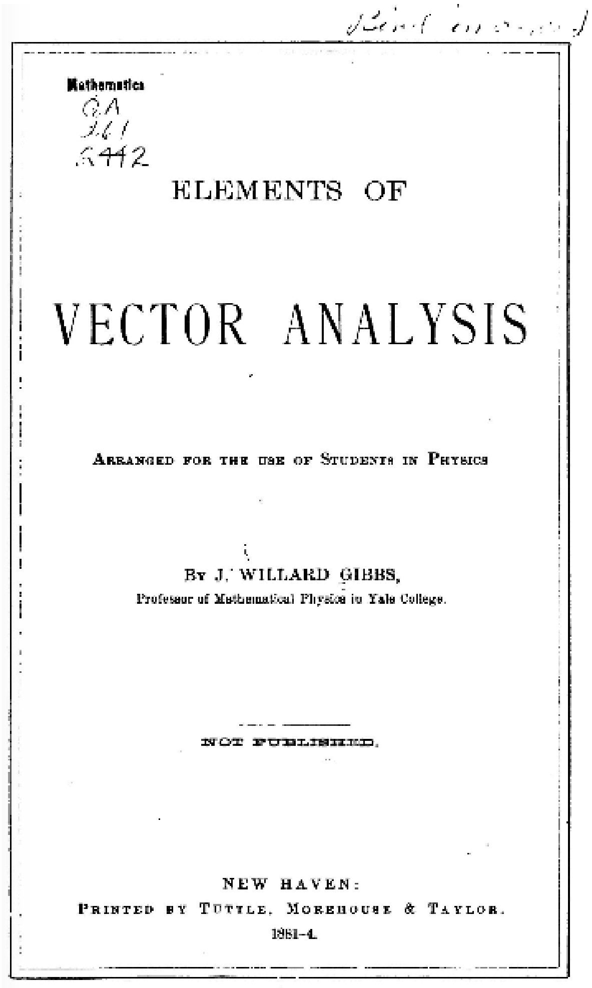
Advanced Field Theory

ELEC-E4730

Lecture 2 — 4 March 2019



Mathematics
QA
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ELEMENTS OF

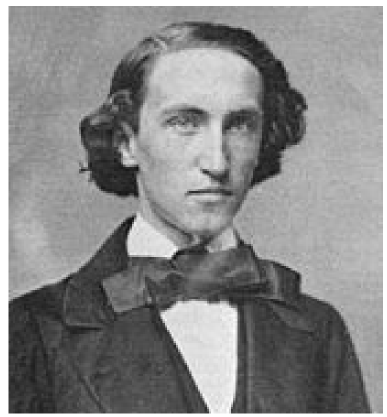
VECTOR ANALYSIS

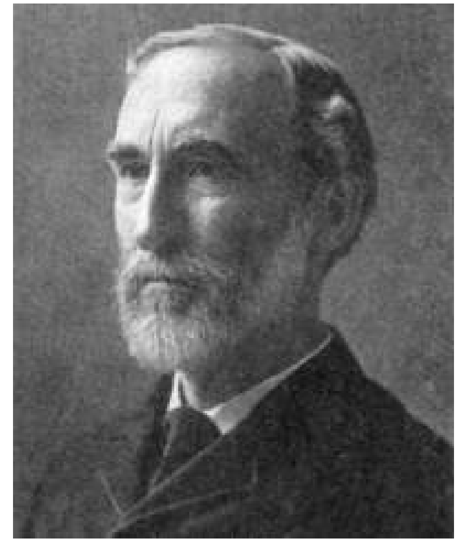
ARRANGED FOR THE USE OF STUDENTS IN PHYSICS

BY J. WILLARD GIBBS,
Professor of Mathematical Physics in Yale College.

NOT PUBLISHED.

NEW HAVEN:
PRINTED BY TUTTLE, MOREHOUSE & TAYLOR.
1881–4.





Josiah Willard Gibbs (1839–1903)

Josiah Willard Gibbs:

Elements of Vector Analysis: Arranged for the Use of Students in Physics

Published privately 1884

106. An expression of the form

$$\alpha\lambda.\rho+\beta\mu.\rho+\text{etc.}$$

evidently represents a linear function of ρ, and may be conveniently written in the form

$$\{\alpha\lambda+\beta\mu+\text{etc.}\}.\rho.$$

The expression $\rho.\alpha\lambda+\rho.\beta\mu+\text{etc.},$

or $\rho.\{\alpha\lambda+\beta\mu+\text{etc.}\},$

also represents a linear function of ρ, which is, in general, different from the preceding, and will be called its *conjugate*.

107. *Def.*—An expression of the form $\alpha\lambda$ or $\beta\mu$ will be called a *dyad*. An expression consisting of any number of dyads united by the signs $+$ or $-$ will be called a *dyadic binomial, trinomial*, etc., as the case may be, or more briefly, a *dyadic*. The latter term will be used so as to include the case of a single dyad. When we desire to express a dyadic by a single letter, the Greek capitals will be used, except such as are like the Roman, and also Δ and Σ. The letter I will also be used to represent a certain dyadic, to be mentioned hereafter.

DYADIC DEFINED FIRST TIME, 1884

Since any linear vector function may be expressed by means of a dyadic (as we shall see more particularly hereafter, see No. 110), the study of such functions, which is evidently of primary importance in the theory of vectors, may be reduced to that of dyadics.

108. *Def.*—Any two dyadics Φ and Ψ are equal,

when $\Phi.\rho=\Psi.\rho$ for all values of ρ,

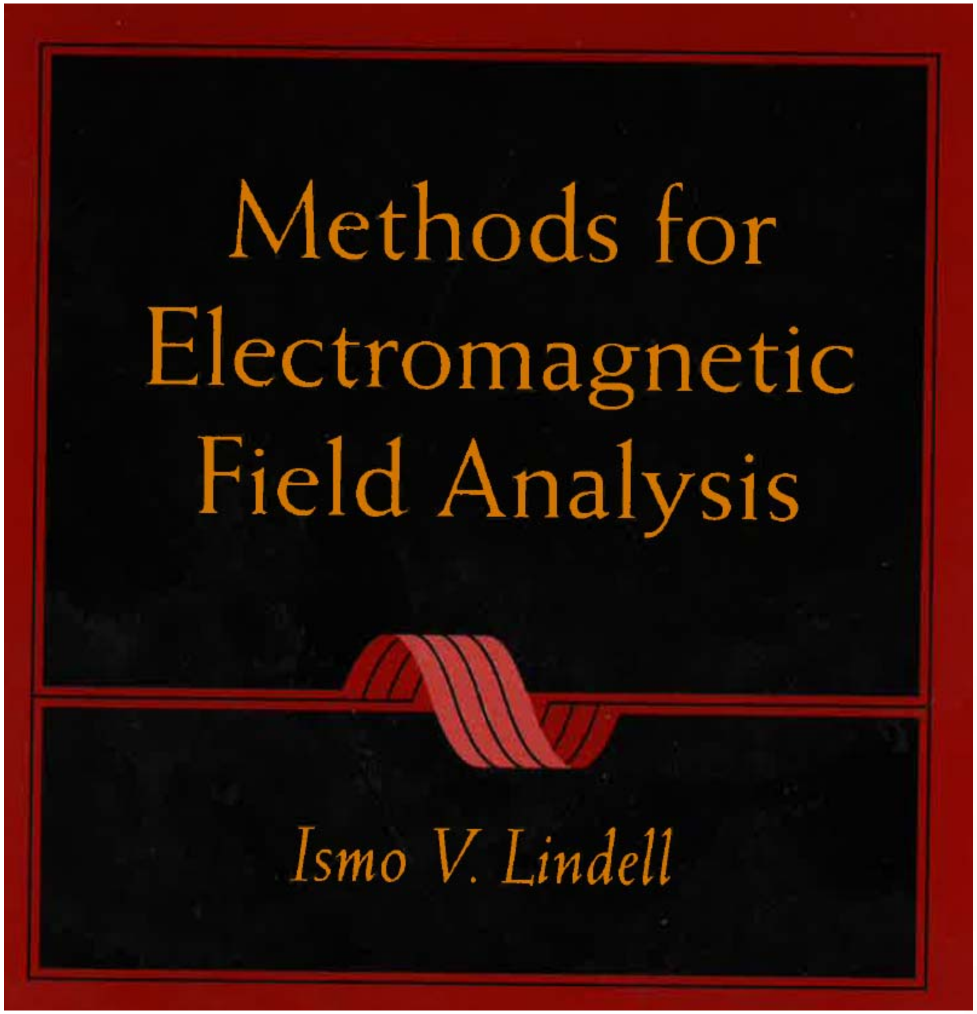
or, when $\rho.\Phi=\rho.\Psi$ for all values of ρ,

or, when $\sigma.\Phi.\rho=\sigma.\Psi.\rho$ for all values of σ and of ρ.

The third condition is easily shown to be equivalent both to the first and to the second. The three conditions are therefore equivalent.

It follows that $\Phi=\Psi$, if $\Phi.\rho=\Psi.\rho$, or $\rho.\Phi=\rho.\Psi$, for three non-

But my critic does not like the notations $|\Phi|$, Φ_s, $\Phi_\times$. His ridicule, indeed, reaches high-water mark in the paragraphs in which he mentions them. Concerning another notation, $\Phi{}^\times_\times\Phi$ (defined in NATURE, vol. xliii. p. 513), he exclaims, "Thus burden after burden, in the form of new notation, is added apparently for the sole purpose of exercising the faculty of memory." He would vastly prefer, it would appear, to write with Hamilton $m\phi'^{-1}$, "when m represents what the unit volume becomes under the influence of the linear operator." But this notation is only apparently compact, since the m requires explanation. Moreover, if a strain were given in what Hamilton calls the standard trinomial form, to write out the formula for the operator on surfaces in that standard form by the use of the expression $m\phi'^{-1}$ would require, it seems to me, ten (if not fifty) times the effort of memory and of ingenuity, which would be required for the same purpose with the use of $\frac{1}{2}\Phi{}^\times_\times\Phi$.



Appendix D

Dyadic identities

Definitions

$$\overline{\overline{A}}{}^2 = \overline{\overline{A}}\cdot\overline{\overline{A}}, \qquad \overline{\overline{A}}{}^{-2} = (\overline{\overline{A}}{}^{-1})^2 = (\overline{\overline{A}}{}^2)^{-1}$$

$$\overline{\overline{A}}{}^{(2)} = \frac{1}{2}\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}, \qquad \overline{\overline{A}}{}^{(-2)} = (\overline{\overline{A}}{}^{-1})^{(2)} = (\overline{\overline{A}}{}^{(2)})^{-1}$$

$$\mathrm{tr}\overline{\overline{A}} = \overline{\overline{A}}:\overline{\overline{I}}$$

$$\mathrm{spm}\overline{\overline{A}} = \mathrm{tr}\overline{\overline{A}}{}^{(2)} = \frac{1}{2}\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}:\overline{\overline{I}}$$

$$\det\overline{\overline{A}} = \frac{1}{6}\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}:\overline{\overline{A}}$$

$$\det\overline{\overline{A}} \neq 0 \quad\leftrightarrow\quad \overline{\overline{A}}\ \text{complete}$$

$$\det\overline{\overline{A}} = 0 \quad\leftrightarrow\quad \overline{\overline{A}}\ \text{planar}$$

$$\overline{\overline{A}}{}^{(2)} = 0 \quad\leftrightarrow\quad \overline{\overline{A}}\ \text{linear.}$$

Identities

$\mathrm{tr}(\bar{\bar{A}}\cdot\bar{\bar{B}})$

$$\overline{\overline{A}}{}^\times_\times\overline{\overline{B}} = \overline{\overline{B}}{}^\times_\times\overline{\overline{A}} = \left[(\overline{\overline{A}}:\overline{\overline{I}})(\overline{\overline{B}}:\overline{\overline{I}}) - \overline{\overline{A}}:\overline{\overline{B}}{}^T\right]\overline{\overline{I}} -$$

$$(\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{B}}{}^T - (\overline{\overline{B}}:\overline{\overline{I}})\overline{\overline{A}}{}^T + [\overline{\overline{A}}\cdot\overline{\overline{B}} + \overline{\overline{B}}\cdot\overline{\overline{A}}]^T$$

$$\overline{\overline{A}}{}^\times_\times\overline{\overline{I}} = (\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}} - \overline{\overline{A}}{}^T$$

$$(\mathbf{a}\times\overline{\overline{I}})\,{}^{\times}_{\times}\overline{\overline{I}} = \mathbf{a}\times\overline{\overline{I}}$$

$$\overline{\overline{S}}\,{}^{\times}_{\times}\overline{\overline{I}} = -\overline{\overline{S}} \quad (\overline{\overline{S}}\ \text{symmetric, trace free})$$

$$(\mathbf{a}\times\overline{\overline{I}})\,{}^{\times}_{\times}(\mathbf{b}\times\overline{\overline{I}}) = \mathbf{ab}+\mathbf{ba}$$

$$\overline{\overline{S}}\,{}^{\times}_{\times}(\mathbf{a}\times\overline{\overline{I}}) = (\overline{\overline{S}}\cdot\mathbf{a})\times\overline{\overline{I}} \quad (\overline{\overline{S}}\ \text{symmetric})$$

$$(\overline{\overline{A}}\times\mathbf{a})\,{}^{\times}_{\times}(\overline{\overline{B}}\times\mathbf{a}) = (\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{B}})\cdot\mathbf{aa}$$

$$(\mathbf{a}\times\overline{\overline{A}})\,{}^{\times}_{\times}(\mathbf{a}\times\overline{\overline{B}}) = \mathbf{aa}\cdot(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{B}})$$

$$(\mathbf{a}\times\overline{\overline{I}})\,{}^{\times}_{\times}(\mathbf{a}\times\overline{\overline{I}}) = 2\mathbf{aa}$$

$$\overline{\overline{A}}\,{}^{\times}_{\times}(\overline{\overline{B}}\,{}^{\times}_{\times}\overline{\overline{C}}) = (\overline{\overline{A}}:\overline{\overline{C}})\overline{\overline{B}} + (\overline{\overline{A}}:\overline{\overline{B}})\overline{\overline{C}} - \overline{\overline{B}}\cdot\overline{\overline{A}}^T\cdot\overline{\overline{C}} - \overline{\overline{C}}\cdot\overline{\overline{A}}^T\cdot\overline{\overline{B}}$$

$$\overline{\overline{I}}\,{}^{\times}_{\times}(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{B}}) = (\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{B}} + (\overline{\overline{B}}:\overline{\overline{I}})\overline{\overline{A}} - (\overline{\overline{A}}\cdot\overline{\overline{B}}+\overline{\overline{B}}\cdot\overline{\overline{A}})$$

$$\overline{\overline{I}}\,{}^{\times}_{\times}(\overline{\overline{I}}\,{}^{\times}_{\times}\overline{\overline{A}}) = \overline{\overline{A}} + (\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}}$$

$$\overline{\overline{I}}\,{}^{\times}_{\times}(\overline{\overline{I}}\,{}^{\times}_{\times}\overline{\overline{I}}) = 4\overline{\overline{I}}$$

$$(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})\,{}^{\times}_{\times}(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}) = 8(\overline{\overline{A}}^{(2)})^{(2)} = 8\det\overline{\overline{A}}\,\overline{\overline{A}}$$

$$(\overline{\overline{A}}^{(2)})^{(2)} = \overline{\overline{A}}\det\overline{\overline{A}}$$

$$\det(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}) = 8\det{}^2\overline{\overline{A}}$$

$\frac{1}{2}(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})\cdot(\overline{\overline{B}}\,{}^{\times}_{\times}\overline{\overline{C}})$
$= (\overline{\overline{A}}\cdot\overline{\overline{B}})\,{}^{\times}_{\times}(\overline{\overline{A}}\cdot\overline{\overline{C}})$

$$(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{B}})\cdot(\overline{\overline{C}}\,{}^{\times}_{\times}\overline{\overline{D}}) = (\overline{\overline{A}}\cdot\overline{\overline{C}})\,{}^{\times}_{\times}(\overline{\overline{B}}\cdot\overline{\overline{D}}) + (\overline{\overline{A}}\cdot\overline{\overline{D}})\,{}^{\times}_{\times}(\overline{\overline{B}}\cdot\overline{\overline{C}})$$

$$(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})\cdot(\overline{\overline{B}}\,{}^{\times}_{\times}\overline{\overline{B}}) = 2(\overline{\overline{A}}\cdot\overline{\overline{B}})\,{}^{\times}_{\times}(\overline{\overline{A}}\cdot\overline{\overline{B}})$$

$$(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{B}})^2 = (\overline{\overline{A}}^2)\,{}^{\times}_{\times}(\overline{\overline{B}}^2) + (\overline{\overline{A}}\cdot\overline{\overline{B}})\,{}^{\times}_{\times}(\overline{\overline{A}}\cdot\overline{\overline{B}})$$

$$(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})^2 = 2(\overline{\overline{A}}^2)\,{}^{\times}_{\times}(\overline{\overline{A}}^2)$$

$$(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{I}})^2 = (\overline{\overline{A}}^2)\,{}^{\times}_{\times}\overline{\overline{I}} + \overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}$$

$$(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})^T\cdot\overline{\overline{A}} = \overline{\overline{A}}\cdot(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})^T = \frac{1}{3}(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}:\overline{\overline{A}})\overline{\overline{I}}$$

$$\overline{\overline{A}}^{(2)T}\cdot\overline{\overline{A}} = \overline{\overline{A}}\cdot\overline{\overline{A}}^{(2)T} = \det\overline{\overline{A}}\,\overline{\overline{I}}$$

$$\overline{\overline{A}}^{-1} = \frac{\overline{\overline{A}}^{(2)T}}{\det\overline{\overline{A}}} = \frac{3(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})^T}{\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}:\overline{\overline{A}}} \quad (\overline{\overline{A}}\ \text{complete})$$

$$\overline{\overline{A}}^{-1} = \frac{(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}^{(2)*})^T}{\overline{\overline{A}}^{(2)}:\overline{\overline{A}}^{(2)*}} \quad (\text{planar inverse})$$

$$\overline{\overline{A}}^{-1}\cdot\overline{\overline{A}} = \overline{\overline{I}} - \frac{\overline{\overline{A}}^{(2)*}\cdot\overline{\overline{A}}^{(2)T}}{\overline{\overline{A}}^{(2)}:\overline{\overline{A}}^{(2)*}} \quad (\overline{\overline{A}}\ \text{planar})$$

$$\overline{\overline{A}}^{-1} = \frac{\overline{\overline{A}}^T\,{}^{\times}_{\times}\mathbf{uu}}{\mathrm{spm}\overline{\overline{A}}} \quad (\text{two}-\text{dimensional inverse})$$

$$\overline{\overline{A}}^{-1}\cdot\overline{\overline{A}} = \overline{\overline{I}}_t = \overline{\overline{I}} - \mathbf{uu} \quad (\overline{\overline{A}}\ \text{two}-\text{dimensional})$$

$$\overline{\overline{A}}:\overline{\overline{B}} = \overline{\overline{B}}:\overline{\overline{A}} = (\overline{\overline{A}}\cdot\overline{\overline{B}}^T):\overline{\overline{I}}$$

$$(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{B}}):\overline{\overline{C}} = \overline{\overline{A}}:(\overline{\overline{B}}\,{}^{\times}_{\times}\overline{\overline{C}}) \quad (\text{with all permutations})$$

$$\mathrm{spm}\overline{\overline{A}} = \mathrm{tr}\overline{\overline{A}}^{(2)} = \frac{1}{2}[(\overline{\overline{A}}:\overline{\overline{I}})^2 - \overline{\overline{A}}:\overline{\overline{A}}^T]$$

$$\mathrm{spm}(\overline{\overline{A}}\cdot\overline{\overline{B}}) = \mathrm{tr}(\overline{\overline{A}}^{(2)}\cdot\overline{\overline{B}}^{(2)}) = \overline{\overline{A}}^{(2)}:\overline{\overline{B}}^{(2)T}$$

$$\det\overline{\overline{A}} = \frac{1}{3}\overline{\overline{A}}^3:\overline{\overline{I}} - \frac{1}{2}(\overline{\overline{A}}^2:\overline{\overline{I}})(\overline{\overline{A}}:\overline{\overline{I}}) + \frac{1}{6}(\overline{\overline{A}}:\overline{\overline{I}})^3$$

$$\det(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}) = 8\det(\overline{\overline{A}}^{(2)}) = 8(\det\overline{\overline{A}})^2$$

$$\det(\overline{\overline{A}}\cdot\overline{\overline{B}}) = \det\overline{\overline{A}}\,\det\overline{\overline{B}}$$

$$\det(\overline{\overline{A}}\cdot\overline{\overline{B}} + \alpha\overline{\overline{I}}) = \det(\overline{\overline{B}}\cdot\overline{\overline{A}} + \alpha\overline{\overline{I}})$$

$$\mathbf{a}\times(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{B}}) = \overline{\overline{B}}\times(\mathbf{a}\cdot\overline{\overline{A}}) + \overline{\overline{A}}\times(\mathbf{a}\cdot\overline{\overline{B}})$$

$$(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{B}})\times\mathbf{a} = (\overline{\overline{A}}\cdot\mathbf{a})\times\overline{\overline{B}} + (\overline{\overline{B}}\cdot\mathbf{a})\times\overline{\overline{A}}$$

$$(\overline{\overline{A}}\cdot\mathbf{a})\times(\overline{\overline{A}}\cdot\mathbf{b}) = \frac{1}{2}(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})\cdot(\mathbf{a}\times\mathbf{b}) = \overline{\overline{A}}^{(2)}\cdot(\mathbf{a}\times\mathbf{b})$$

$$(\mathbf{a}\cdot\overline{\overline{A}})\times(\mathbf{b}\cdot\overline{\overline{A}}) = \frac{1}{2}(\mathbf{a}\times\mathbf{b})\cdot(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}) = (\mathbf{a}\times\mathbf{b})\cdot\overline{\overline{A}}^{(2)}$$

$\mathrm{adj}\,\overline{\overline{A}} = \overline{\overline{A}}^{(2)T} = \frac{1}{2}(\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}})^T$

$\det(\overline{\overline{A}}+\overline{\overline{B}}) = \det\overline{\overline{A}} + \det\overline{\overline{B}} + \overbrace{\mathrm{tr}(\mathrm{adj}\,\overline{\overline{A}}\cdot\overline{\overline{B}})}^{\frac{1}{2}\overline{\overline{A}}\,{}^{\times}_{\times}\overline{\overline{A}}:\overline{\overline{B}}} + \mathrm{tr}(\overline{\overline{A}}\cdot\mathrm{adj}\,\overline{\overline{B}})$