7. Regularity by the direct methods in the calculus of variations

7.1. De Giorgi class.

In this section we show that minimizers of the p-Dirichlet integral satisfy a Caccioppoli type energy estimate on distribution sets.

Let $\Omega \subset X$ be an open set. A function $u \in N^{1,p}(\Omega)$ belongs to the De Giorgi class $DG(\Omega)$ if there exists a constant C such that for every $k \in \mathbb{R}$, $x \in \Omega$ and $0 < r < R < \frac{\operatorname{diam}(X)}{3}$ with $B(x,R) \subset \Omega$ we have

$$\int_{A_x(k,r)} g_u^p \, d\mu \le \frac{C}{(R-r)^p} \int_{A_x(k,R)} (u-k)^p \, d\mu,$$

where

$$A_x(k,r) = \{ y \in B(x,r) : u(y) > k \}.$$

In the rest of the discussion we drop the subscript x from $A_x(k,r)$ as $x \in \Omega$ is fixed. Observe that the condition above is equivalent to

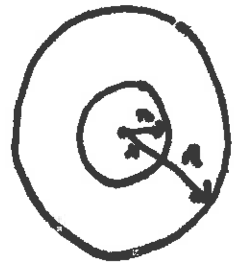
$$\int_{B(x,r)} g_{(u-k)_+}^p \, d\mu \le \frac{C}{(R-r)^p} \int_{B(x,R)} (u-k)_+^p \, d\mu,$$

where $(u-k)_+ = \max\{u-k, 0\}$.

Assume that u is a minimizer of the p-Dirichlet integral in Ω. Let $B(x,R) \subset \Omega$ with $0 < r < R < \dfrac{\operatorname{diam}(X)}{3}$.

Let

$$\eta(y) = \begin{cases} 1, & d(x,y) < r, \\ \dfrac{R - d(y,x)}{R - r}, & r \le d(x,y) \le R, \\ 0, & d(x,y) > R. \end{cases}$$



Then $0 \le \eta \le 1$, $\eta = 1$ on $B(x,r)$ and $\eta = 0$ on $X \setminus B(x,R)$.

Let

$$v = u - \eta \max\{u-k, 0\}.$$

Then

$$u - v = \eta \max\{u-k,0\} \in N_0^{1,p}(A(k,R))$$

and thus

$$\int_{\Omega \setminus A(k,R)} g_u^p \, d\mu + \int_{A(k,R)} g_u^p \, d\mu = \int_{\Omega} g_u^p \, d\mu$$

$$\le \int_{\Omega} g_v^p \, d\mu = \int_{\Omega \setminus A(k,R)} g_u^p \, d\mu + \int_{A(k,R)} g_v^p \, d\mu.$$

$v = u$ in $\Omega \setminus A(k,R)$

This implies

$$\int_{A(k,r)} g_u^p \, d\mu \le \int_{A(k,R)} g_u^p \, d\mu \le \int_{A(k,R)} g_v^p \, d\mu.$$

Note that

$$v = u - \eta(u-k) = (1-\eta)(u-k) + k \quad \text{on } A(k,R)$$

and thus by 3.22 we have

$$g_v \le (u-k)\, g_\eta + (1-\eta)\, g_u .$$

This implies

$$\int_{A(k,r)} g_u^p \, d\mu \le 2^p \int_{A(k,R)} \left((u-k)^p g_\eta^p + (1-\eta)^p g_u^p \right) d\mu$$

$$\le \frac{C}{(R-r)^p} \int_{A(k,R)} (u-k)^p \, d\mu + C \int_{A(k,R)\setminus A(k,r)} g_u^p \, d\mu .$$

$1-\eta = 0$ in $B(x,r)$

By filling the hole by adding the term

$$C \int_{A(k,r)} g_u^p \, d\mu$$

to the both sides we obtain

$$(1+C) \int_{A(k,r)} g_u^p \, d\mu \le C \int_{A(k,R)} g_u^p \, d\mu + \frac{C}{(R-r)^p} \int_{A(k,R)} (u-k)^p \, d\mu .$$

This implies

$$\int_{A(k,r)} g_u^p \, d\mu \le \theta \int_{A(k,R)} g_u^p \, d\mu + \frac{C}{(R-r)^p} \int_{A(k,R)} (u-k)^p \, d\mu ,$$

where $\theta = \frac{C}{C+1} < 1$.

This implies

$$\int_{A(k,r)} g_u^p \, d\mu \le \theta \int_{A(k,R)} g_u^p \, d\mu + \frac{C}{(R-r)^p} \underbrace{\int_{A(k,S)} (u-k)^p \, d\mu}_{=\gamma}$$

whenever $0 < r < R \le S < \frac{\operatorname{diam}(X)}{3}$. Let $f : (0,S] \to [0,\infty)$,

$$f(r) = \int_{A(k,r)} g_u^p \, d\mu .$$

Since $f(r) \le \int_{\Omega} g_u^p \, d\mu < \infty$ for every $0 < r \le S$, we conclude that f is a bounded function. Moreover, it satisfies

$$f(r) \le \theta f(R) + \gamma (R-r)^{-p}$$

for every $0 < r < R \le S$. By Lemma 4.18 in the NPDE course

$$f(r) \le C \gamma (R-r)^{-p}$$

for every $0 < r < R \le S$. Thus

$$\int_{A(k,r)} g_u^p \, d\mu \le C (R-r)^{-p} \int_{A(k,S)} (u-k)^p \, d\mu ,$$

where we may choose $R = S$. ▨

This shows that $u \in DG(\Omega)$. Since $-u$ is a minimizer of the p-Dirichlet integral as well, we also have $-u \in DG(\Omega)$.

7.2. <u>De Giorgi class and boundedness</u>. Assume that $u \in DG_p(\Omega)$. Let $0 < \frac{R}{2} < r < R \le \frac{\operatorname{diam}(X)}{3}$ such that $B(x,R) \subset \Omega$. Then

$$g_{(u-k)_+} \le g_u \chi_{A(k,R)}$$

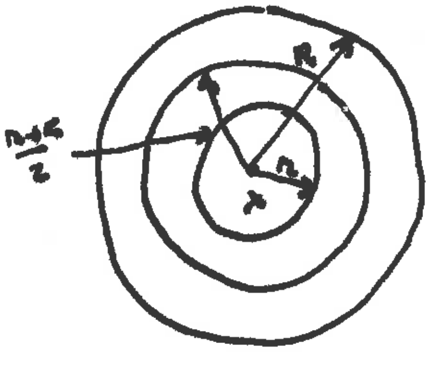
almost everywhere in $B(x,R)$. Since $u \in DG(\Omega)$, we have

$$\int_{B(x,\frac{r+R}{2})} g^p_{(u-k)_+}\,d\mu \le \frac{C}{(R-\frac{r+R}{2})^p} \int_{B(x,R)} (u-k)_+^p\,d\mu$$

$$= \frac{2C}{(R-r)^p} \int_{B(x,R)} (u-k)_+^p\,d\mu.$$

Let η be a $\frac{C}{R-r}$-Lipschitz cutoff function with $0 \le \eta \le 1$, $\eta = 0$ in $X \setminus B(x,\frac{r+R}{2})$ and $\eta = 1$ in $B(x,r)$. Let

$$v = \eta(u-k)_+.$$



By 2.22 we have

$$g_v \le g_{(u-k)_+}\eta + (u-k)_+ g_\eta$$

$$\le g_{(u-k)_+} + \frac{C}{R-r}(u-k)_+.$$

Thus

$$\int_{B(x,\frac{r+R}{2})} g_v^p\,d\mu \le 2^p \int_{B(x,\frac{r+R}{2})} g^p_{(u-k)_+}\,d\mu + 2^p \int_{B(x,\frac{r+R}{2})} \frac{C}{(R-r)^p}(u-k)_+^p\,d\mu$$

$$\le \frac{C}{(R-r)^p} \int_{B(x,R)} (u-k)_+^p\,d\mu.$$

Since $v=\eta(u-k)_+ \in N^{1,p}(B(x,\frac{r+R}{2}))$, by Remark 6.4 there exists $q>p$ such that

$$
\begin{aligned}
\left(\fint_{B(x,r)} (u-k)_+^q \, d\mu\right)^{\frac{p}{q}} &\le C \left(\fint_{B(x,\frac{r+R}{2})} |v|^q \, d\mu\right)^{\frac{p}{q}} \\
&\le C \left(\frac{r+R}{2}\right)^p \left(\fint_{B(x,\frac{r+R}{2})} g_v^p \, d\mu\right) \\
&\le C R^p \frac{1}{(R-r)^p} \fint_{B(x,R)} (u-k)_+^p \, d\mu .
\end{aligned}
$$

By Hölder's inequality

$$
\begin{aligned}
\fint_{B(x,r)} (u-k)_+^p \, d\mu &\le \left(\fint_{B(x,r)} (u-k)_+^q \, d\mu\right)^{\frac{p}{q}} \left(\frac{\mu(A(k,r))}{\mu(B(x,r))}\right)^{1-\frac{p}{q}} \\
&\le C \frac{R^p}{(R-r)^p} \left(\frac{\mu(A(k,r))}{\mu(B(x,r))}\right)^{1-\frac{p}{q}} \fint_{B(x,R)} (u-k)_+^p \, d\mu .
\end{aligned}
\qquad (*)
$$

Let $h<k$. Then

$$
\begin{aligned}
(k-h)^p \mu(A(k,r)) &= \int_{A(k,r)} (k-h)^p \, d\mu \\
&\underset{\substack{u>k \\ \text{in } A(k,r)}}{\le} \int_{A(k,r)} (u-h)^p \, d\mu \\
&\underset{A(k,r)\subset A(h,r)}{\le} \int_{A(h,R)} (u-h)^p \, d\mu .
\end{aligned}
\qquad (**)
$$

Let

$$u(h,r) = \left(\fint_{B(x,r)} (u-h)_+^p \, d\mu \right)^{\frac{1}{p}}.$$

Then by $(*)$

$$u(k,r) \le c \, \frac{R}{R-r} \left(\frac{\mu(A(k,r))}{\mu(B(x,r))} \right)^{\frac{1}{p}-\frac{1}{q}} u(h,R),$$

where by $(**)$

$$\begin{aligned}
\mu(A(k,r)) &\le \frac{1}{(k-h)^p} \int_{A(k,r)} (u-h)^p \, d\mu \\
&= \frac{\mu(B(x,r))}{(k-h)^p} \fint_{B(x,r)} (u-h)_+^p \, d\mu \\
&\underset{\text{doubling}}{\le} c \, \frac{\mu(B(x,R))}{(k-h)^p} \fint_{B(x,R)} (u-h)_+^p \, d\mu \\
&= c \, \frac{\mu(B(x,R))}{(k-h)^p} \, u(h,R)^p
\end{aligned}$$

which implies

$$u(k,r) \le c \, \frac{R}{R-r} (k-h)^{-\theta} u(h,R)^{1+\theta} \qquad (***)$$

with $\theta = 1 - \frac{p}{q} > 0$.

<u>Claim</u>: If we choose $d^{\theta} = c \, 2^{\frac{(1+\theta)^2}{\theta}+1} u(k_0,R)^{\theta}$, where $k_0 \in \mathbb{R}$, c and θ are as in $(***)$, then

$$u(k_0+d, \tfrac{R}{2}) = 0.$$

<u>Proof of the claim:</u> Let

$$k_m = k_0 + d(1-2^{-m}) \quad \text{and} \quad r_m = \frac{R}{2} + 2^{-m-1}R, \quad m = 0,1,2,\dots$$

Then $k_0 \le k_m < k_0 + d$ and $k_m \nearrow k_0 + d$, $\frac{R}{2} < r_m \le R$ and $r_m \searrow \frac{R}{2}$. We show that

$$u(k_m, r_m) \le 2^{-\mu m} u(k_0, R)$$

for every $m = 0,1,2,\dots$ with $\mu = \frac{1+\theta}{\theta}$.

$\boxed{m=0}$ Clear, since $r_0 = R$.

$\boxed{m \Rightarrow m+1}$ By (***) we have

$$u(k_{m+1}, r_{m+1}) \le c \frac{r_m}{r_m - r_{m+1}} (k_{m+1} - k_m)^{-\theta} u(k_m, r_m)^{1+\theta}$$

$$\le c \frac{R}{2^{-m-2}R} (2^{-m-1} d)^{-\theta} u(k_0, R)^{1+\theta} \, 2^{-\mu m(1+\theta)}$$

the induction assumption

$$= 2^{-\mu(m+1)} u(k_0, R).$$

substitute ~~θ~~ d^{θ}.

This implies

$$\lim_{m\to\infty} u(k_m, r_m) \le \lim_{m\to\infty} 2^{-\mu m} u(k_0, R) = 0.$$

On the other hand

$$0 \le u(k_0+d, \tfrac{R}{2}) = \left(\fint_{B(x,\frac{R}{2})} (u-(k_0+d))_+^p \, d\mu \right)^{\frac{1}{p}}$$

$$\le \left(\frac{\mu(B(x,r_m))}{\mu(B(x,\frac{R}{2}))} \fint_{B(x,r_m)} (u-k_m)_+^p \, d\mu \right)^{\frac{1}{p}}$$

$B(x,\frac{R}{2}) \subset B(x,r_m)$

$k_m < k_0 + d$

$$\le c \left(\fint_{B(x,r_m)} (u-k_m)_+^p \, d\mu \right)^{\frac{1}{p}}$$

$B(x,r_m) \subset B(x,R)$, μ doubling

$$= c\, u(k_m, r_m) \to 0 \text{ as } m \to \infty.$$

This implies that $u(k_0+d, \frac{R}{2}) = 0$.

7.3. <u>The weak $\left\{\begin{matrix}\text{maximum principle} \\ \text{Harnack inequality}\end{matrix}\right.$</u>. Let $\Omega \subset X$ be open, $B(x,R) \subset \Omega$ with $0 < R \le \frac{\operatorname{diam}(X)}{3}$ and $k_0 \in \mathbb{R}$. If $u \in DG(\Omega)$, then exists a constant c such that

$$\operatorname*{ess\,sup}_{B(x,\frac{R}{2})} u \le k_0 + c \left(\fint_{B(x,R)} (u-k_0)_+^p \, d\mu \right)^{\frac{1}{p}}.$$

The constant c depends only on the constant in the De Giorgi condition, the doubling constant and the constants in the Poincaré inequality. In particular, it is independent of $B(x,r)$.

Proof: By the claim on page 7/7, we have

$$u\left(k_0+d, \frac{R}{2}\right) = \left(\fint_{B(x,\frac{R}{2})} (u-(k_0+d))_+^p \, d\mu \right)^{\frac{1}{p}} = 0$$

which implies

$$\mu\left(\left\{ y \in B(x,\tfrac{R}{2}) : u(y) > k_0 + d \right\}\right) = 0$$

and thus

$$\operatorname*{ess\,sup}_{B(x,\frac{R}{2})} u \le k_0 + d.$$

$$\le k_0 + C\, u(k_0, R)$$

$\uparrow$ the formula for d on page 7/7

$$= k_0 + C \left(\fint_{B(x,R)} (u-k_0)_+^p \, d\mu \right)^{\frac{1}{p}}. \qquad \square$$

Assume, in addition, that $-u \in DG(\Omega)$. Then

$$\operatorname*{ess\,sup}_{B(x,\frac{R}{2})} |u| \le \operatorname*{ess\,sup}_{B(x,R)} u + \operatorname*{ess\,sup}_{B(x,R)} (-u)$$

$$\le C\left(\fint_{B(x,R)} u_+^p \, d\mu \right)^{\frac{1}{p}} + C \left(\fint_{B(x,R)} (-u)_+^p \, d\mu \right)^{\frac{1}{p}}$$

$$\le C \left(\fint_{B(x,R)} |u|^p \, d\mu \right)^{\frac{1}{p}}.$$

7.4. De Giorgi class and Hölder continuity. The goal of this section is to prove De Giorgi's theorem, which states that functions in De Giorgi's class are locally Hölder continuous.

We assume that X is complete and that it supports a p-Poincaré inequality

$$\fint_B |u-u_B|\,d\mu \le C \operatorname{diam}(B)\left(\fint_{\lambda B} g_u^p\,d\mu\right)^{\frac{1}{p}}$$

for every ball $B\subset X$, see 3.1. By a result of Keith and Zhong [Annals of Math 2008] there exists $q<p$ such that

$$\fint_B |u-u_B|\,d\mu \le C \operatorname{diam}(B)\left(\fint_{\lambda B} g_u^q\,d\mu\right)^{\frac{1}{q}}.$$

This shows that a Poincaré inequality is a self-improving property.

By Theorem 3.8 there exists $t>q$ such that

$$\left(\fint_B |u-u_B|^t\,d\mu\right)^{\frac{1}{t}} \le C \operatorname{diam}(B)\left(\fint_{\lambda' B} g_u^q\,d\mu\right)^{\frac{1}{q}}.$$

It is possible to choose t such that $1<q<p<t$.

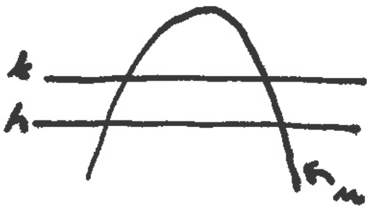
Assume that $u\in DG(\Omega)$ and let $0<r<R<\dfrac{\operatorname{diam}(X)}{3\lambda'}$ be such that $B(x,2\lambda' R)\subset\Omega$. Assume that

$$\mu(A(h,R))\le\gamma\,\mu(B(x,R))$$

for some γ with $0<\gamma<1$.

Let $k > h$ and

$$v(y) = \min\{u(y), k\} - \min\{u(y), h\}.$$



Since $u \in N^{1,p}(\Omega)$, we have $v \in N^{1,p}(\Omega)$. By Lemma 6.3

$$(k-h)\mu(A(k,R)) = \int_{A(k,R)} v \, d\mu \le \int_{B(x,R)} |v| \, d\mu$$

$u > k \Rightarrow \min\{u,k\} = k$ and $\min\{u,h\} = h$

$$\le \mu(B(x,R))^{1-\frac{1}{q}} \Big(\int_{B(x,R)} |v|^q \, d\mu \Big)^{\frac{1}{q}}$$

Hölder

$$\le c R \mu(B(x,R))^{1-\frac{1}{q}} \Big(\int_{B(x,\lambda' R)} g_v^q \, d\mu \Big)^{\frac{1}{q}}$$

Lemma 6.3

$A = \{ y \in B(x,R) : |v(y)| > 0 \}$

$\mu(A) \le \mu(A(h,R)) \le \gamma \mu(B(x,R))$

$$= c R \mu(B(x,R))^{1-\frac{1}{q}} \Big(\int_{A(h,\lambda' R) \setminus A(k,\lambda' R)} g_v^q \, d\mu \Big)^{\frac{1}{q}}$$

$g_v = g_v \chi_{\{h < u < k\}}$ μ-a.e.

$$\le c R \mu(B(x,R))^{1-\frac{1}{q}} \Big(\int_{A(h,\lambda' R)} g_v^p \, d\mu \Big)^{\frac{1}{p}} \big(\mu(A(h,\lambda' R)) - \mu(A(k,\lambda' R)) \big)^{\frac{1}{q} - \frac{1}{p}}$$

Hölder, $q < p$

Since $u \in DG(\Omega)$, we have

$$\int_{A(h,\lambda' R)} g_v^p \, d\mu \underset{\substack{\uparrow \\ g_v \le g_u \text{ in } A(h,\lambda' R)}}{\le} \int_{A(h,\lambda' R)} g_u^p \, d\mu$$

$$\underset{\substack{\uparrow \\ \mu \in DG(\Omega),}}{\le} \frac{c}{\left(2\lambda' R - \lambda' R\right)^p} \int_{A(h,2\lambda' R)} (u-h)^p \, d\mu$$

This gives

$$(k-h)\mu(A(k,R)) \le c\,\mu(B(x,R))^{1-\frac{1}{q}} \cdot \Big(\int_{A(h,2\lambda' R)} (u-h)^p \, d\mu\Big)^{\frac{1}{p}} \big(\mu(A(h,\lambda' R)) - \mu(A(k,\lambda' R))\big)^{\frac{1}{q}-\frac{1}{p}}. \quad (*)$$

We denote

$$m(R) = \operatorname*{ess\,inf}_{B(x,R)} u \quad \text{and} \quad M(R) = \operatorname*{ess\,sup}_{B(x,R)} u.$$

By the results in 7.3 we have $M(R) < \infty$.

7.5. <u>Lemma</u>. Assume that $u \in DG(\Omega)$ is locally bounded from below. Let $M = M(2\lambda' R)$, $m = m(2\lambda' R)$ and $k_0 = \frac{M+m}{2}$. If $\mu(A(k_0,R)) \le \gamma\, \mu(B(x,R))$ for some $0<\gamma<1$, then

$$\lim_{k \to M} \mu(A(k,R)) = 0.$$

Proof: Let

$$k_i = M - 2^{-(i+1)}(M-m), \quad i = 0, 1, 2, \dots$$

Then $k_i \uparrow M$ as $i \to \infty$ and $k_0 = \frac{M+m}{2}$. Note that

$$M - k_{i-1} = 2^{-i}(M-m)$$

and

$$k_i - k_{i-1} = 2^{-(i+1)}(M-m).$$

(*) implies

$$(k_i - k_{i-1})\,\mu(A(k_i, R)) \le c\,\mu(B(x,R))^{1-\frac{1}{q}}$$

$$\cdot \Big(\int_{A(k_{i-1}, 2\lambda' R)} (u - k_{i-1})^p \, d\mu\Big)^{\frac{1}{p}} \Big(\mu(A(k_{i-1}, \lambda' R)) - \mu(A(k_i, \lambda' R))\Big)^{\frac{1}{q} - \frac{1}{p}},$$

and ~~thus~~ since $u - k_{i-1} \le M - k_{i-1}$ on $A(k_{i-1}, 2\lambda' R)$, we have

$$2^{-(i+1)}(M-m)\,\mu(A(k_i, R)) \le c\,\mu(B(x,R))^{1-\frac{1}{q}+\frac{1}{p}}$$

$$\cdot\, 2^{-i}(M-m)\Big(\mu(A(k_{i-1}, \lambda' R)) - \mu(A(k_i, \lambda' R))\Big)^{\frac{1}{q} - \frac{1}{p}}.$$

Note that if $l \ge i$, then $\mu(A(k_l, R)) \le \mu(A(k_i, R))$ and thus

$$\mu(A(k_l, R)) \le c\,\mu(B(x,R))^{1-\frac{1}{q}+\frac{1}{p}}$$

$$\cdot \Big(\mu(A(k_{i-1}, \lambda' R)) - \mu(A(k_i, \lambda' R))\Big)^{\frac{1}{q} - \frac{1}{p}}.$$

This gives

$$l\,\mu(A(k_l,R))^{\frac{pq}{p-q}} = \sum_{i=1}^{l} \mu(A(k_l,R))^{\frac{pq}{p-q}}$$

independent of i

$$\le C\,\mu(B(x,R))^{\frac{pq}{p-q}\cdot\left(1-\frac{p-q}{pq}\right)} \cdot \left(\mu(A(k_0,\lambda' R)) - \mu(A(k_l,\lambda' R))\right)$$

telescoping series

$$\le C\,\mu(B(x,R))^{\frac{pq}{p-q}-1} \cdot \mu(B(x,\lambda' R))$$

$$\le C\,\mu(B(x,R))^{\frac{pq}{p-q}}$$

μ doubling

and consequently

$$\lim_{l\to\infty} \mu(A(k_l,R)) \le \lim_{l\to\infty} \frac{\mu(B(x,R))^{\frac{pq}{p-q}}}{l} = 0$$

The claim follows, since $\mu(A(k,R))$ is a decreasing function of k.

$\square$

We denote

$$\operatorname{osc}(u, B(x,R)) = M(R) - m(R) = \operatorname*{ess\,sup}_{B(x,R)} u - \operatorname*{ess\,inf}_{B(x,R)} u$$

7.6. Theorem. Assume that both u and $-u$ belong to $DG(\Omega)$. Then

$$\operatorname{osc}(u, B(x,\lambda' r)) \le 4^{\alpha}\left(\frac{r}{R}\right)^{\alpha} \operatorname{osc}(u, B(x,\lambda' R))$$

for some $0<\alpha<1$ whenever $0<r<R<\frac{\operatorname{diam}(X)}{3\lambda'}$ with $B(x,2\lambda' R)\subset\Omega$.
In particular, u is locally Hölder continuous in Ω.

Proof: Let

$$k_0 = \frac{M+m}{2},$$

where M and m are as in Lemma 7.5. If

$$\mu(A(k_0,R)) > \frac{\mu(B(x,R))}{2},$$

then

$$\mu(\{y\in B(x,R): -u(y) > -k_0\}) \le \frac{\mu(B(x,R))}{2}$$

and we may consider $-u$ instead of u in the discussion below. Therefore, without loss of generality, we may assume that

$$\mu(A(k_0,R)) \le \frac{\mu(B(x,R))}{2}.$$

By the weak maximum principle (in 7.3) with k_0 replaced by

$$k_i = M - 2^{-i-1}(M-m), \quad i=0,1,2,\dots,$$

we have

$$M\left(\frac{\lambda^i R}{2}\right) \le k_i + C\left(M(2\lambda^i R) - k_i\right)\left(\frac{\mu(A(k_i,R))}{\mu(B(x,R))}\right)^{\frac{1}{p}}.$$

By Lemma 7.5, for large enough i, we have

$$C\left(\frac{\mu(A(k_i,R))}{\mu(B(x,R))}\right)^{\frac{1}{p}} < \frac{1}{2}.$$

This gives

$$M\left(\frac{\lambda^i R}{2}\right) \le k_i + \frac{1}{2}\Big(\underbrace{M(2\lambda^i R)}_{=M} - k_i\Big)$$

$$= M(2\lambda^i R) - 2^{-i-1}\left(M(2\lambda^i R) - m(2\lambda^i R)\right)$$

the definition of k_i

$$+ \frac{1}{2}\cdot 2^{-i-1}\left(M(2\lambda^i R) - m(2\lambda^i R)\right)$$

$$= M(2\lambda^i R) - 2^{-i-2}\left(M(2\lambda^i R) - m(2\lambda^i R)\right)$$

and thus

$$M\left(\frac{\lambda^i R}{2}\right) - m\left(\frac{\lambda^i R}{2}\right) \le M\left(\frac{\lambda^i R}{2}\right) - m(2\lambda^i R)$$

$$\le M(2\lambda^i R) - m(2\lambda^i R) - 2^{-i-2}\left(M(2\lambda^i R) - m(2\lambda^i R)\right)$$

$$= \left(M(2\lambda^i R) - m(2\lambda^i R)\right)\left(1 - 2^{-i-2}\right).$$

This gives

$$\operatorname{osc}\left(u, B\left(x, \frac{\lambda' R}{2}\right)\right) \le \gamma \operatorname{osc}(u, B(x, 2\lambda' R))$$

with $\gamma = 1 - 2^{-i'-2} < 1$. To complete the proof, we iterate this estimate. Let $j \in \mathbb{N}$ such that

$$4^{j-1} \le \frac{R}{r} < 4^{j}.$$

Then

$$\operatorname{osc}(u, B(x, \lambda' r)) \le \gamma^{j-1} \left(\operatorname{osc}(u, B(x, 4^{j-1}\lambda' r))\right)$$

$$\le \gamma^{j-1} \operatorname{osc}(u, B(x, \lambda' R))$$

$4^{j-1} r \le R$

$$\le \frac{1}{\gamma}\left(\frac{r}{R}\right)^{\alpha} \operatorname{osc}(u, B(x, \lambda' R))$$

$$\gamma^{j-1} = 4^{\log_4 \gamma^{j-1}} = 4^{(j-1)\log_4 \gamma}$$

$$= 4^{j \log_4 \gamma}\, 4^{-\log_4 \gamma}$$

$$= \frac{1}{\gamma} 4^{j \log_4 \gamma}$$

$$\le \frac{1}{\gamma}\left(\frac{R}{r}\right)^{\log_4 \gamma} = \frac{1}{\gamma}\left(\frac{r}{R}\right)^{\alpha},$$

$\log_4 \gamma < 0$

$\alpha = -\log_4 \gamma > 0$

We have $\alpha < 1$ by choosing i large enough.

This implies that u is locally Hölder continuous. To see this, let $y, z \in B(x, r)$. Then

$$|u(y) - u(z)| \leq \operatorname{osc}(u, B(x, 2|y-z|))$$

$$\leq C \left(\frac{|y-z|}{R}\right)^{\alpha} \operatorname{osc}(u, B(x, \lambda' R))$$

$2|y-z| < 4r < \lambda' R$

$$\leq C \left(\frac{|y-z|}{R}\right)^{\alpha} \operatorname{ess\,sup}_{B(x, \lambda' R)} |u|$$

$$\leq C \left(\frac{|y-z|}{R}\right)^{\alpha} \left(\fint_{B(x, 2\lambda' R)} |u|^p \, d\mu\right)^{\frac{1}{p}}.$$

The weak maximum principle in 7.3

$$\boxed{\text{THE END}}$$