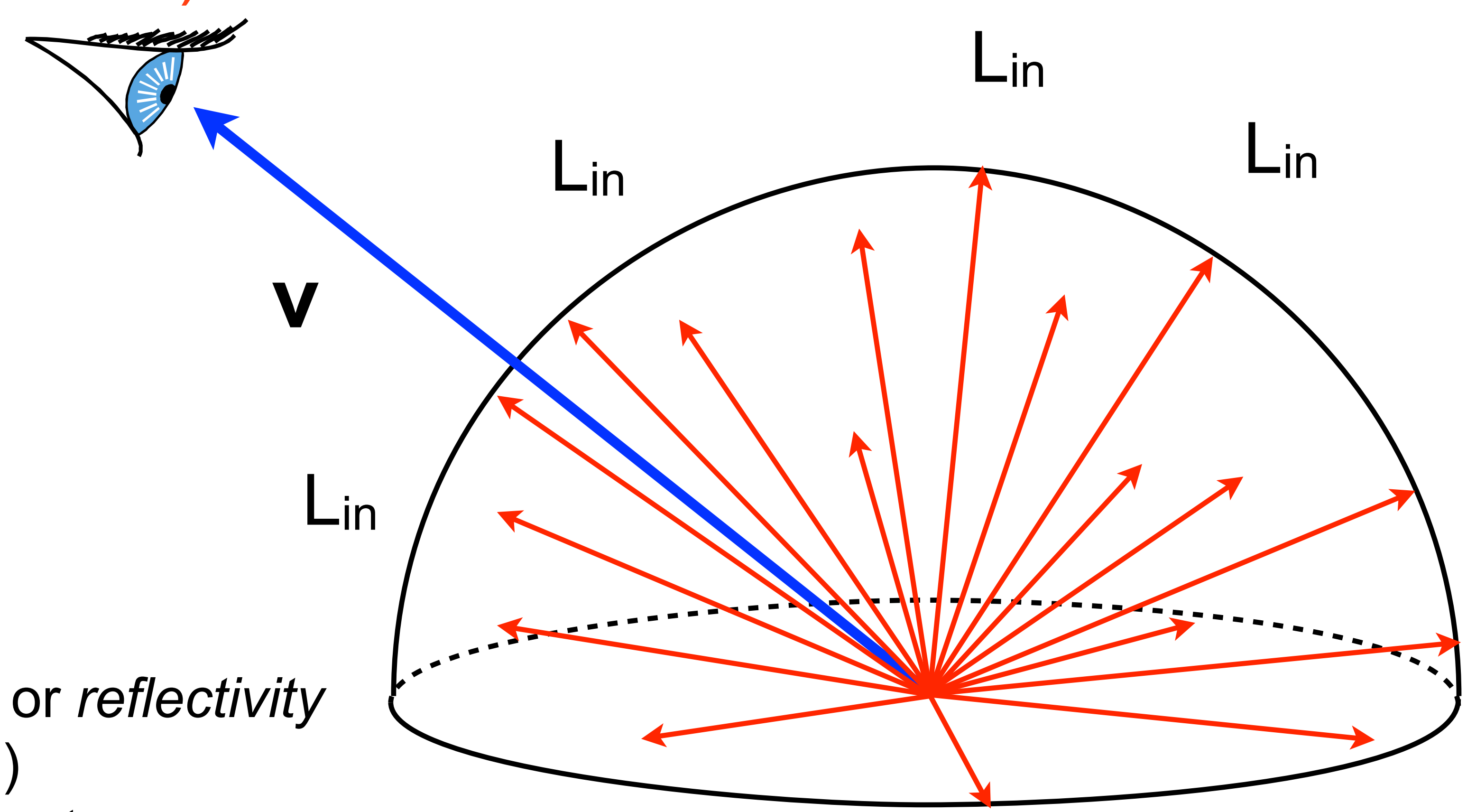
$$L_{\text{out}}(x) = \frac{\rho(x)}{\pi} \int_{\Omega} L_{\text{in}}(x, \omega)\, \cos\theta \, \mathrm{d}\omega$$

outgoing light (diffuse => independent of direction v)

albedo/pi

incident radiance

cosine term



$\rho(x)$

is the albedo or *reflectivity* (between 0,1) of the surface at x

Sum (integrate) over every direction on the hemisphere, modulate incident illumination by cosine, albedo/pi

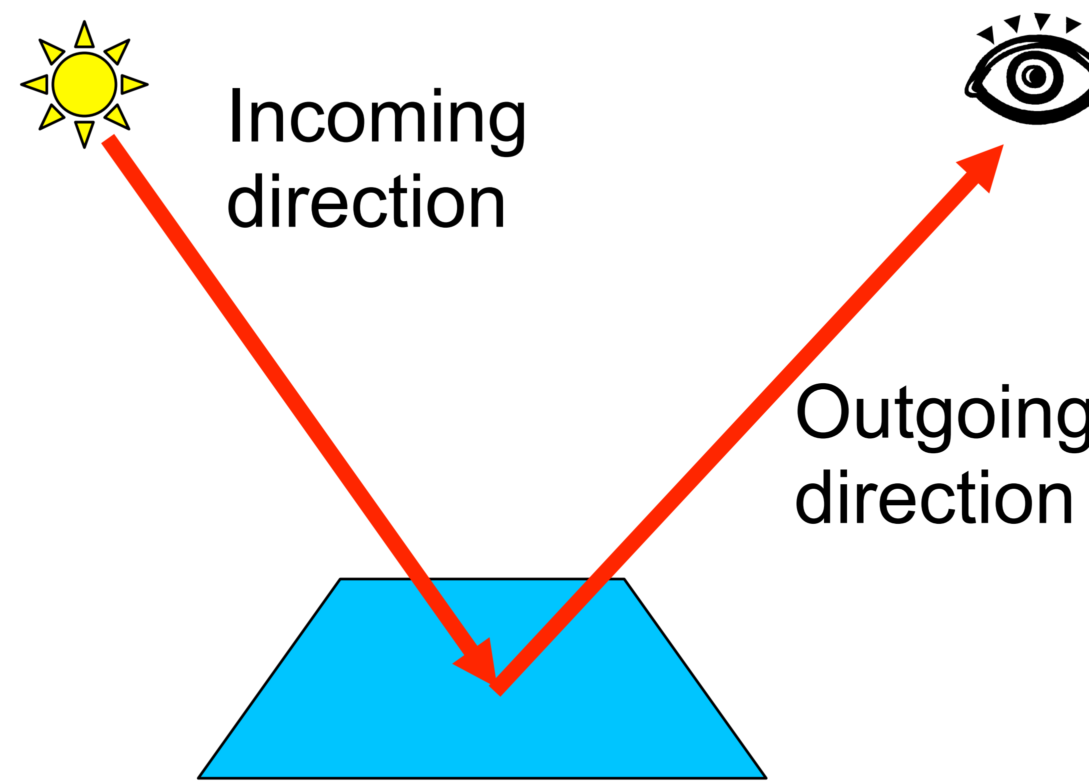
Last Time: Diffuse Reflectance Only

None of these surfaces are diffuse!

Luxology user dreamweaver

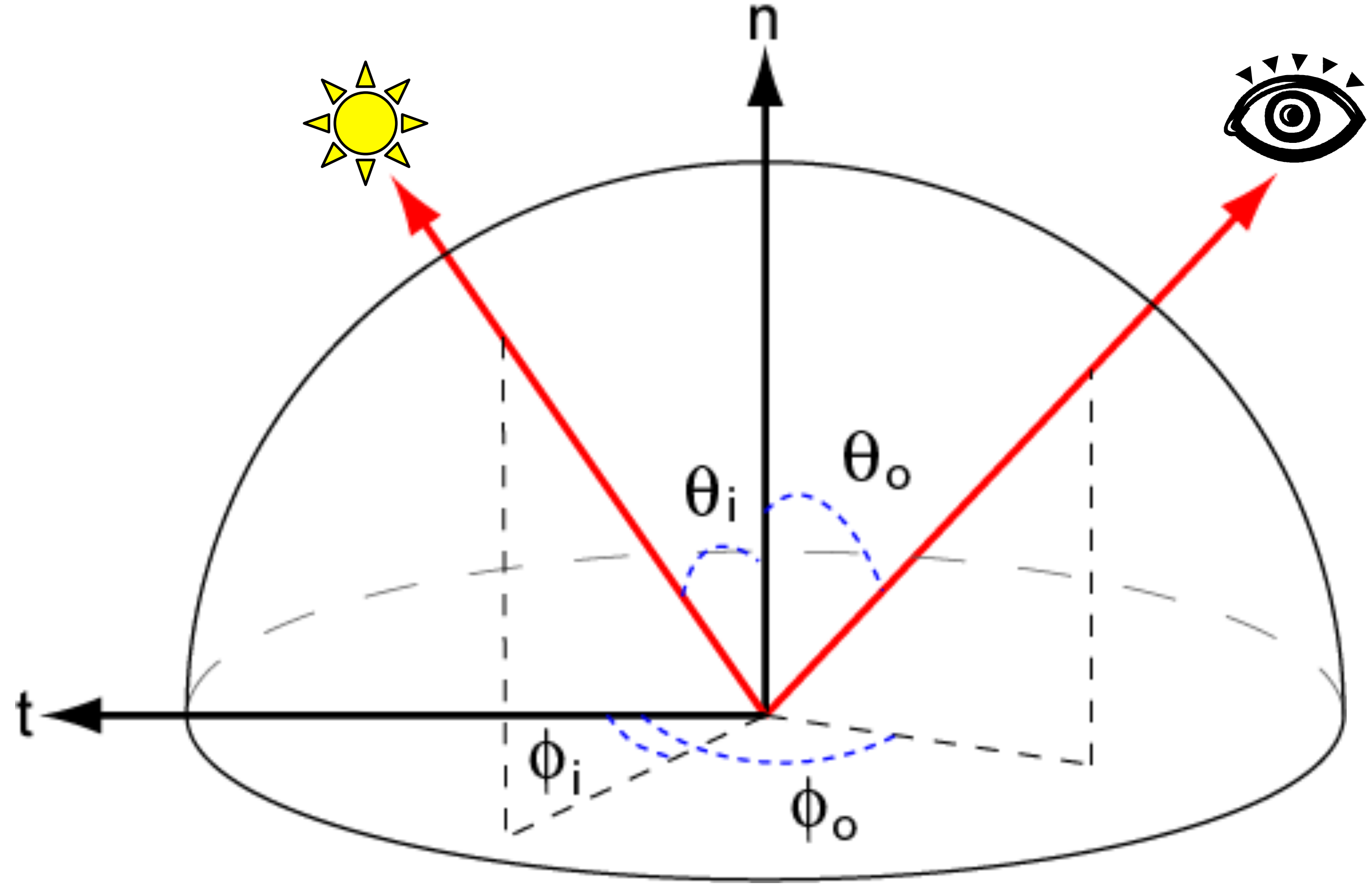
Quantifying Reflection – BRDF

- Bidirectional Reflectance Distribution Function
- "Ratio of light coming from one direction that gets reflected in another direction"
  - Pure reflection, assumes no light scatters into the material
- Focuses on angular aspects, not spatial variation of the material
- **How many dimensions?**



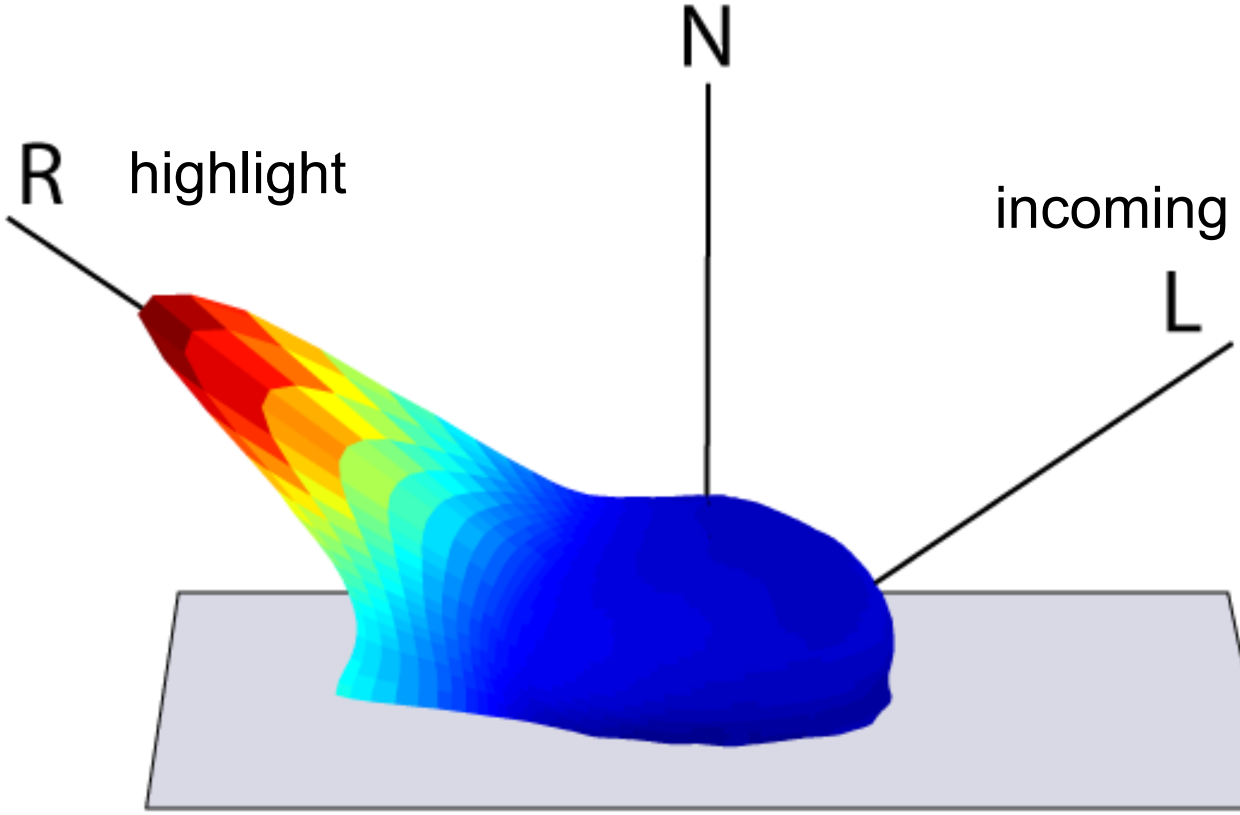
BRDF f_r

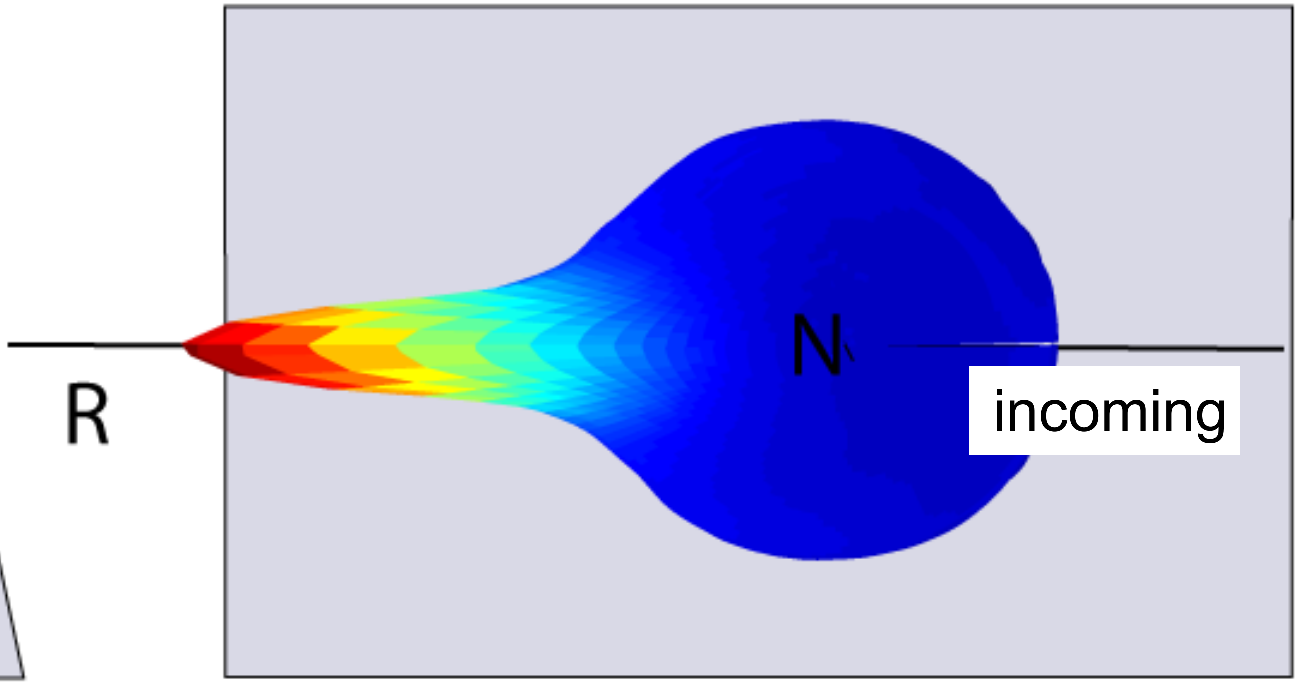
- Bidirectional Reflectance Distribution Function
  - 4D: 2 angles for each direction
  - BRDF = $f_r(\theta_i, \phi_i; \theta_o, \phi_o)$
  - Or just two unit vectors: BRDF = $f_r(\mathbf{l}, \mathbf{v})$
    - **l** = light direction
    - **v** = view direction



2D Slice at Constant Incidence

- For a fixed incoming direction **l**, view dependence is a 2D spherical function
  - Here a moderate glossy component towards mirror direction R



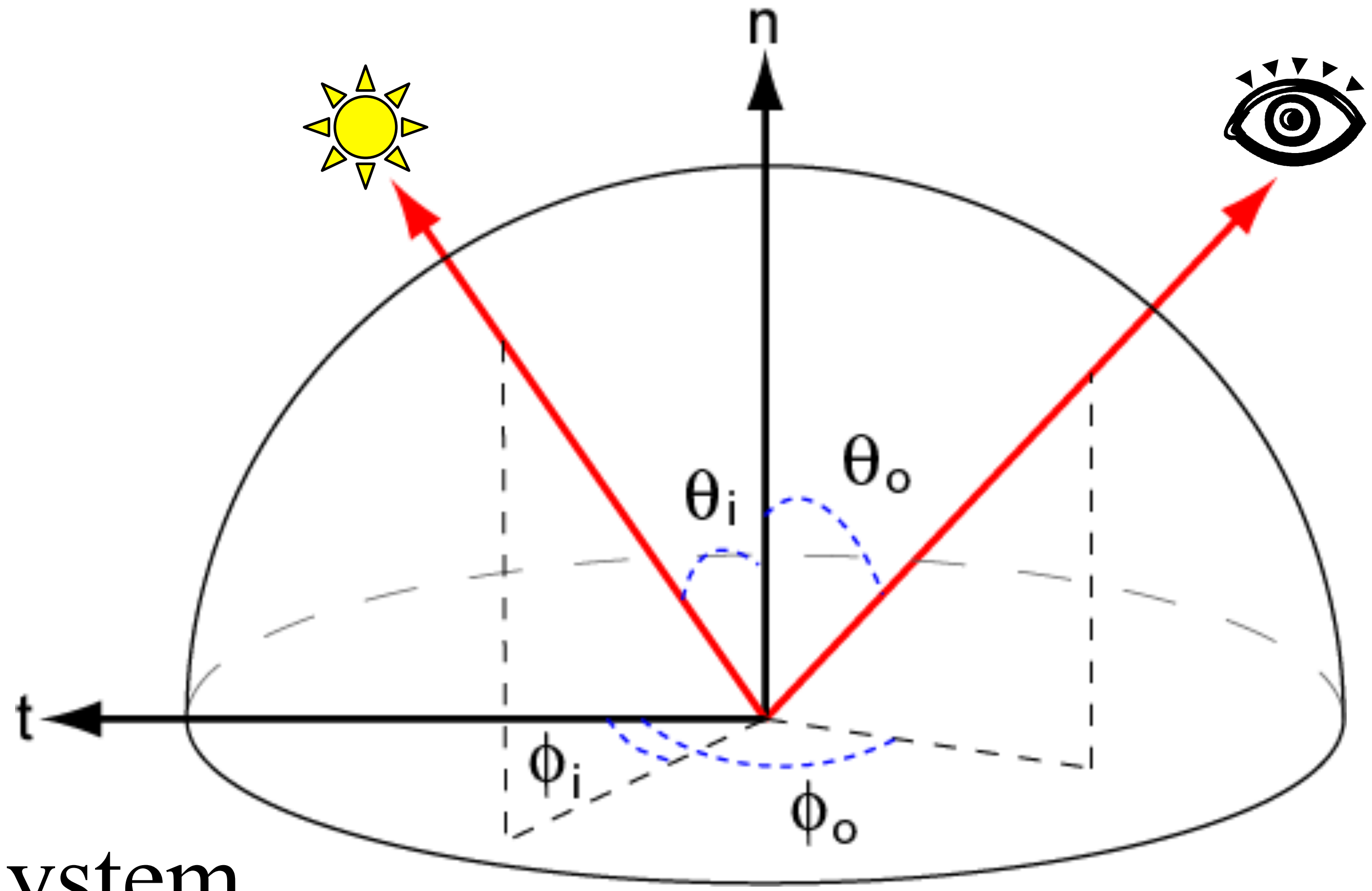


Example: Plot of “PVC” BRDF at 55° incidence

BRDF f_r

- Bidirectional Reflectance Distribution Function
  - 4D: 2 angles for each direction
  - BRDF = $f_r(\theta_i, \phi_i; \theta_o, \phi_o)$
  - Or just two unit vectors: BRDF = $f_r(\mathbf{l}, \mathbf{v})$
    - $\mathbf{l}$ = light direction
    - $\mathbf{v}$ = view direction
  - The BRDF is aligned with the surface; the vectors **l** and **v** must be in a local coordinate system

Mirror BRDF: Infinitely thin and tall spike ("Dirac delta") in mirror direction



BRDF Definition, For Real This Time

- Relates **incident differential irradiance** from every direction **to outgoing radiance**. How?

Reflectance Equation

$$L(x \to \mathbf{v}) =$$

outgoing radiance

$$\int_\Omega f_r(x, \mathbf{l} \to \mathbf{v}) \boxed{L(x \leftarrow \mathbf{l}) \cos\theta} \mathrm{d}\mathbf{l}$$

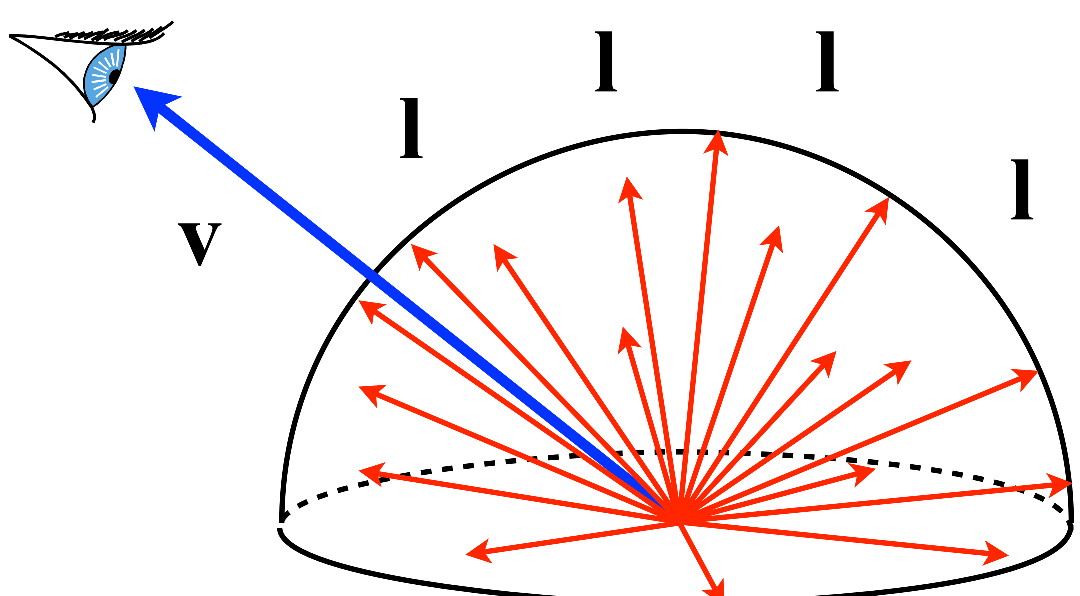
L_in * cos = incident differential irradiance

integral over hemisphere

BRDF

incoming radiance

cosine of incident angle



Compare to Diffuse Case

$$L(x \to \mathbf{v}) =$$

$$\int_\Omega \boxed{f_r(x, \mathbf{l} \to \mathbf{v})} \boxed{L(x \leftarrow \mathbf{l}) \ \cos\theta} \mathrm{d}\mathbf{l}$$

L_in * cos = incident differential irradiance

BRDF

$$L_{\text{out}}(x) = \boxed{\frac{\rho(x)}{\pi}} \int_\Omega \boxed{L_{\text{in}}(x, \omega) \ \cos\theta} \mathrm{d}\omega$$

Diffuse BRDF

$$L_{\mathrm{out}}(x) = \frac{\rho(x)}{\pi} \int_{\Omega} L_{\mathrm{in}}(x, \omega)\, \cos\theta \, \mathrm{d}\omega$$

- Diffuse reflectance independent of outgoing angle
- Hence, the diffuse BRDF is

$$f_r(x) = \frac{\rho}{\pi}$$

  – (ρ is the albedo, remember)

- Note: no cosine, it's included in the reflectance eq.!

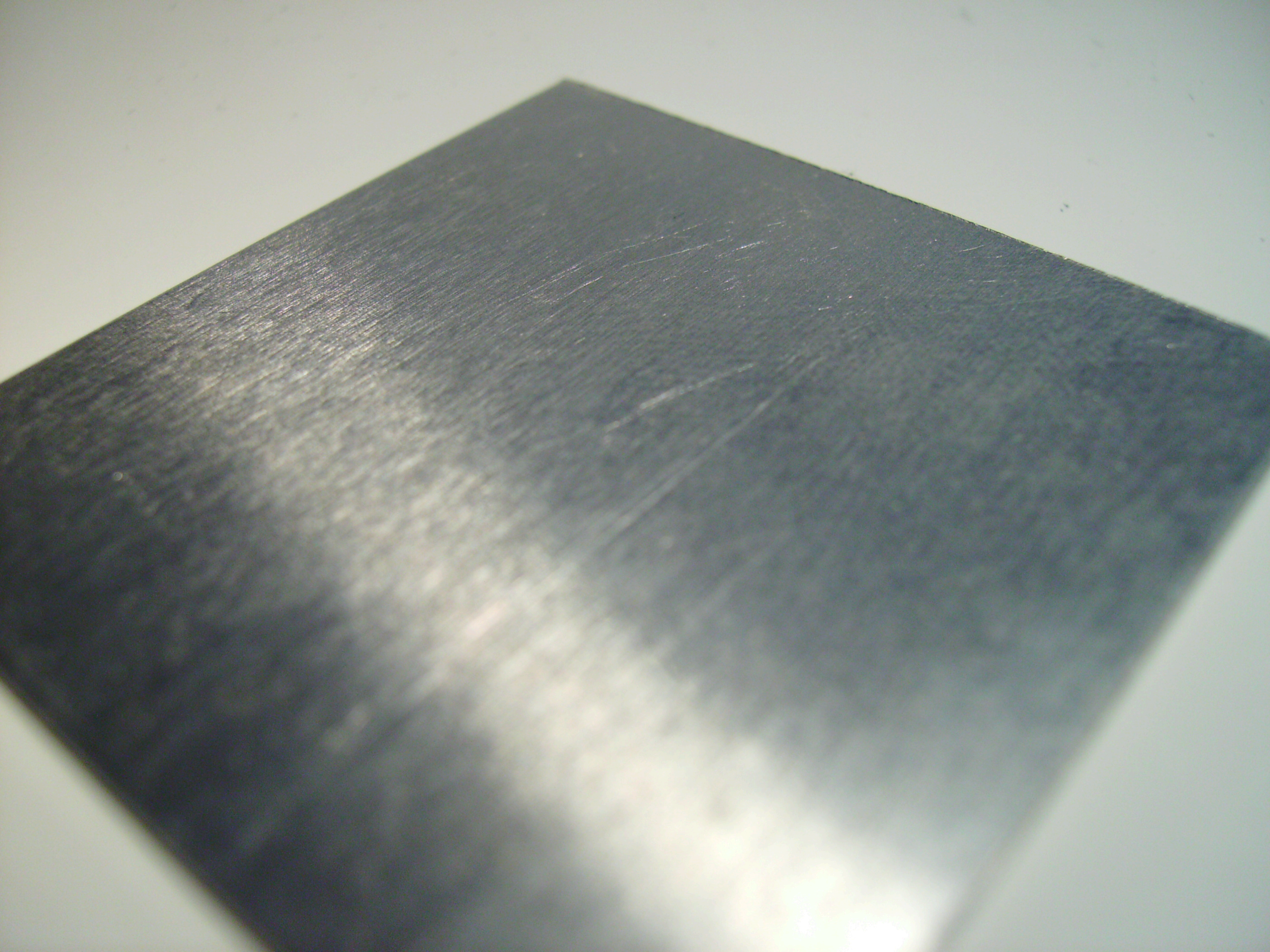
BRDF Properties

- Reciprocity: $f_r(\mathbf{l} \rightarrow \mathbf{v}) = f_r(\mathbf{v} \rightarrow \mathbf{l})$
- Energy conservation: $\int f_r(\mathbf{l} \rightarrow \mathbf{v}) \cos\theta_v \, \mathrm{d}\mathbf{v} \leq 1$
  - **Intuitive**: the BRDF tells you how a single beam of incident illumination from direction **l** is spread into all reflected directions **v**; you can't have more energy coming out than going in.
  - But also, due to reciprocity, the same must hold if you swap the incident and outgoing directions.
- Non-negativity: $f_r(\mathbf{l} \rightarrow \mathbf{v}) \geq 0$

Isotropic vs. Anisotropic

- When keeping **l** and **v** fixed, if rotation of surface around the normal doesn't change the reflection, the material is called isotropic
- Surfaces with strongly oriented microgeometry elements are anisotropic
- Examples:
  - brushed metals,
  - hair, fur, cloth, velvet

Westin et.al 92



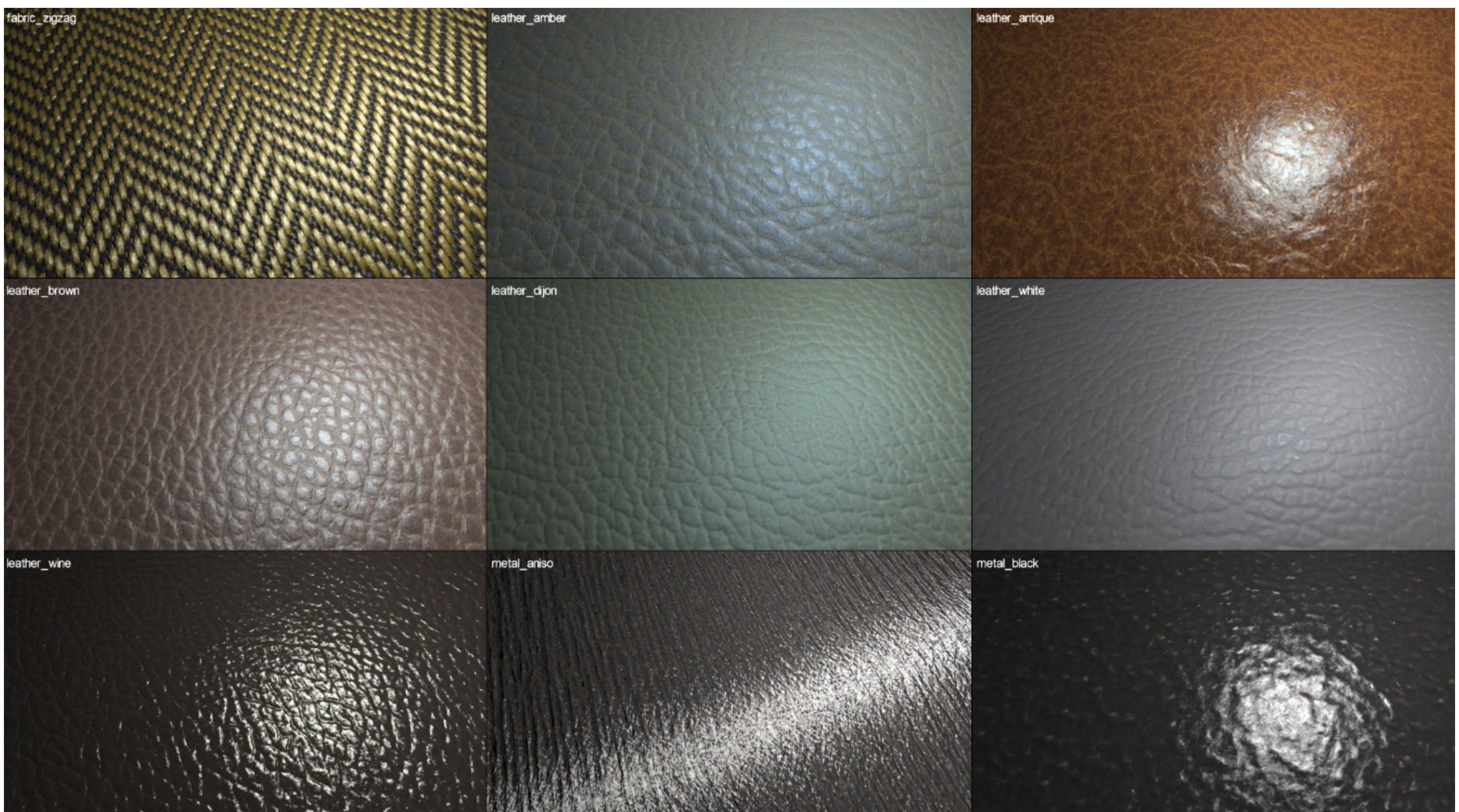
Hmmh

- The BRDF is a 4D function for a single surface point
- When you make it vary over surfaces, you add two more dimensions
  - The Spatially Varying BRDF (SVBRDF) is 6D!

Spatially Varying Reflectance

- Very, very, VERY important for realistic surface appearance
- VIDEO

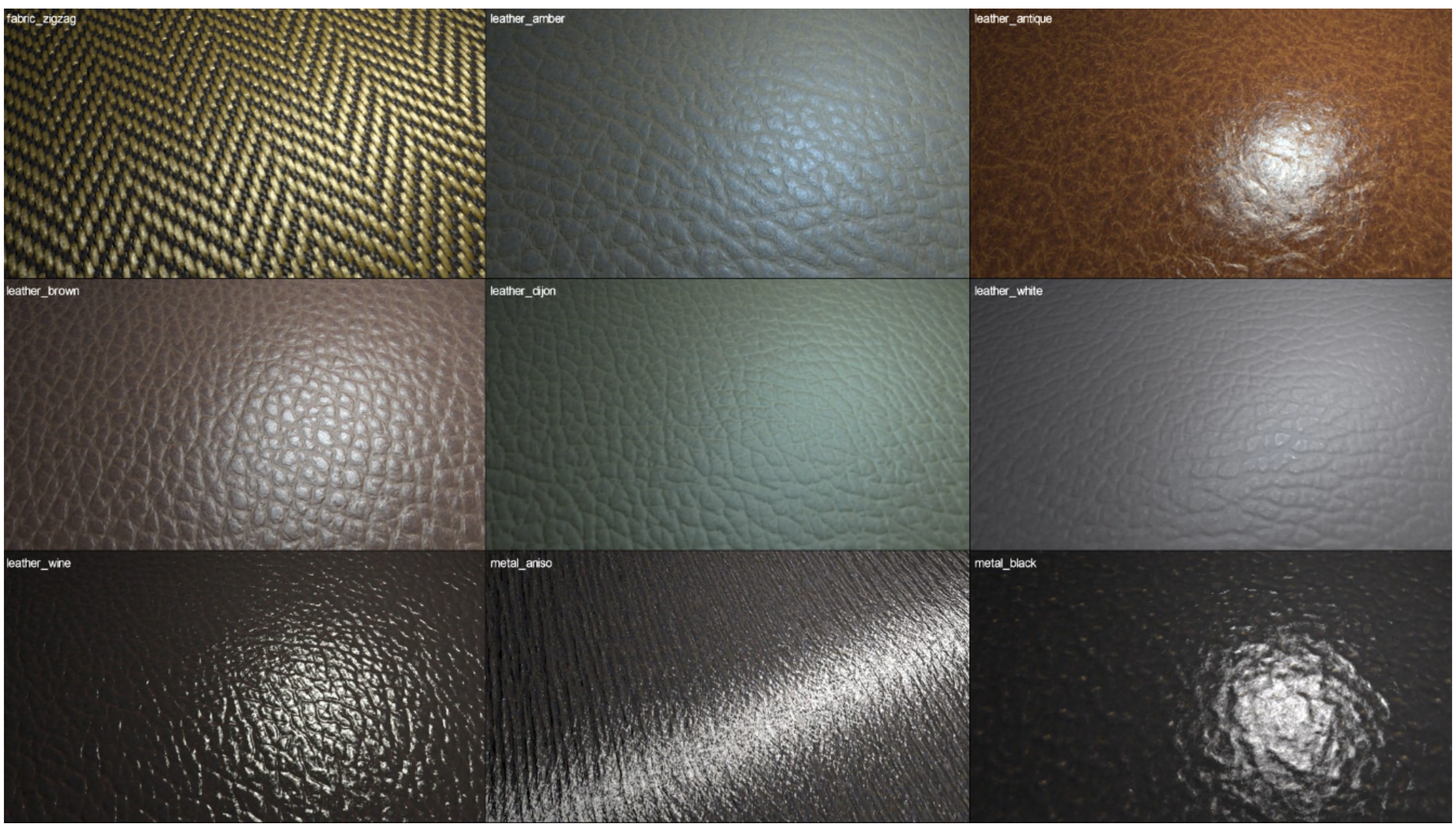
Aittala, Weyrich, Lehtinen 2015



Spatially Varying Reflectance

- You can find these SVBRDF material models online and use them in your assignments!

Aittala, Weyrich, Lehtinen 2015



Parametric BRDF Models

- BRDFs can be measured from real data
  - But storage and computation using arbitrary 4D or 6D functions is unwieldy, must do something smarter

Parametric BRDF Models

- BRDFs can be measured from real data
  - But storage and computation using arbitrary 4D or 6D functions is unwieldy, must do something smarter
- **Solution: parametric models**
  - What this means: use a small set of (hopefully intuitive) parameters that determine reflectance at each point
- We've seen one model already: diffuse reflectance determined by one parameter, the albedo
  - Well, 3 actually (RGB)

Parametric BRDF Models

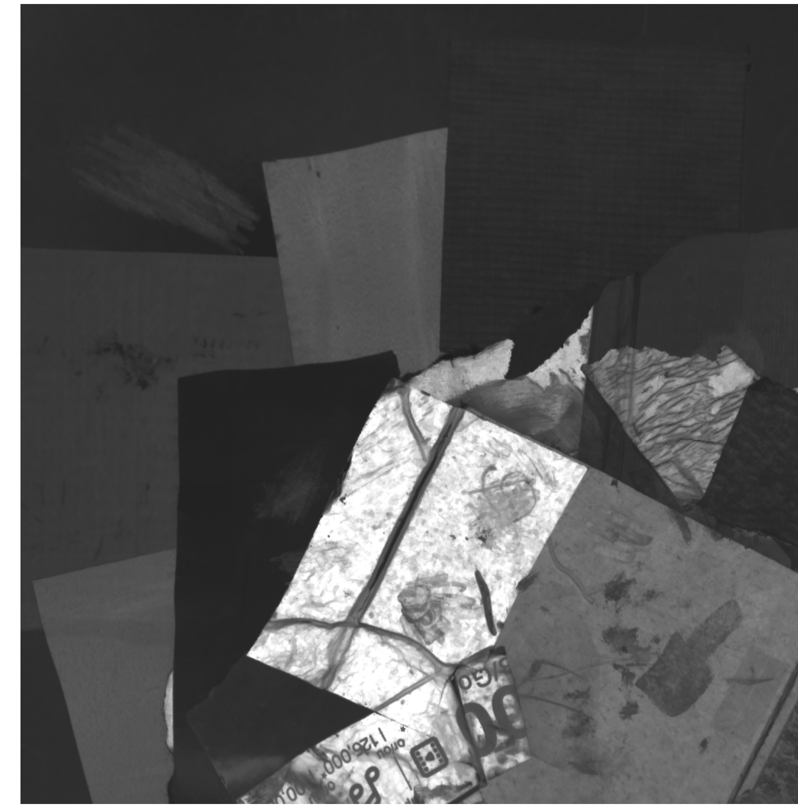
- Parametric BRDF models represent the relationship between incident and outgoing light by some mathematical formula with tunable parameters
  - The appearance can then be tuned by setting parameters
    - "Color", "Shininess", "anisotropy", etc.
  - Many ways of coming up with these
  - Can models with measured data (examples later)

- Popular models: Diffuse, Blinn-Phong, Cook-Torrance, Lafortune, Ward, Oren-Nayar, etc.

Parametric SVBRDF Example

Diffuse albedo (color)

Specular albedo (color)



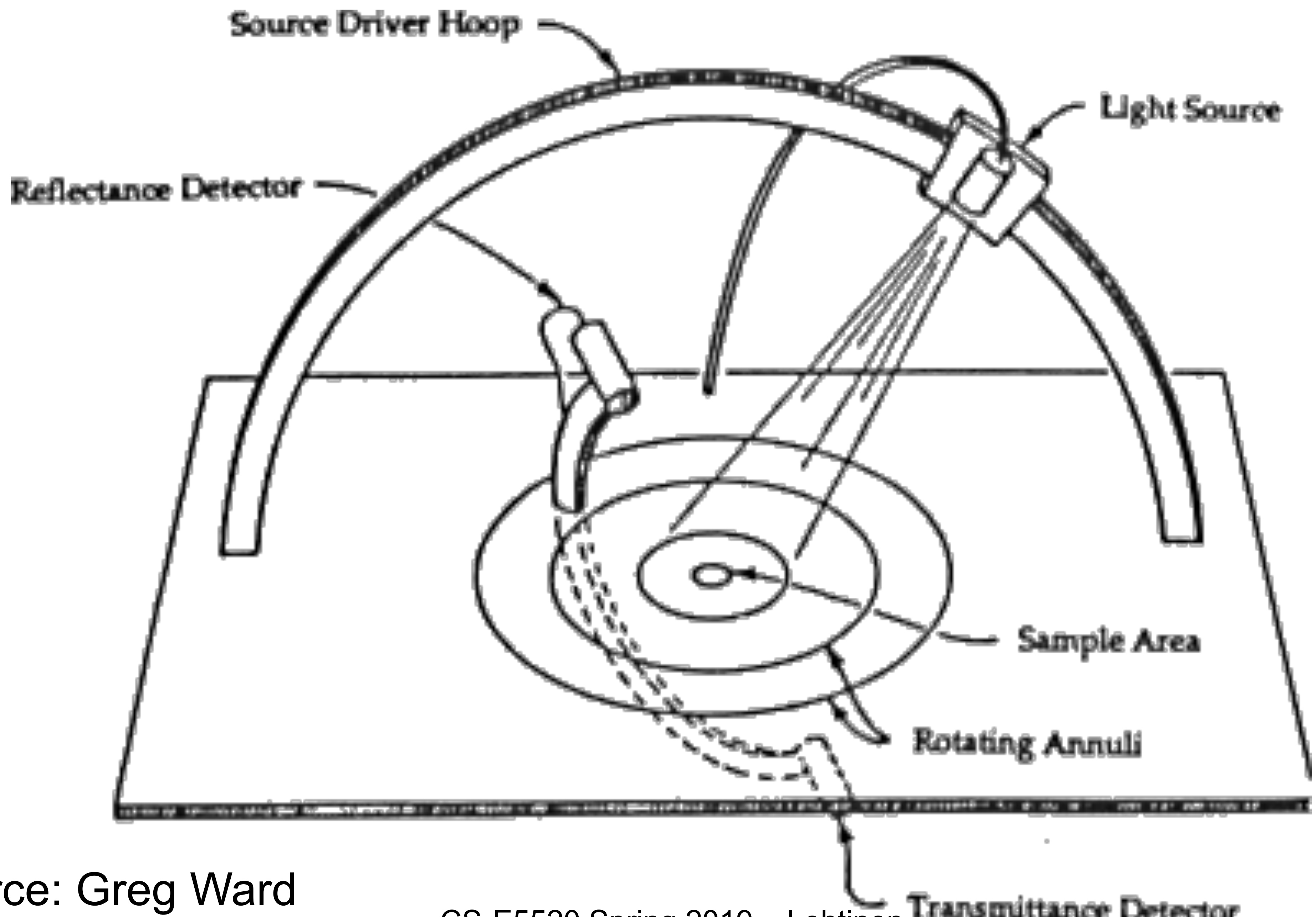
Glossiness

These are just parameters to a Fresnel-modulated Blinn-Phong model!

Surface normal

How do we obtain BRDFs?

- One possibility: Gonioreflectometer
  - 4 degrees of freedom



Source: Greg Ward

How do we obtain BRDFs?

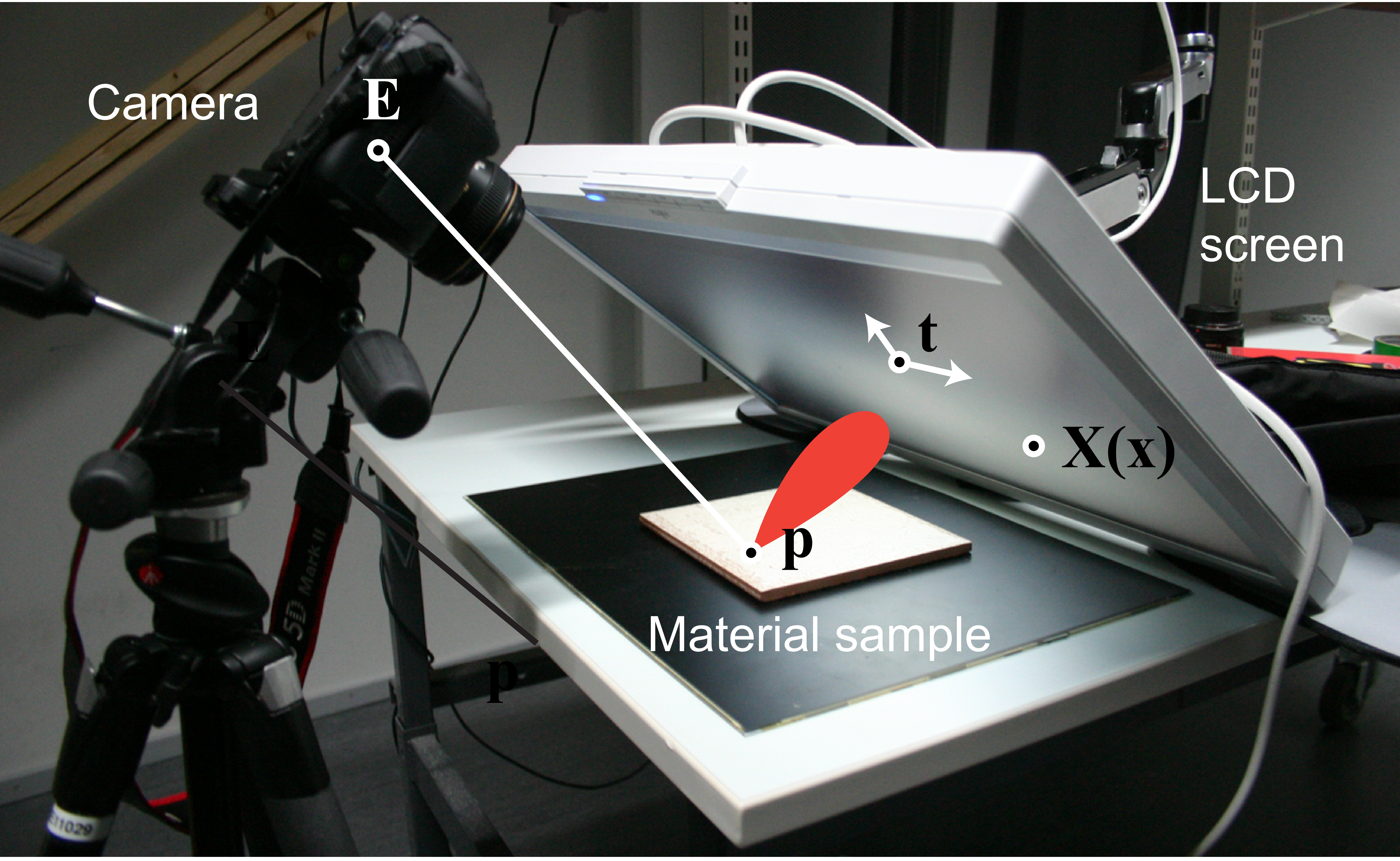
Matusik et al.

Image-Based Acquisition

- See W. Matusik et al. for how
  - A Data-Driven Reflectance Model, SIGGRAPH 2003
  - The data is available from MERL

State of The Art

Aittala, Weyrich, Lehtinen, *Practical SVBRDF Capture in the Frequency Domain*, SIGGRAPH 2013



Even less effort…

with some restrictions on what materials can be captured

- SIGGRAPH 2015, http://tinyurl.com/TwoShotSVBRDF

Two-Shot SVBRDF Capture for Stationary Materials

Miika Aittala — Aalto University

Tim Weyrich — University College London

Jaakko Lehtinen — Aalto University, NVIDIA

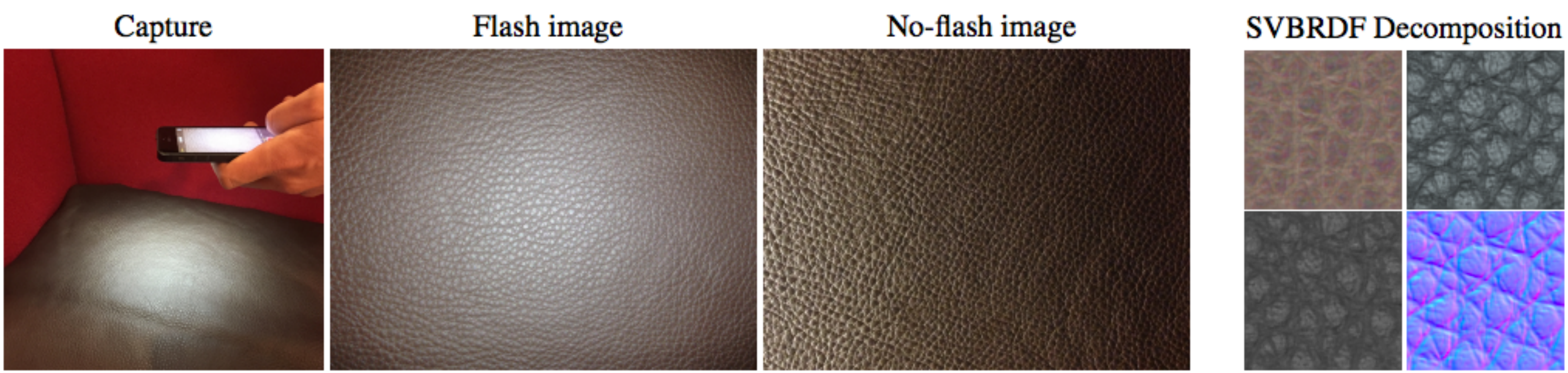


Figure 1: *Given an flash-no-flash image pair of a "textured" material sample, our system produces a set of spatially varying BRDF parameters (an SVBRDF, right) that can be used for relighting the surface. The capture (left) happens in-situ using a mobile phone.*

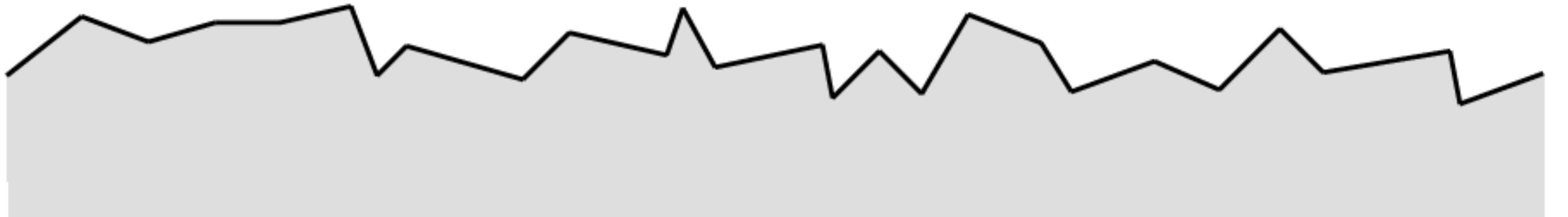
Questions?

Microfacet Theory

- Example
  - Think of water surface as lots of tiny mirrors (microfacets)
  - “Bright” pixels are
    - Microfacets aligned with the vector between sun and eye
    - But not the ones in shadow
    - And not the ones that are occluded

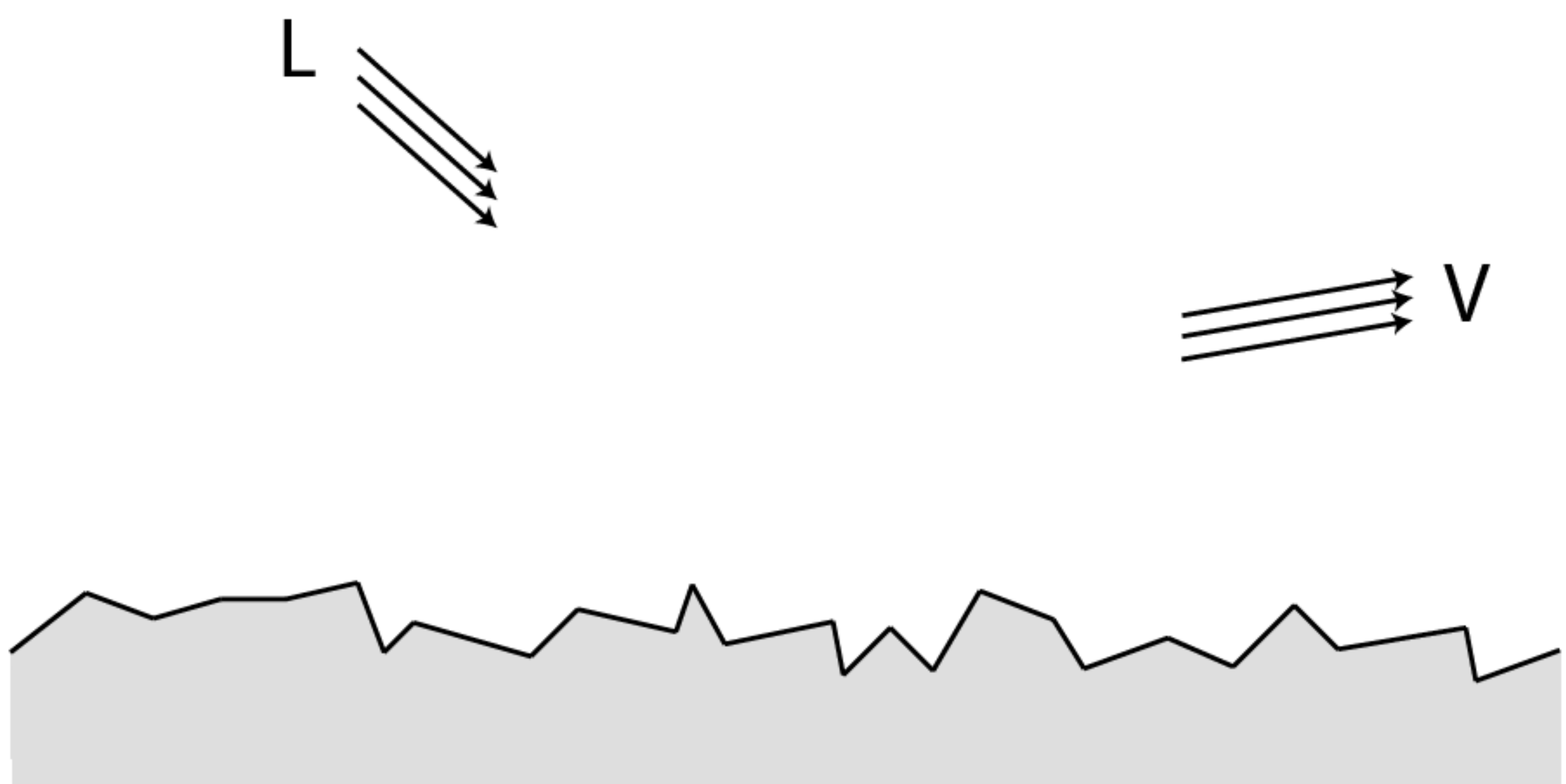
Microfacet Theory

- Model surface by tiny mirrors
  [Torrance & Sparrow 1967]



Microfacet Theory

- Value of BRDF at (L,V) is a product of
  - number of mirrors oriented halfway between L and V



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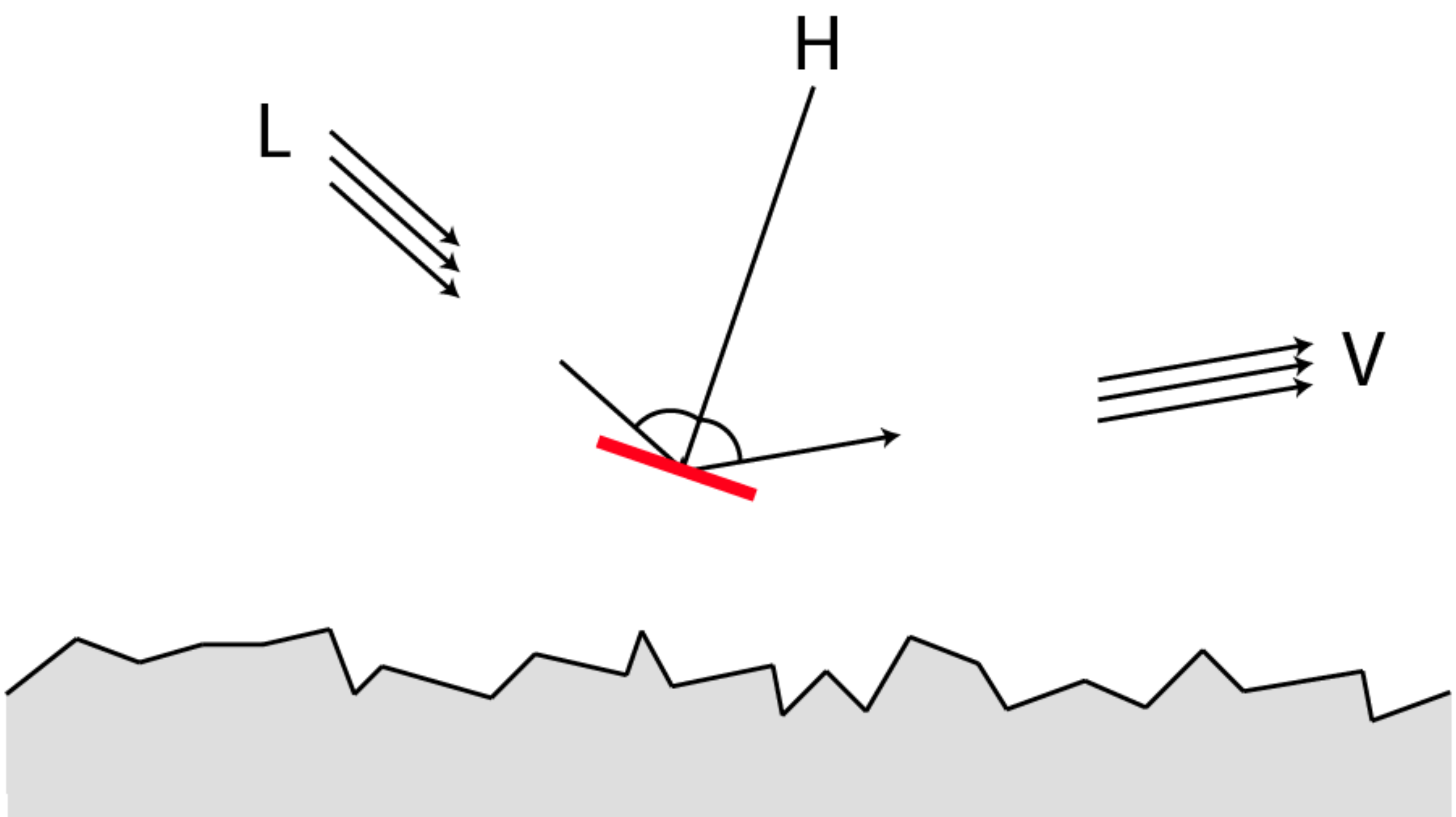

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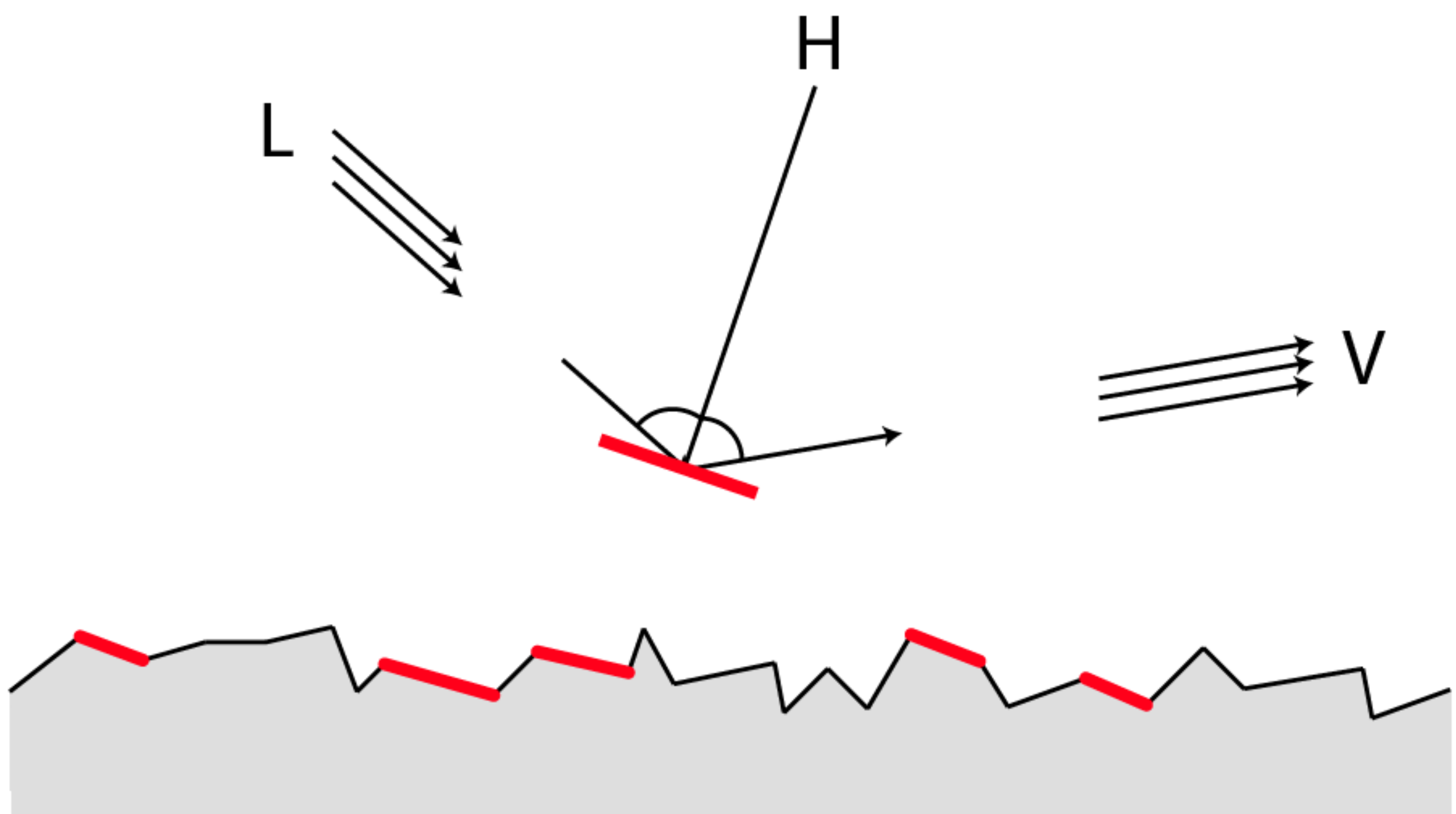
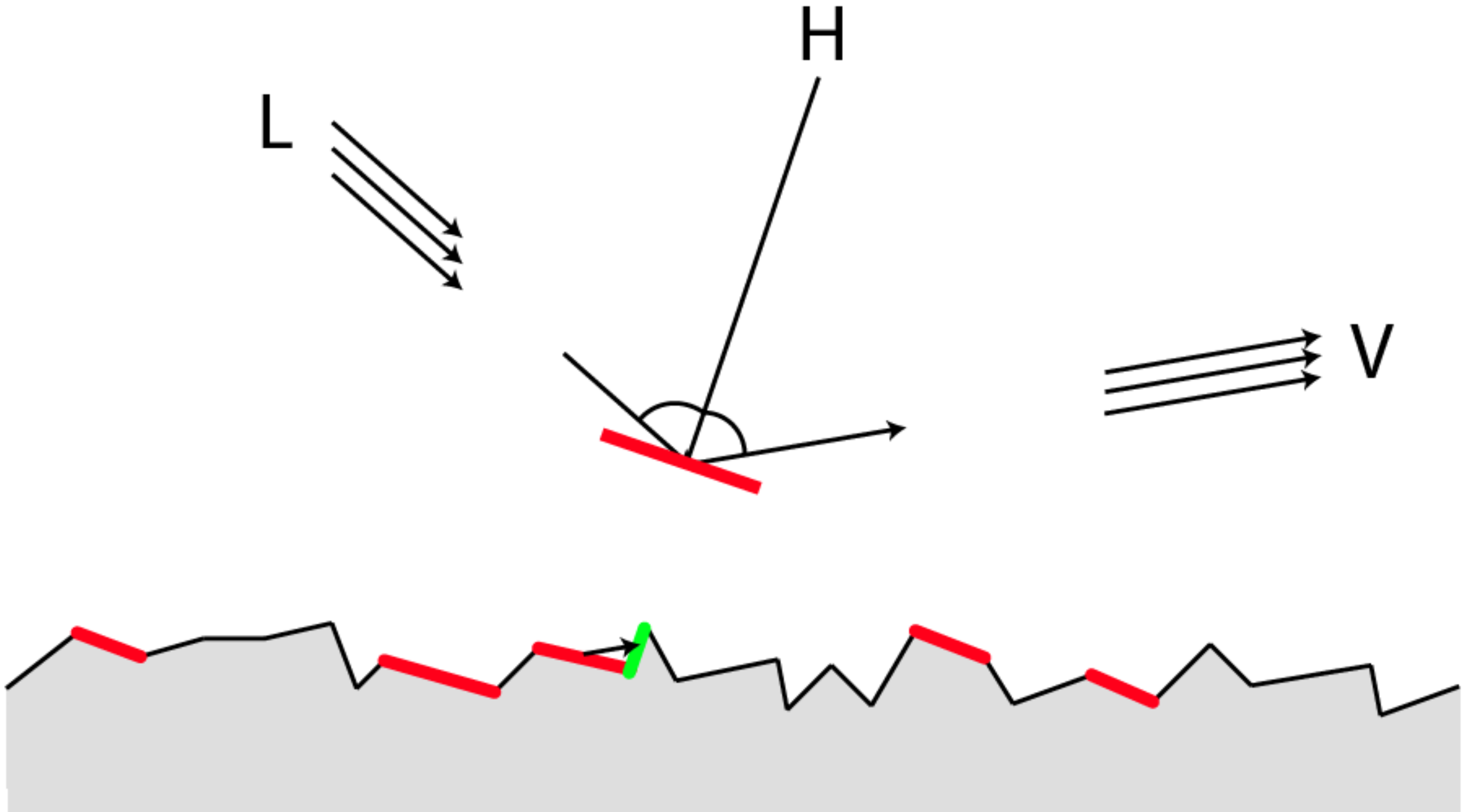


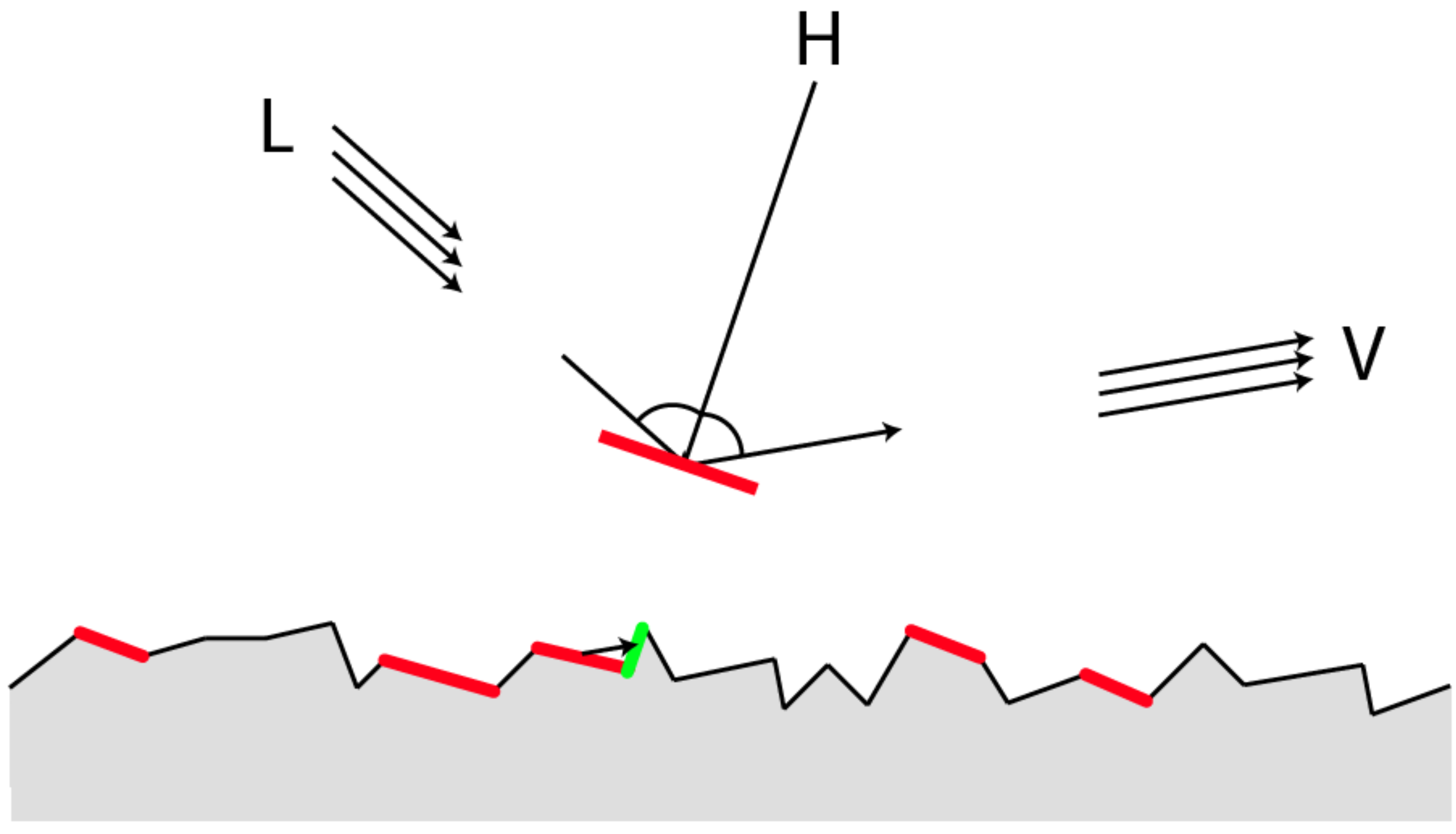
Microfacet Theory

- Value of BRDF at (L,V) is a product of
  - number of mirrors oriented halfway between L and V
  - ratio of the un(shadowed/masked) mirrors



Microfacet Theory

- Value of BRDF at (L,V) is a product of
  - number of mirrors oriented halfway between L and V
  - ratio of the un(shadowed/masked) mirrors
  - Fresnel coefficient

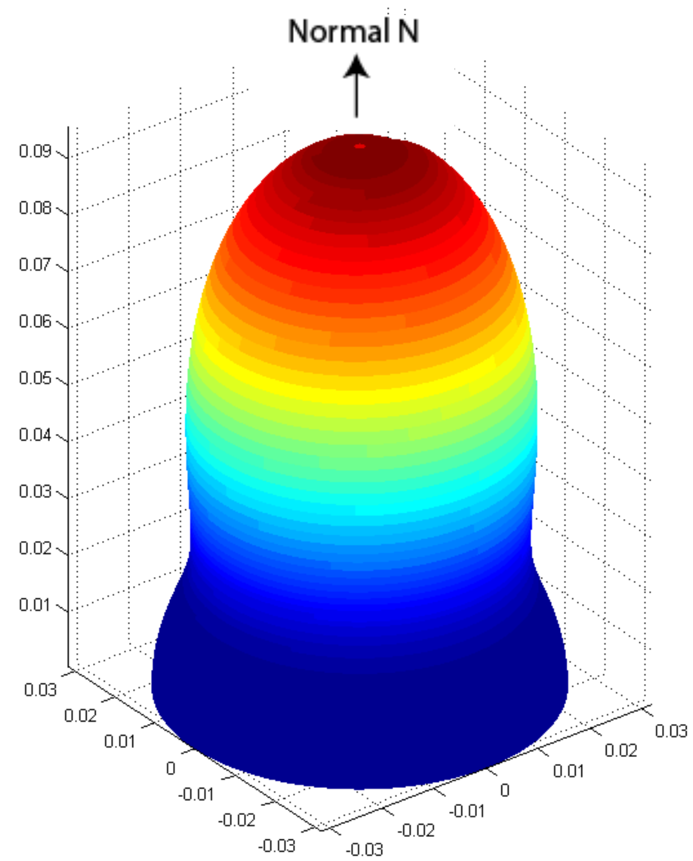


Microfacet Theory-based Models

- Develop BRDF models by imposing simplifications [Torrance-Sparrow 67], [Blinn 77], [Cook-Torrance 81], [Ashikhmin et al. 2000]

- Model the distribution D(**h**) of microfacet normals
  - Also, statistical models for shadows and masking

- As always, $\mathbf{h} = \dfrac{\mathbf{l}+\mathbf{v}}{\|\mathbf{l}+\mathbf{v}\|}$



spherical plot of a Gaussian-like p(H)

General Microfacet BRDF (Cook-Torrance)

- Sum of Diffuse and Specular terms:

$$f_r = \frac{\rho_d}{\pi} + \frac{\rho_s}{\pi} \frac{F(\mathbf{l} \cdot \mathbf{h})\, D(\mathbf{h})\, G(\mathbf{l}, \mathbf{v})}{(\mathbf{n} \cdot \mathbf{l})(\mathbf{n} \cdot \mathbf{v})}$$

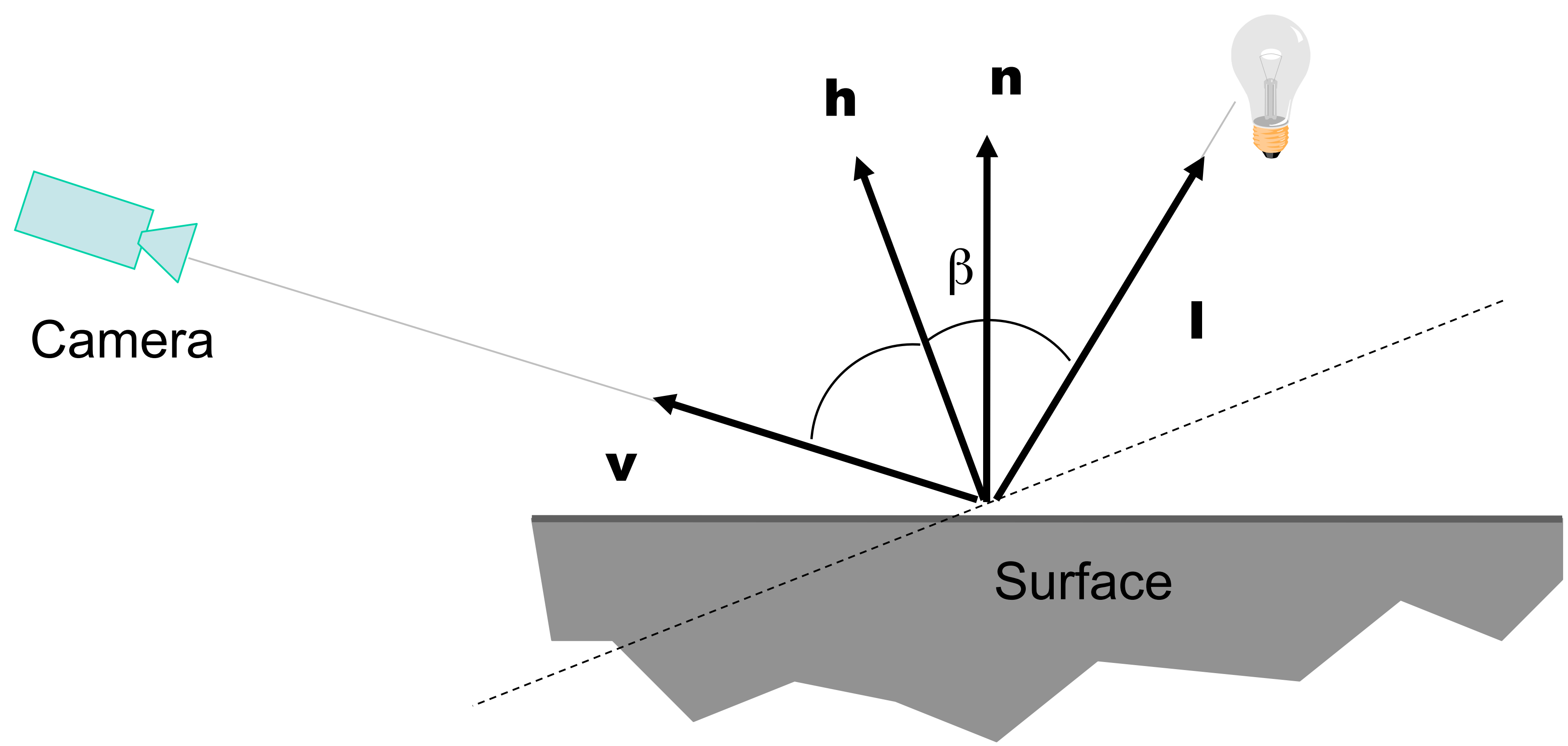
- F is the Fresnel term that accounts for increasing reflection towards grazing angle
- D is the microfacet distribution (common models include Gaussian, Blinn-Phong, Beckmann
  - Shifted Gamma is the new king of the hill
- G is the geometric (shadowing, masking) term
- See linked papers for details

Blinn-Torrance Variation of Phong

- Uses the "halfway vector" **h** between **l** and **v**.

$$D(\mathbf{h}) = N_q \, (\mathbf{n} \cdot \mathbf{h})^q$$

$$\boldsymbol{h} = \frac{\boldsymbol{l} + \boldsymbol{v}}{\|\boldsymbol{l} + \boldsymbol{v}\|}$$



$$N_q = \frac{n+1}{2\pi}$$

is a normalization factor

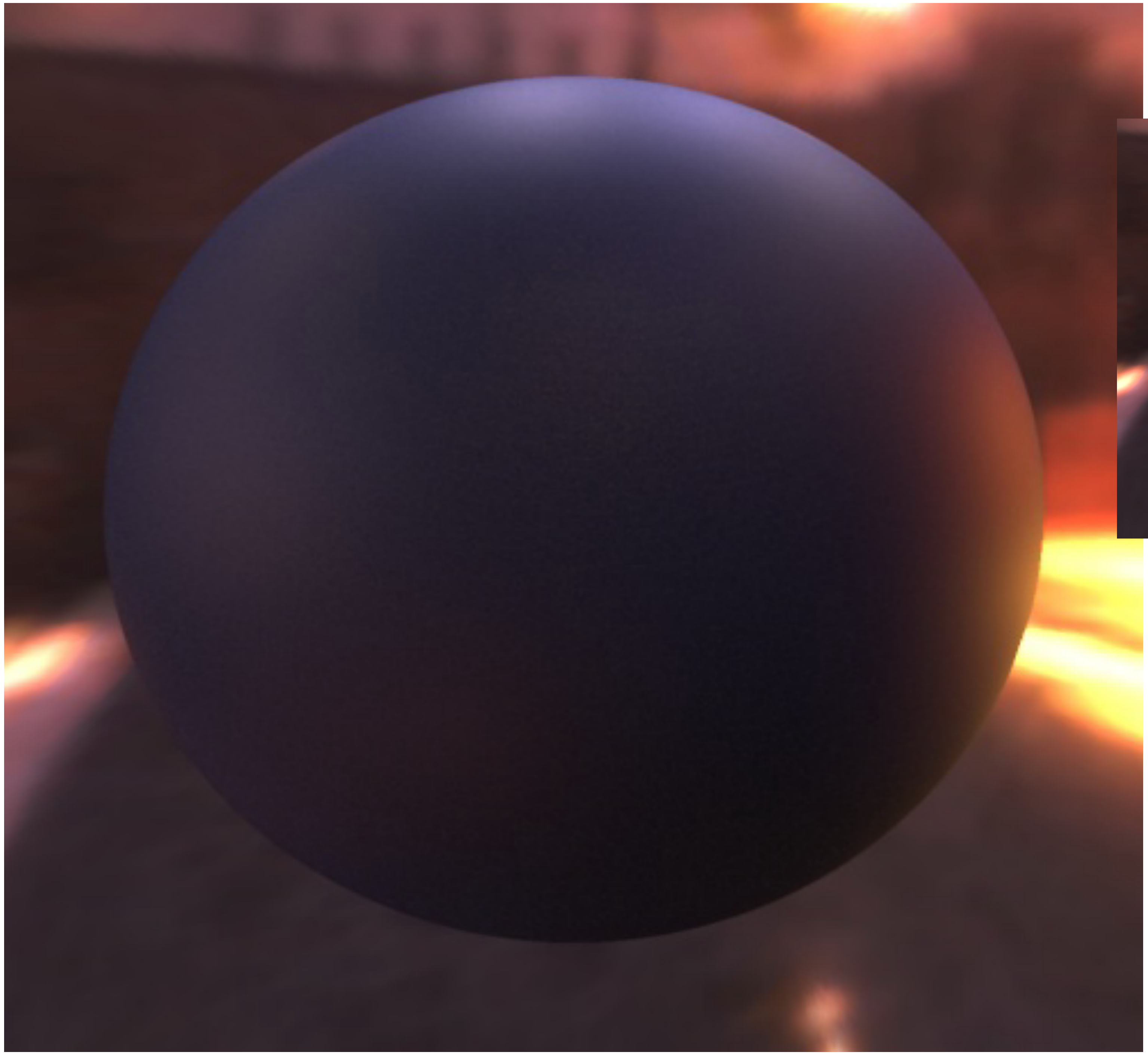
Geometric (Shadowing, Masking) Term

- Can be computed from microfacet distribution by integration
- Cook and Torrance used a heuristic formula

$$G = \min\left\{1, \frac{2(\mathbf{N}\cdot\mathbf{H})(\mathbf{N}\cdot\mathbf{V})}{(\mathbf{V}\cdot\mathbf{H})}, \frac{2(\mathbf{N}\cdot\mathbf{H})(\mathbf{N}\cdot\mathbf{L})}{(\mathbf{V}\cdot\mathbf{H})}\right\}$$

- Current models are more well-founded than this, see e.g. this paper

BRDF Examples: see Ngan et al.

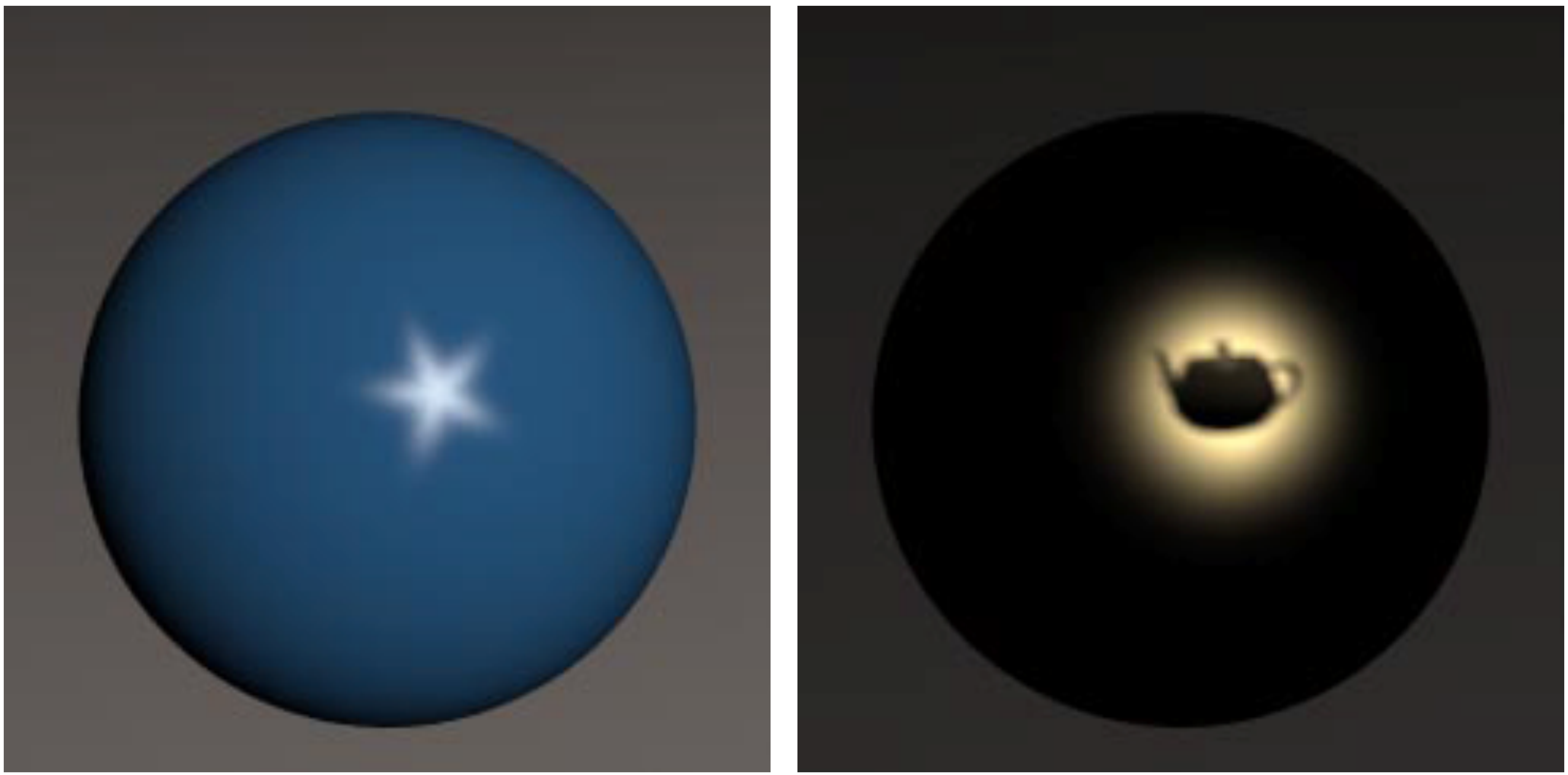


Lighting

Material – Dark blue paint

Questions?

- “Designer BRDFs” by Ashikhmin et al.



Reflectance

- Careful optimization + milling allows one to create a surface that reflects light in such funky ways
- Weyrich, Peers, Matusik, Rusinkiewicz SIGGRAPH 2009, Fabricating Microgeometry for Custom Surface Reflectance

Fabricating Microgeometry for Custom Surface Reflectance

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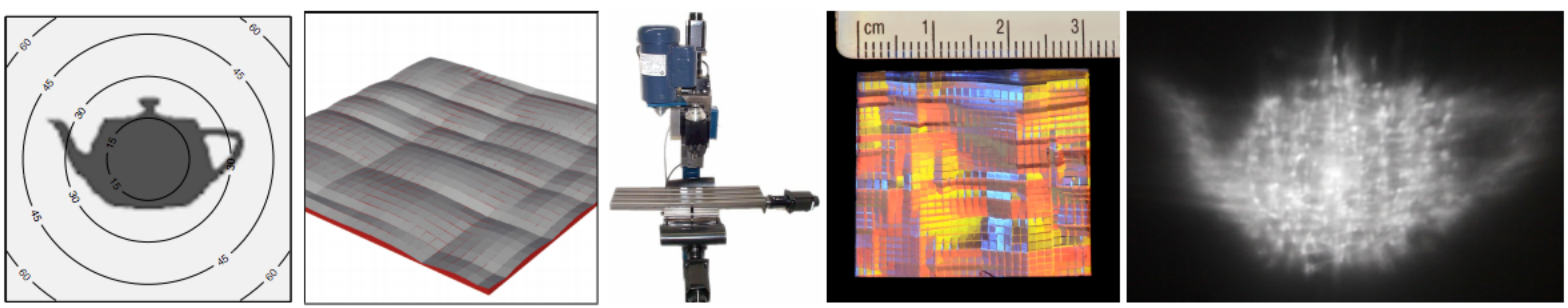


***Figure 1:** From left: a user-designed highlight is converted to an optimized microfacet height field. A computer-controlled milling machine is used to manufacture the surface (30 × 30 facets, each approximately 1 mm × 1 mm), which exhibits the desired reflectance.*

Pure Reflection (BRDF)

BRDF: Light reflects off exactly the same point

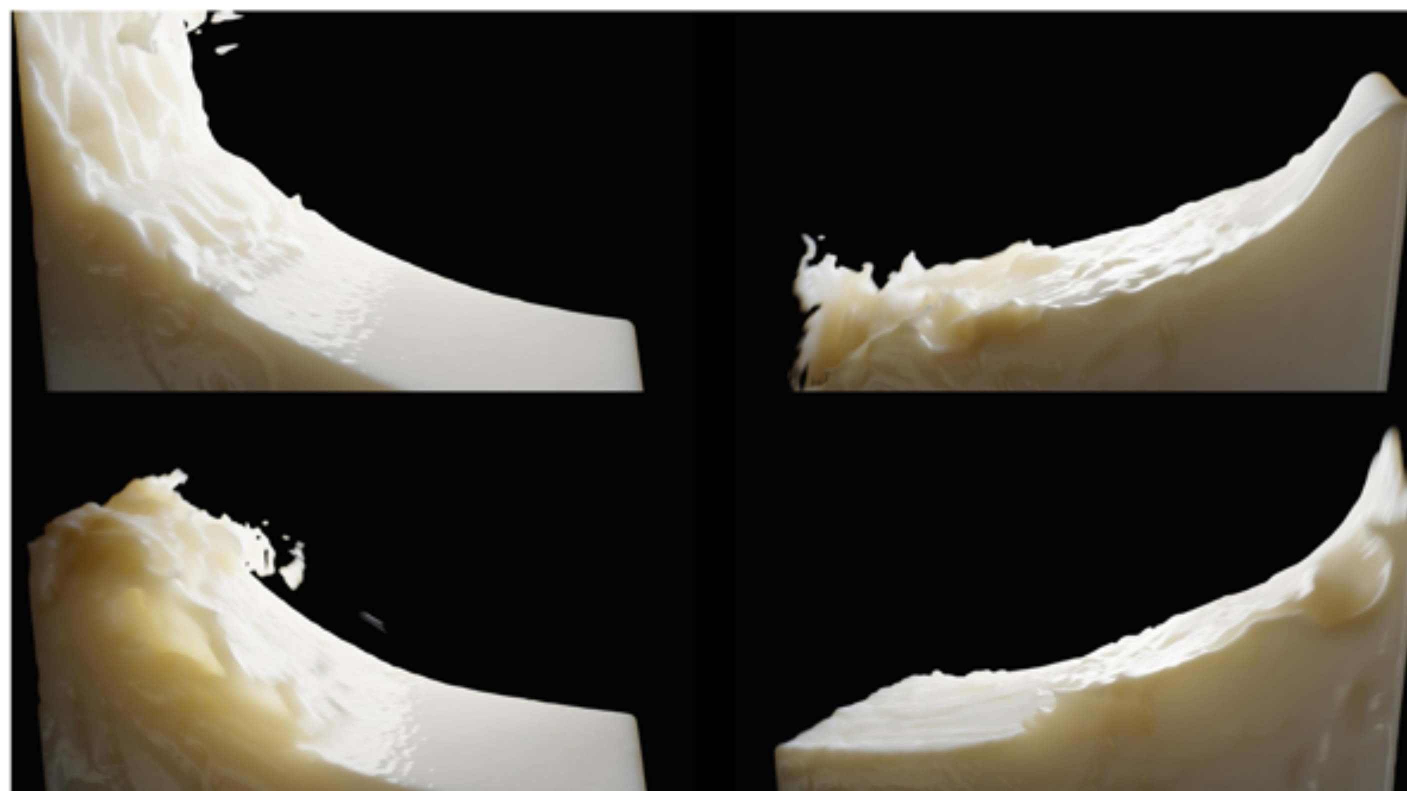
Subsurface Scattering (BSSRDF)

Some light enters material, exits at another point
BSSRDF = Bidirectional Surface Scattering Distribution Function

(See Henrik's paper linked to the title)

Subsurface State of the Art: Weta Digital

See Eugene's paper



BRDF vs. BSSRDF

Jensen et al. SIGGRAPH 2001

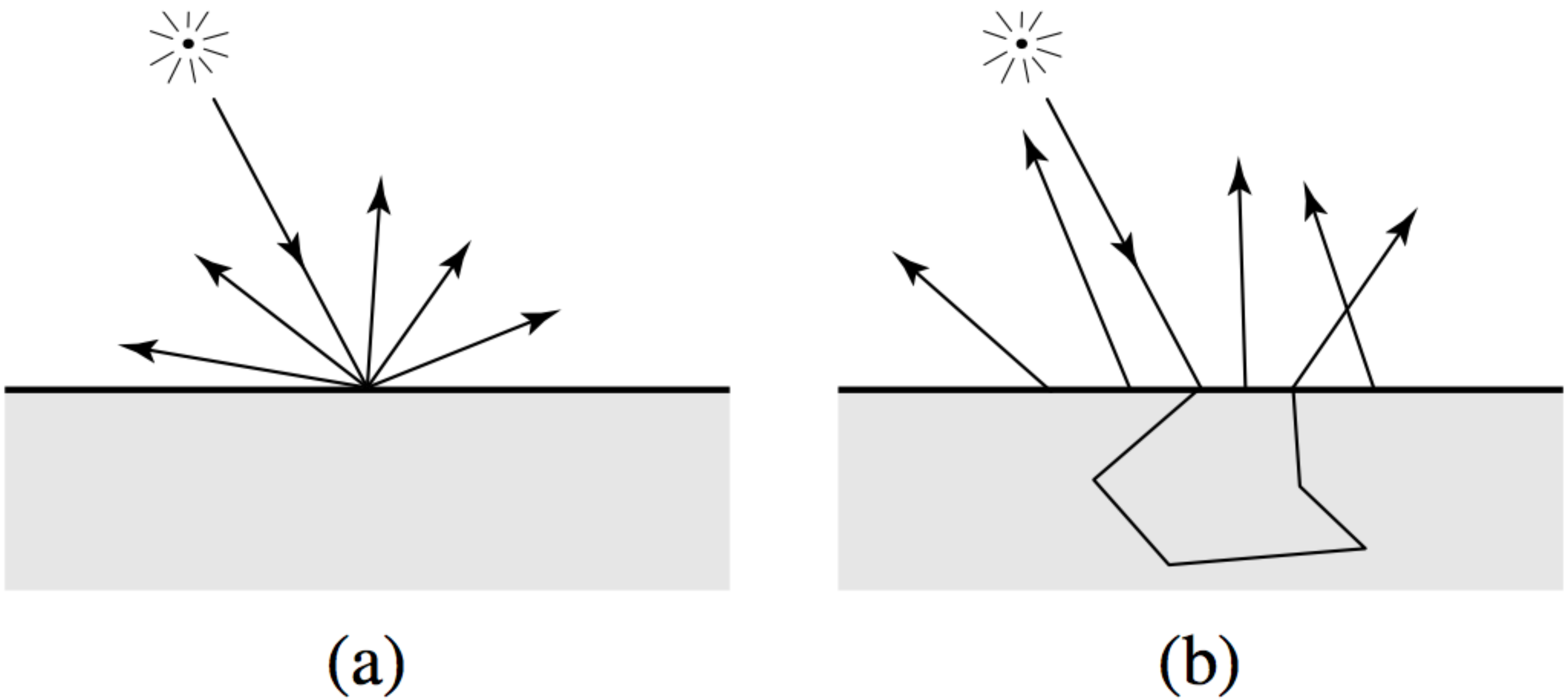


Figure 1: Scattering of light in (a) a BRDF, and (b) a BSSRDF.

BSSRDF Definition

- Relates differential irradiance *at all points* and all directions to outgoing radiance *at every other point* and all outgoing directions
  - 8D! Ouch!

$$L(x \to \mathbf{v}) = \int_A \int_\Omega L(y \leftarrow \mathbf{l}) f_r(x, y, \mathbf{l}, \mathbf{v}) \, \cos\theta \, \mathrm{d}\mathbf{l} \, \mathrm{d}A_y$$

- To get outgoing light at point x, integrate over all other points y and all incident directions at those points
  - Crazy complicated! Must do something smarter, i.e., cache incident illumination, assume diffuse scattering, etc. (See Henrik)

Questions?

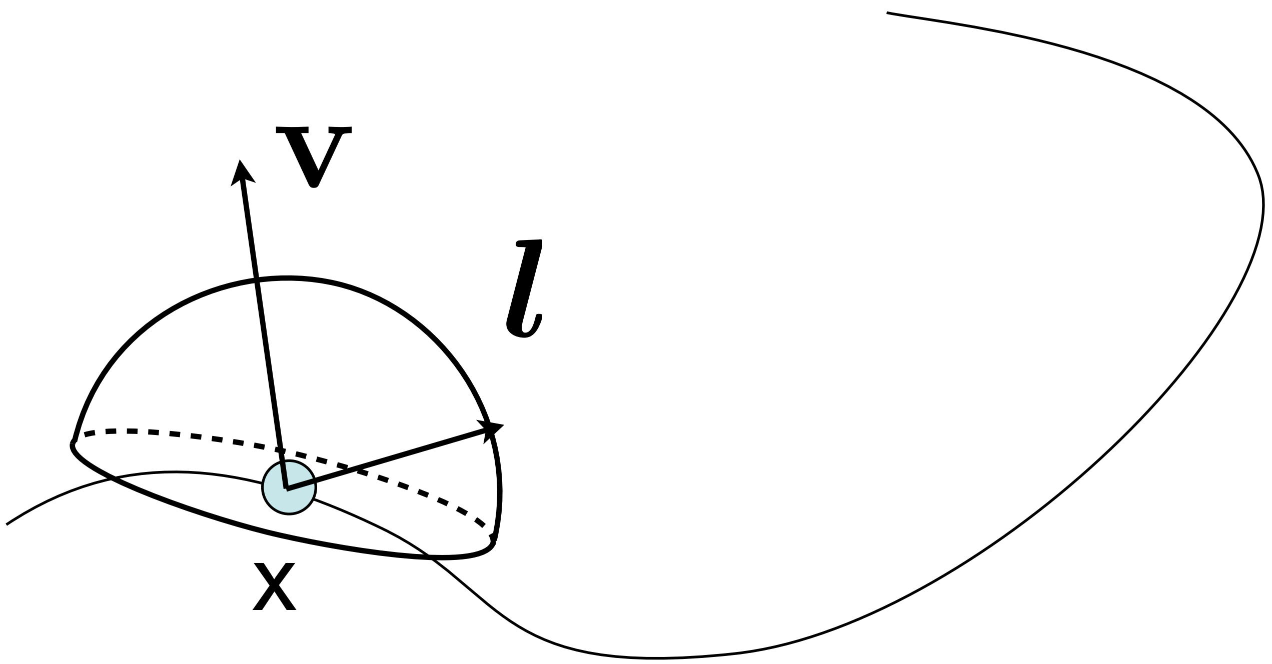
Markus Otto/Winzenrender, Rendered using Maxwell

The Way To Global Illumination

$$L(x \rightarrow \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$

reflectance equation

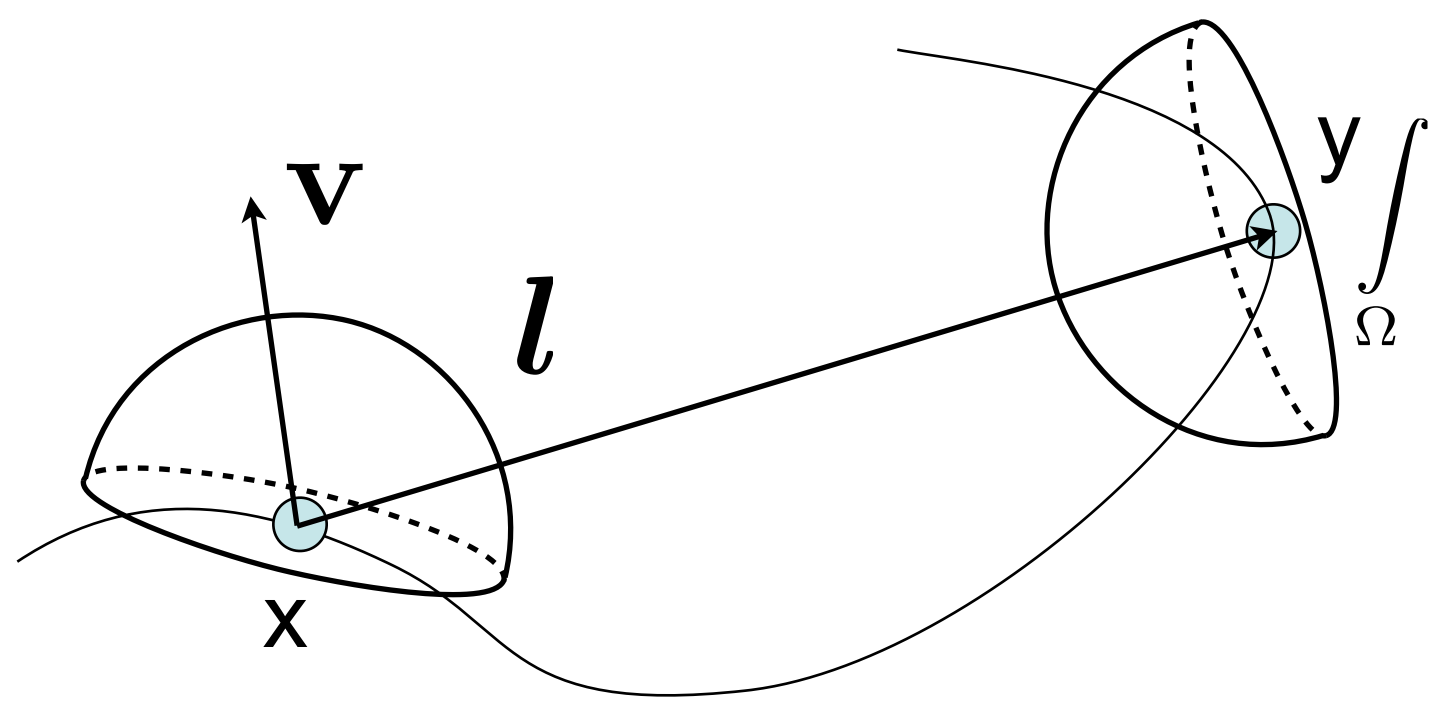
- Where does incident L come from?



The Way To Global Illumination

$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\ \cos\theta\, \mathrm{d}\mathbf{l}$$

- Where does incident L come from?
  - It is the light reflected towards x from the surface point y in direction $\boldsymbol{l}$ ==> must compute similar integral for every $\boldsymbol{l}$!
    - Recursive!



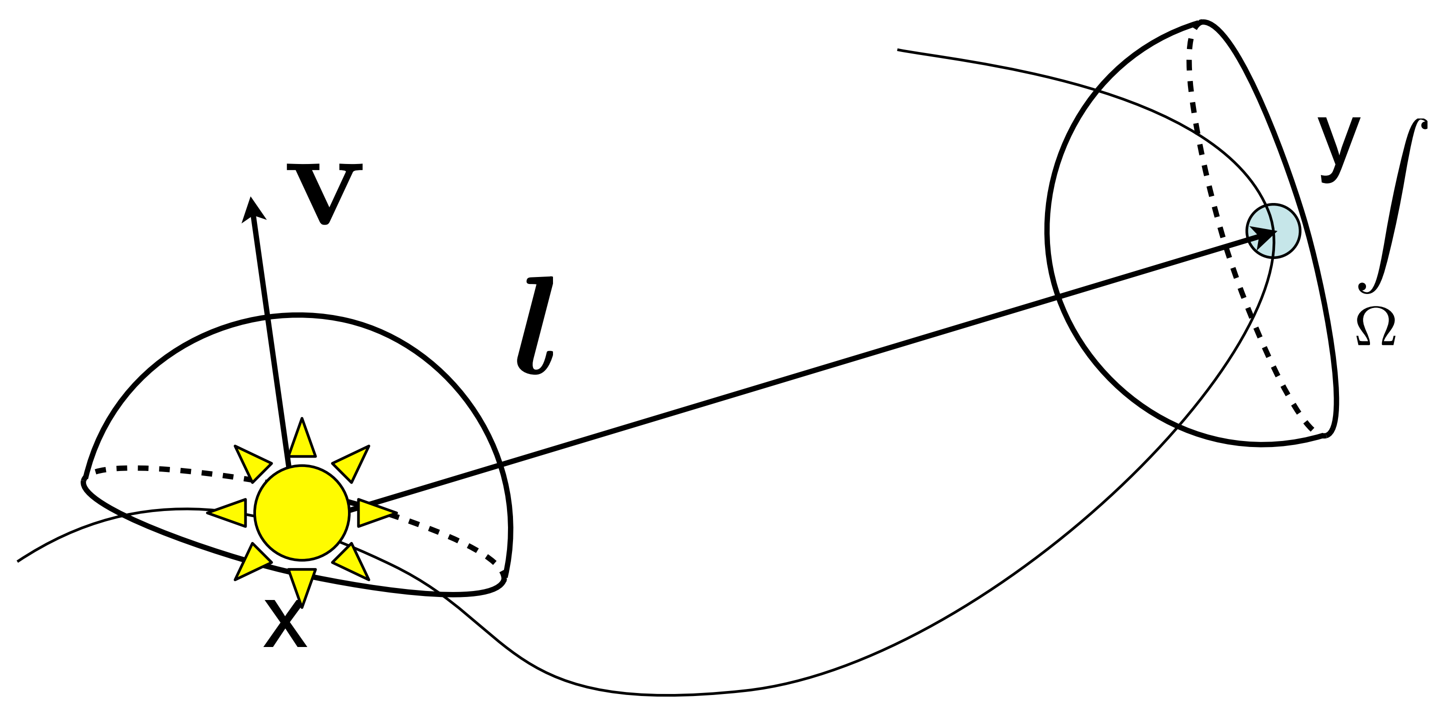
Rendering Equation

$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\ \cos\theta\, \mathrm{d}\mathbf{l}$$

$$\boxed{+\ E(x \to \mathbf{v})}$$

- Where does incident L come from?
  - It is the light reflected towards x from the surface point y in direction $\boldsymbol{l}$ ==> must compute similar integral for every $\boldsymbol{l}$!
    - Recursive!

- ...and if x happens to be on a light source, we add its emitted contribution E



The Rendering Equation

$$L(x \rightarrow \mathbf{v}) = \int_{\Omega} L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$

$$+ E(x \rightarrow \mathbf{v})$$

- Let’s bask in its glory for a moment

The Rendering Equation

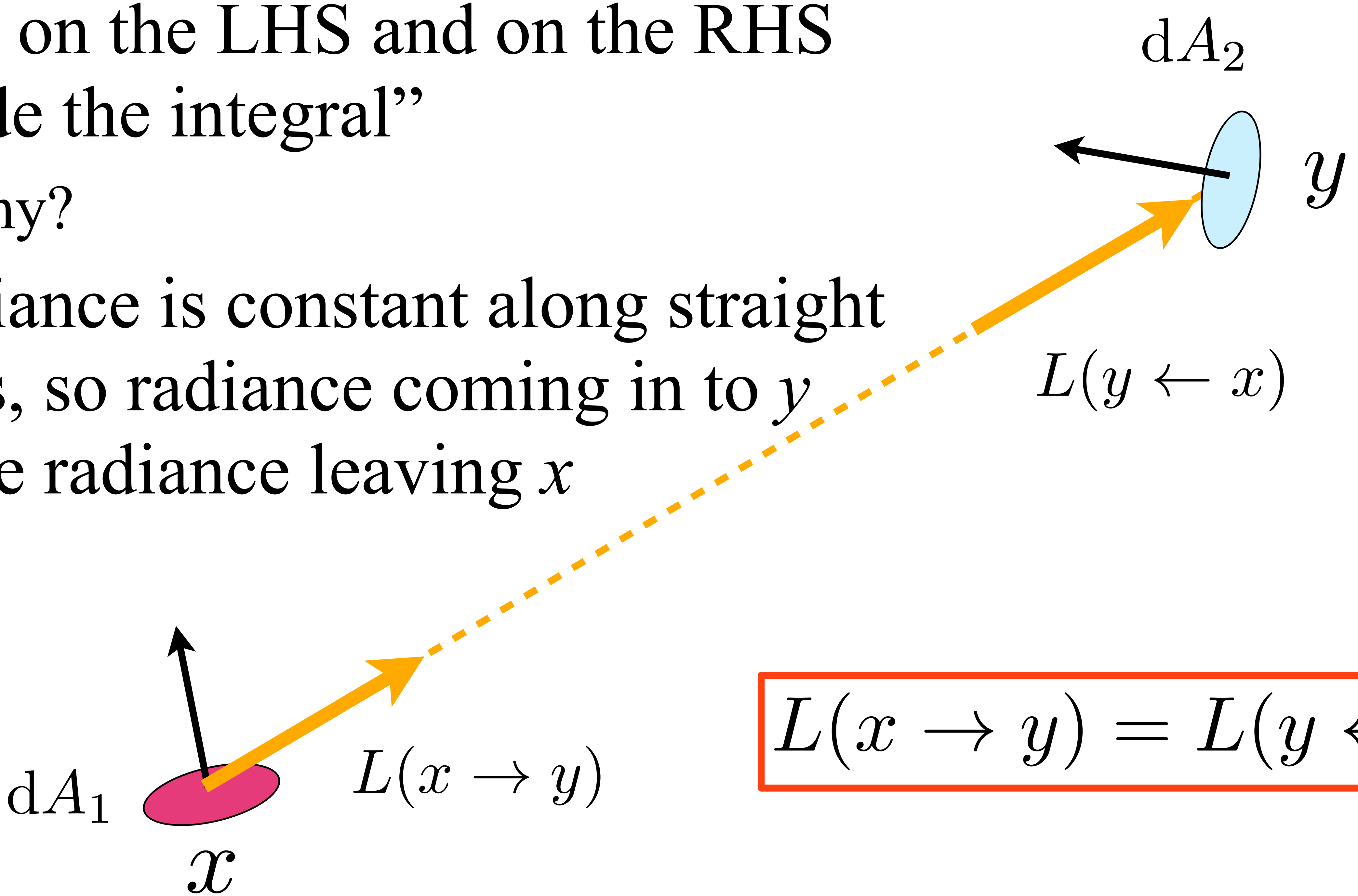
$$L(x \rightarrow \mathbf{v}) = \int_{\Omega} L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$

$$+ \; E(x \rightarrow \mathbf{v})$$

- The rendering equation describes the appearance of the scene, including direct and indirect illumination
  - An "integral equation", the unknown solution function L is both on the LHS (left-hand side) and on the RHS inside the integral

Hmmh..

- “the unknown solution function L is both on the LHS and on the RHS inside the integral”
  - Why?
- Radiance is constant along straight lines, so radiance coming in to *y* is the radiance leaving *x*



$$\boxed{L(x \rightarrow y) = L(y \leftarrow x)}$$

The Rendering Equation

$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\ \cos\theta\, \mathrm{d}\mathbf{l}$$
$$+\ E(x \to \mathbf{v})$$

- The rendering equation describes the appearance of the scene, including direct and indirect illumination
  - An "integral equation", the unknown solution function L is both on the LHS and on the RHS inside the integral
    - More precisely: a "Fredholm equation of the 2nd kind"
  - Originally described by Kajiya and Immel et al. in 1986

  - **Take a class in Functional Analysis to learn more!**

The Rendering Equation

- The unknown in this equation is the *function* $L(x \rightarrow \mathbf{v})$ defined for all points x and all directions $\mathbf{v}$
- Analytic (exact) solution is impossible in all cases of practical interest
- Lots of ways to solve approximately
  - Monte Carlo techniques use random samples for evaluating the integrals
  - Finite element methods (FEM) discretize the solution using basis functions
    - Radiosity, wavelets, precomputed radiance transfer, etc.
  - Topic of next lecture!

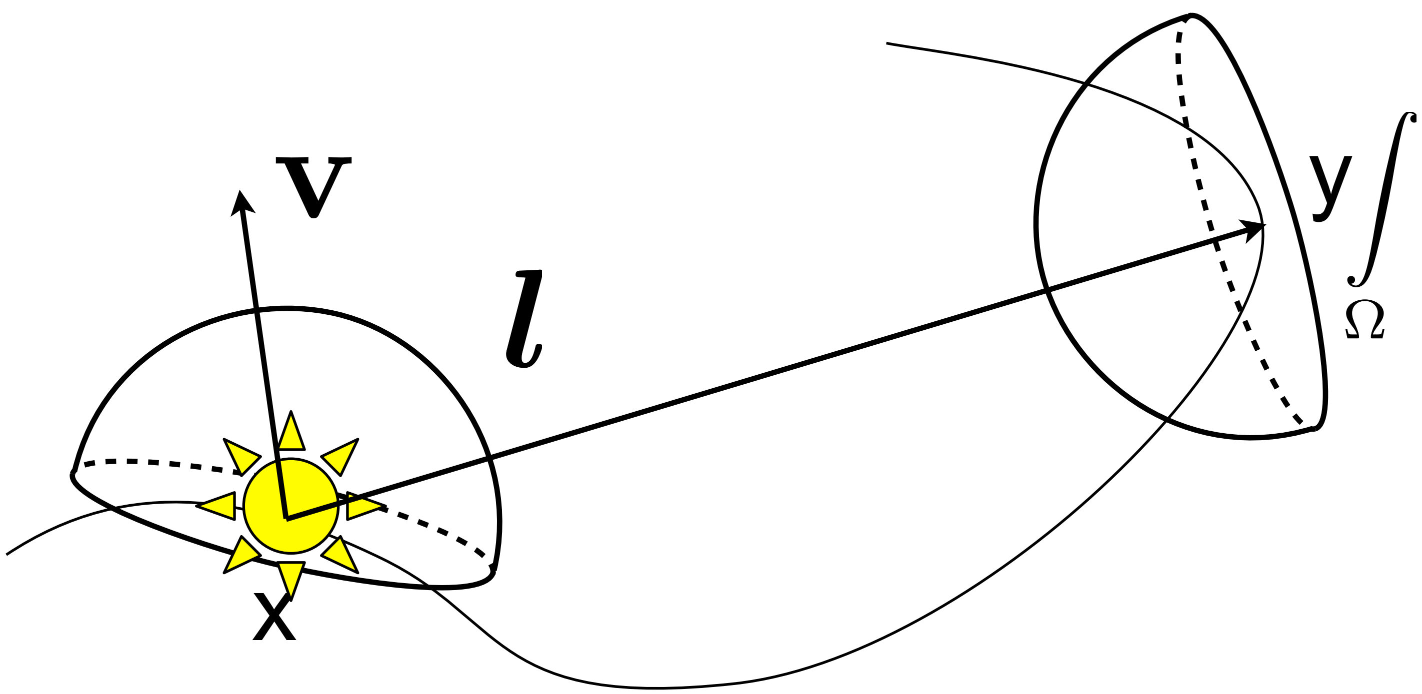
Questions?

Stack Studios, Rendered using Maxwell

The Rendering Equation

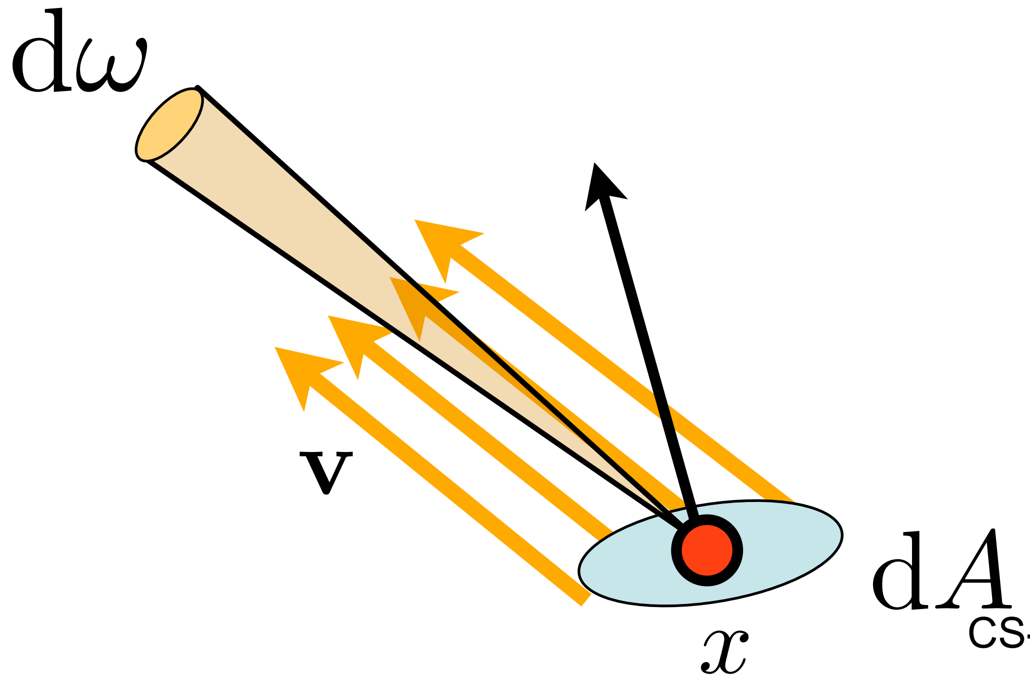
$$L(x \rightarrow \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v})\, \cos\theta \,\mathrm{d}\mathbf{l} + E(x \rightarrow \mathbf{v})$$

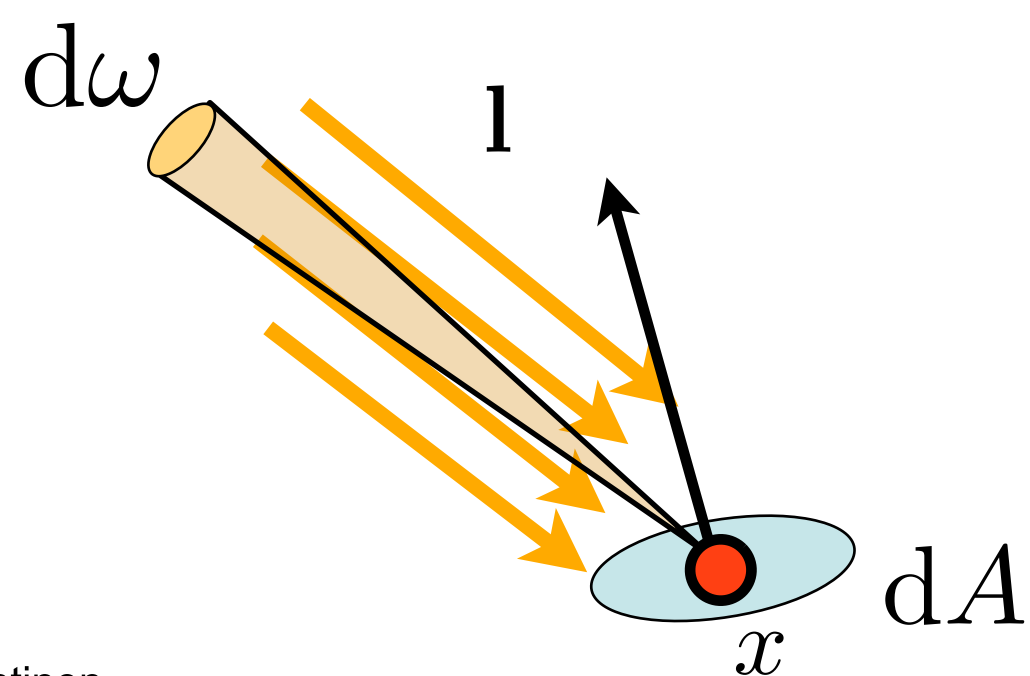
- Recursive!
  - To know incident radiance at *x*, must know outgoing radiance at *all points y seen by x*



Recap: Radiance Notation

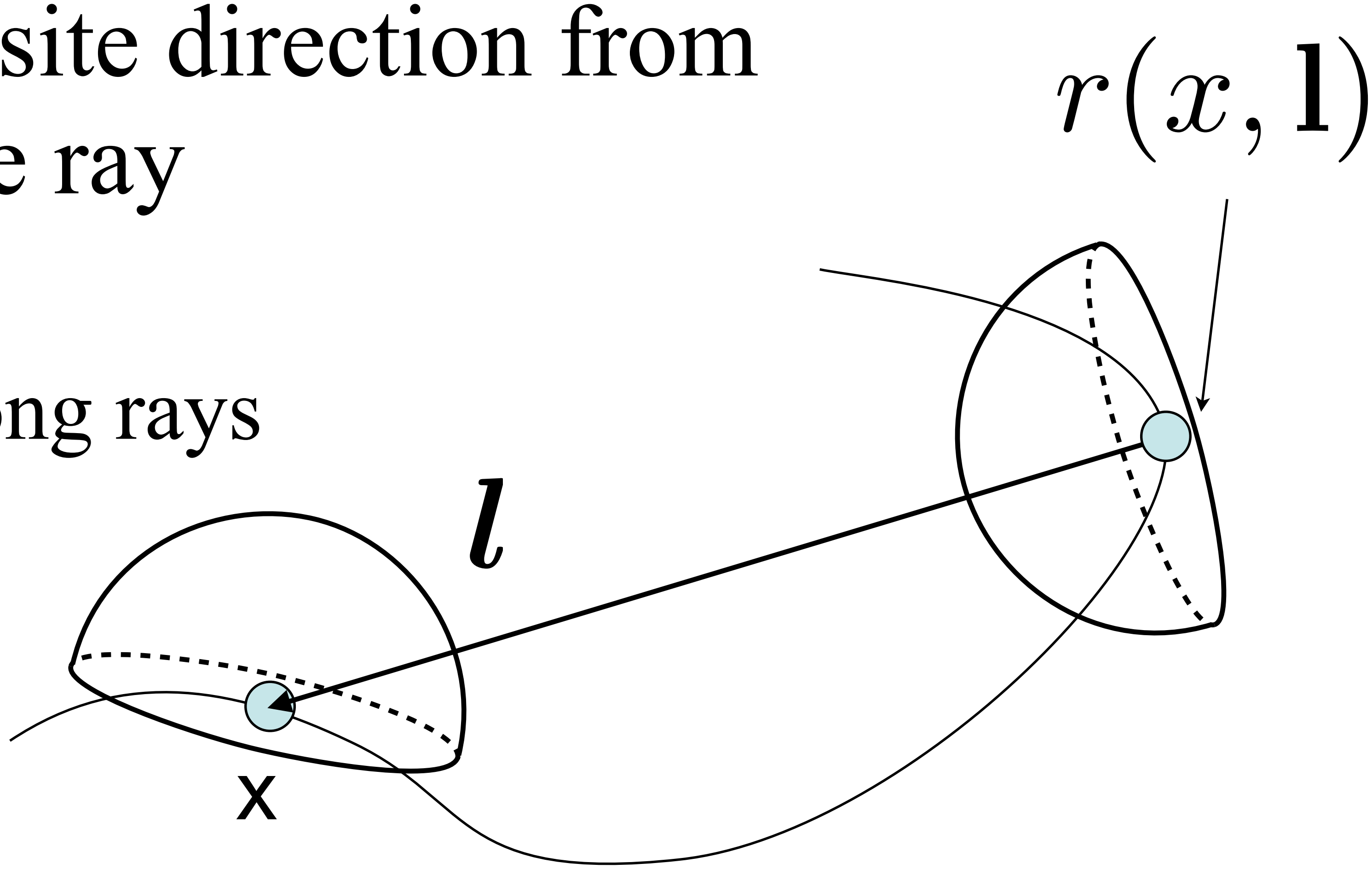
- $L(x \to \mathbf{v})$ denotes radiance leaving dA located at point x towards direction $\mathbf{v}$
  - Alternative notation: $L_{\text{out}}(x, \mathbf{v})$
- $L(x \leftarrow \mathbf{l})$ denotes radiance impinging on dA located at point x from direction $\mathbf{l}$
  - Alternative notation: $L_{\text{in}}(x, \mathbf{l})$





Operator Formulation 1

- "The lighting incident to x from $\mathbf{l}$ is the light exiting to the opposite direction from the point r(x,$\mathbf{l}$) where the ray from x towards $\mathbf{l}$ hits"
  - Constancy of radiance along rays
  - "Ray-cast function" $r(x,\mathbf{l})$ returns point hit by ray from x towards $\mathbf{l}$

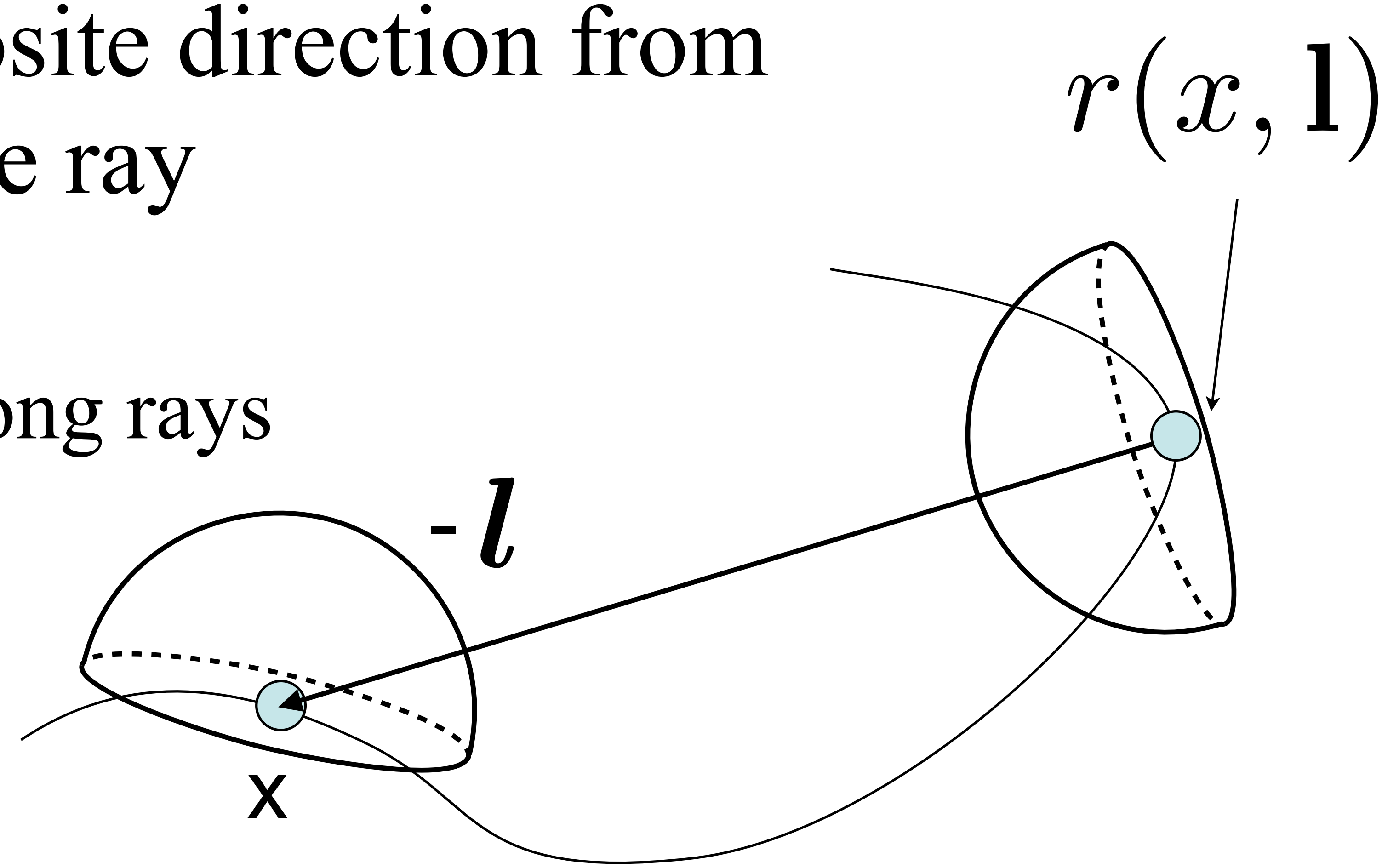


Operator Formulation 1

- Let's define the *propagation operator* G

$$L_{\mathrm{in}}(x, \mathbf{l}) = (\mathcal{G} L_{\mathrm{out}}) \stackrel{\mathrm{def}}{=} L_{\mathrm{out}}(r(x, \mathbf{l}) \rightarrow -\mathbf{l})$$

- "The lighting incident to *x* from **l** is the light exiting to the opposite direction from the point r(*x*,**l**) where the ray from *x* towards **l** hits"
  - Constancy of radiance along rays
  - "Ray-cast function" *r*(*x*,**l**) returns point hit by ray from *x* towards **l**



Operator Formulation 1

- Let's define the *propagation operator* G

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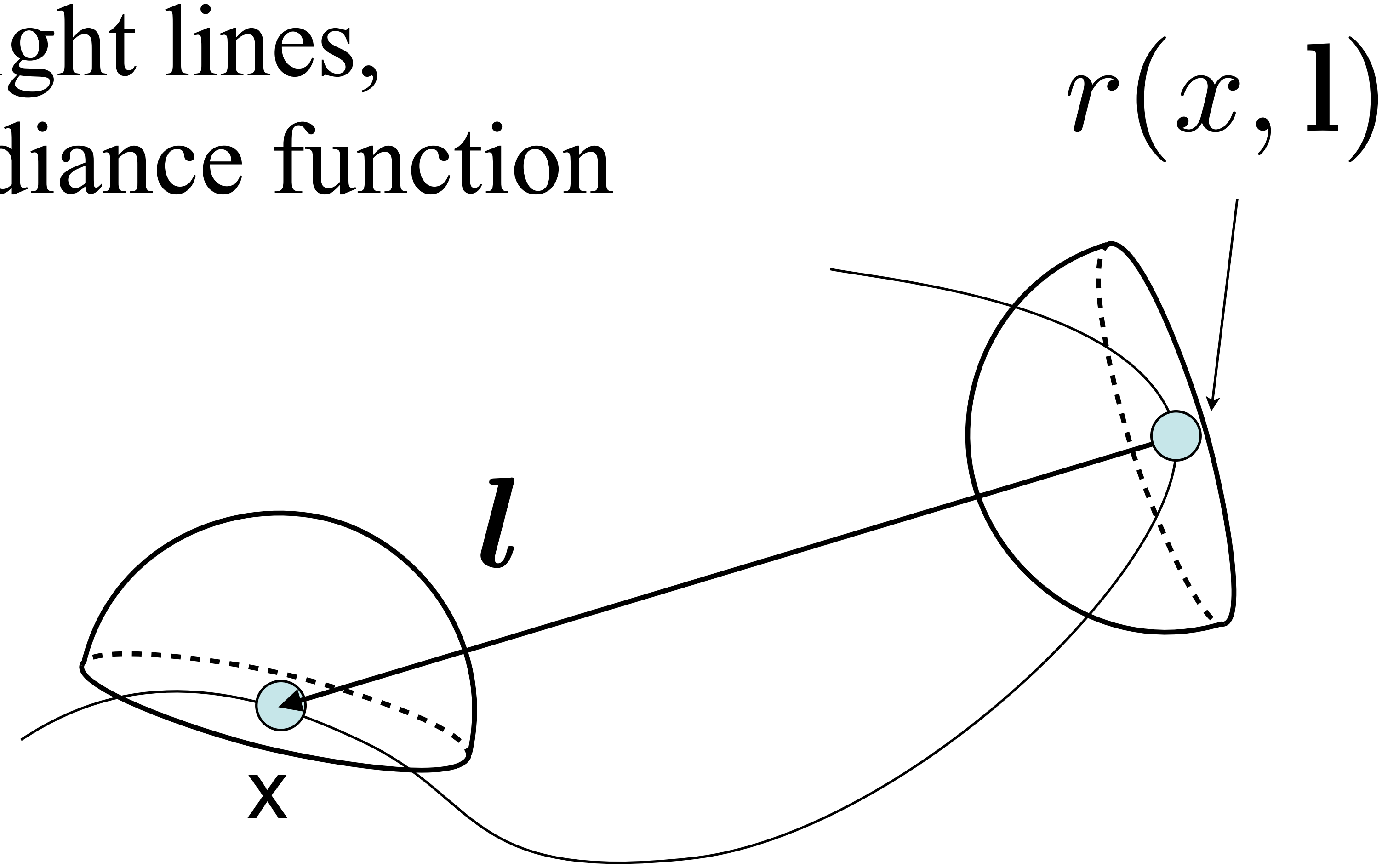
- G takes an outgoing radiance function, propagates it along straight lines, produces an incident radiance function



Operator Formulation cont'd

- ..and the *local reflection operator* R

$$L_{\text{out}}(x, \mathbf{v}) = (\mathcal{R} L_{\text{in}}) =$$

$$\int_{\Omega} L_{\text{in}}(x, \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$

- Takes incident radiance function (defined for all points and directions), produces outgoing radiance function (defined for all points and directions)
- This is just another way of writing the reflectance integral you saw already

These operators are not complicated

Take in one function, do something to it, return another

runes.nu, rendered using Maxwell

Operator Form of Rendering Eq.

$$L_{\text{out}}(x, \mathbf{v}) = \int_\Omega L_{\text{in}}(x, \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\; \cos\theta_{\text{in}}\, \mathrm{d}\mathbf{l} + E_{\text{out}}(x, \mathbf{v})$$

- Propagation + reflectance operators

$$L_{\text{out}} = \mathcal{R} L_{\text{in}}$$

$$L_{\text{in}} = \mathcal{G} L_{\text{out}}$$

Operator Form of Rendering Eq.

$$L_{\mathrm{out}}(x, \mathbf{v}) = \int_{\Omega} L_{\mathrm{in}}(x, \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\, \cos\theta_{\mathrm{in}}\, \mathrm{d}\mathbf{l} + E_{\mathrm{out}}(x, \mathbf{v})$$

- Propagation + reflectance operators

$$L_{\mathrm{out}} = \mathcal{R} L_{\mathrm{in}}$$
$$L_{\mathrm{in}} = \mathcal{G} L_{\mathrm{out}}$$

- Let's put them together (I is identity):

$$\boxed{L_{\mathrm{out}} = \mathcal{R}\mathcal{G} L_{\mathrm{out}} + E \Leftrightarrow (\mathcal{I} - \mathcal{T}) L_{\mathrm{out}} = E}$$

Operator Form of Rendering Eq.

$$L_{\text{out}} = \mathcal{R}\mathcal{G}L_{\text{out}} + E$$

- Let's call RG, propagation followed by reflection, the *transport operator* T
- Looks a lot like a linear system Ax=b, doesn't it?
  - Well, it *is* a linear system.
    Just with functions instead of vectors.
  - Easy to verify linearity: T(aX+bY) = aTX + bTY for any functions X,Y and scalars a,b

$$\boxed{(\mathcal{I} - \mathcal{T})L_{\text{out}} = E}$$

Consequence of Linearity

$$(\mathcal{I} - \mathcal{T})L_{\text{out}} = E$$

$$\Leftrightarrow L_{\text{out}} = (\mathcal{I} - \mathcal{T})^{-1}\, E$$

- This is kind of a deep result, although simple: *the lighting solution is linear w.r.t. the emission.*
  - I.e., solution is a linear function of the emission.

Consequence of Linearity, Pt 2

$$(\mathcal{I} - \mathcal{T})L_{\text{out}} = E$$

$$\Leftrightarrow L_{\text{out}} = (\mathcal{I} - \mathcal{T})^{-1}\, E$$

- Light is additive, i.e., we can break emission into parts

$$L_{\text{out}} = (\mathcal{I} - \mathcal{T})^{-1}\,(E_1 + E_2)$$

$$= \boxed{(\mathcal{I} - \mathcal{T})^{-1}\, E_1} + \boxed{(\mathcal{I} - \mathcal{T})^{-1}\, E_2}$$

“Neumann Series”

See link by clicking title!

$$\Leftrightarrow L_{\text{out}} = (\mathcal{I} - \mathcal{T})^{-1} E$$

- The Neumann series says

$$(\mathcal{I} - \mathcal{T})^{-1} = \sum_{i=0}^{\infty} \mathcal{T}^i$$

$$= \mathcal{I} + \mathcal{T} + \mathcal{T}^2 + \mathcal{T}^3 + \ldots$$

- I.e. $\boxed{L_{\text{out}} = E + \mathcal{T}E + \mathcal{T}\mathcal{T}E + \ldots}$

Neumann Series

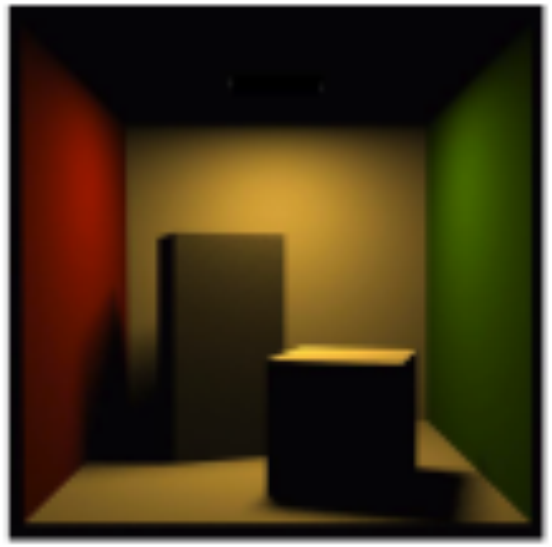
$$\boxed{L_{\text{out}} = E + \mathcal{T}E + \mathcal{T}\mathcal{T}E + \ldots}$$

- The lighting solution is the sum of
  - emitted light E,
  - light reflected once TE,
  - light reflected twice TTE, etc.

- Monte Carlo methods compute these integrals probabilistically

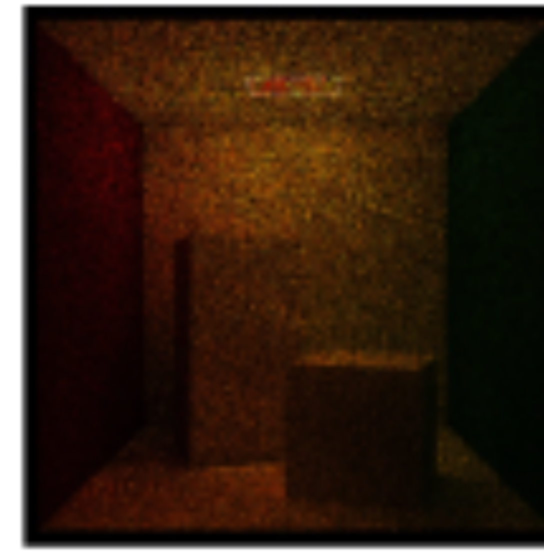
Note: Pat uses L_e instead of E, K instead of T

L_e

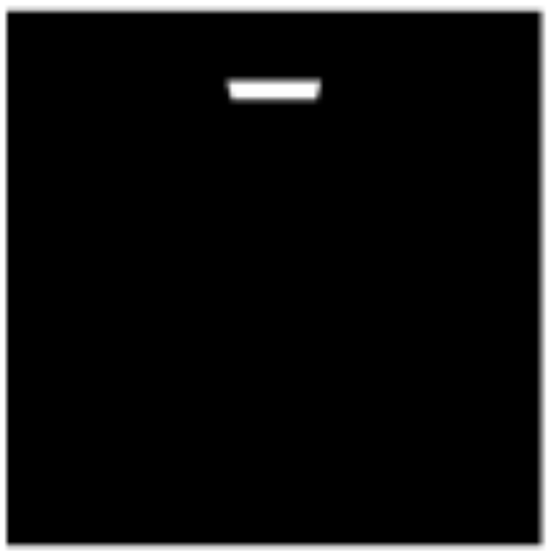


$K \circ L_e$

$K \circ K \circ L_e$

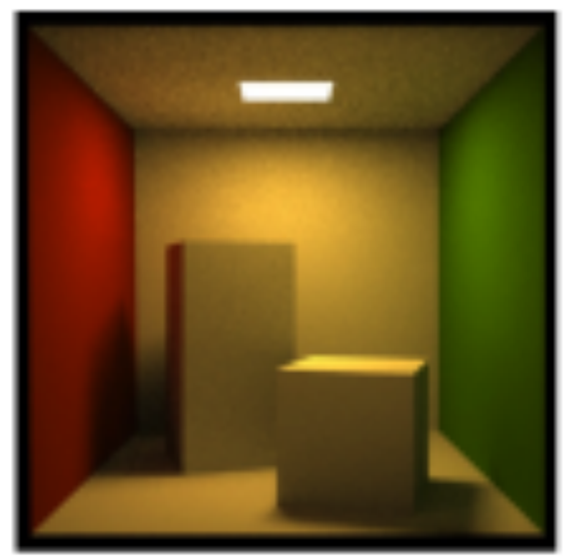


$K \circ K \circ K \circ L_e$

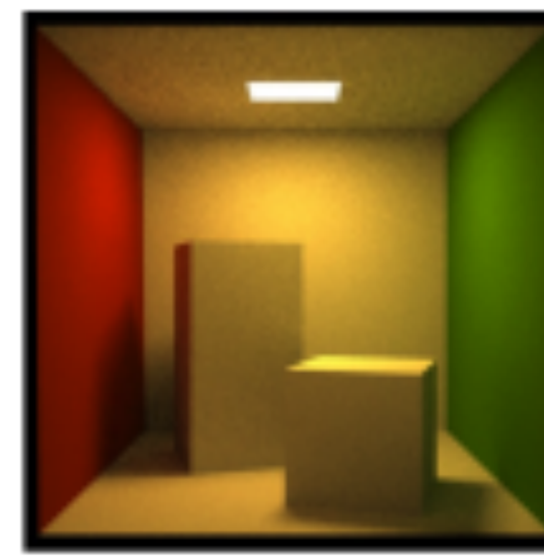


L_e

$L_e + K \circ L_e$

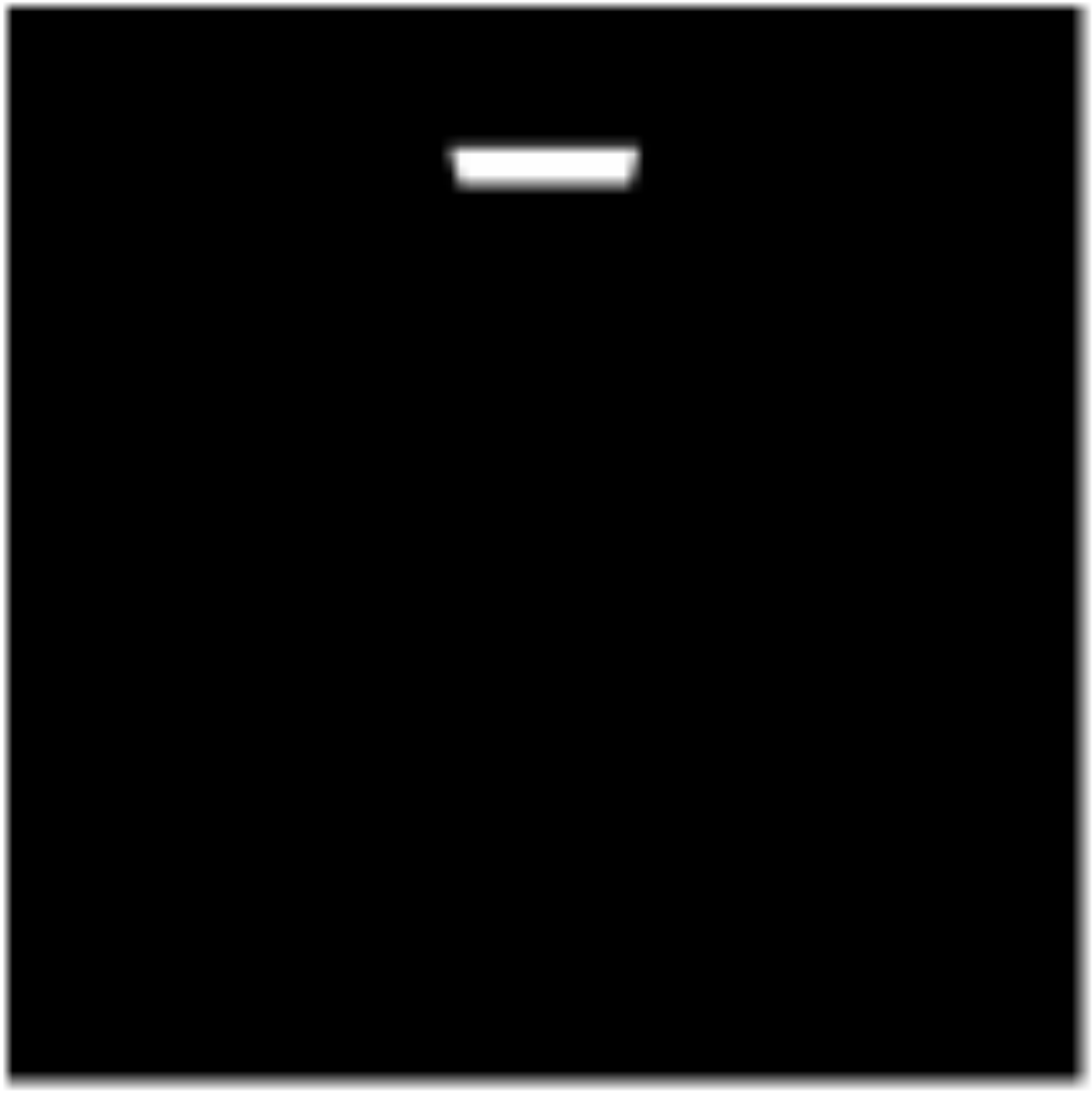


$L_e + \cdots K^2 \circ L_e$

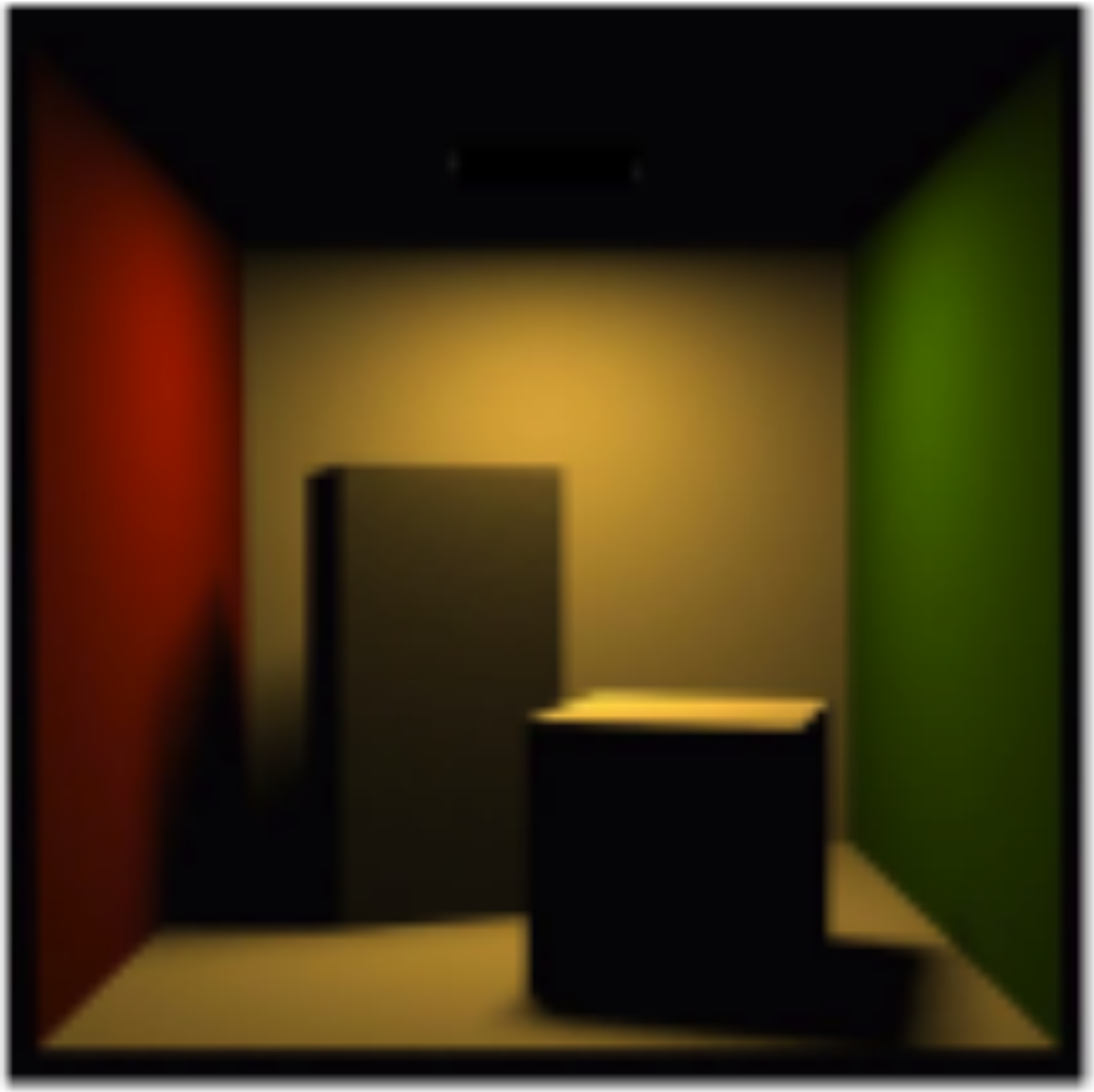


$L_e + \cdots K^3 \circ L_e$

E = Emitted Radiance (Light sources)

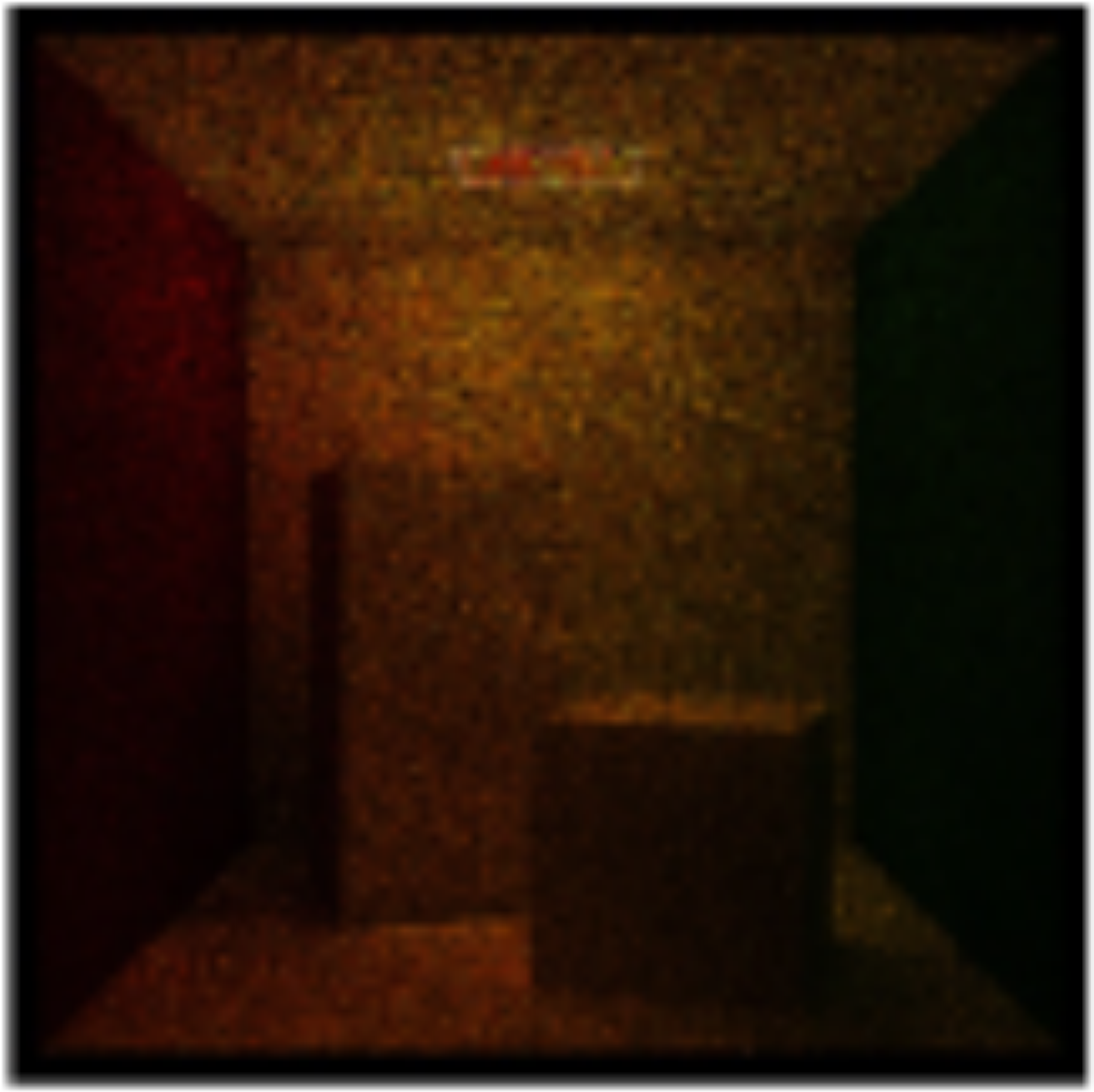


TE = Direct Lighting



TTE = First Indirect Bounce

TTTE = Second Indirect Bounce



Questions?