Aalto University
School of Science
Department of Computer Science
Combinatorics of Efficient Computations

Approximation Algorithms

Lecture 2: Set Cover & Shortest SuperString

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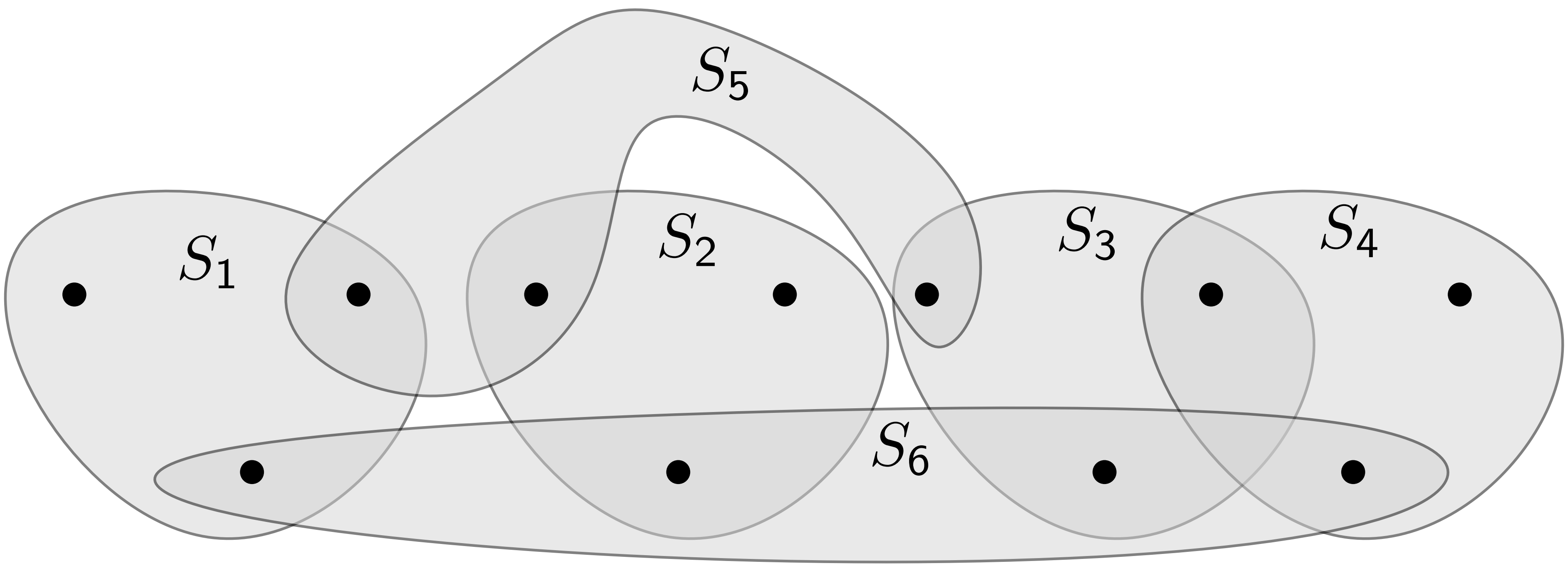
2019

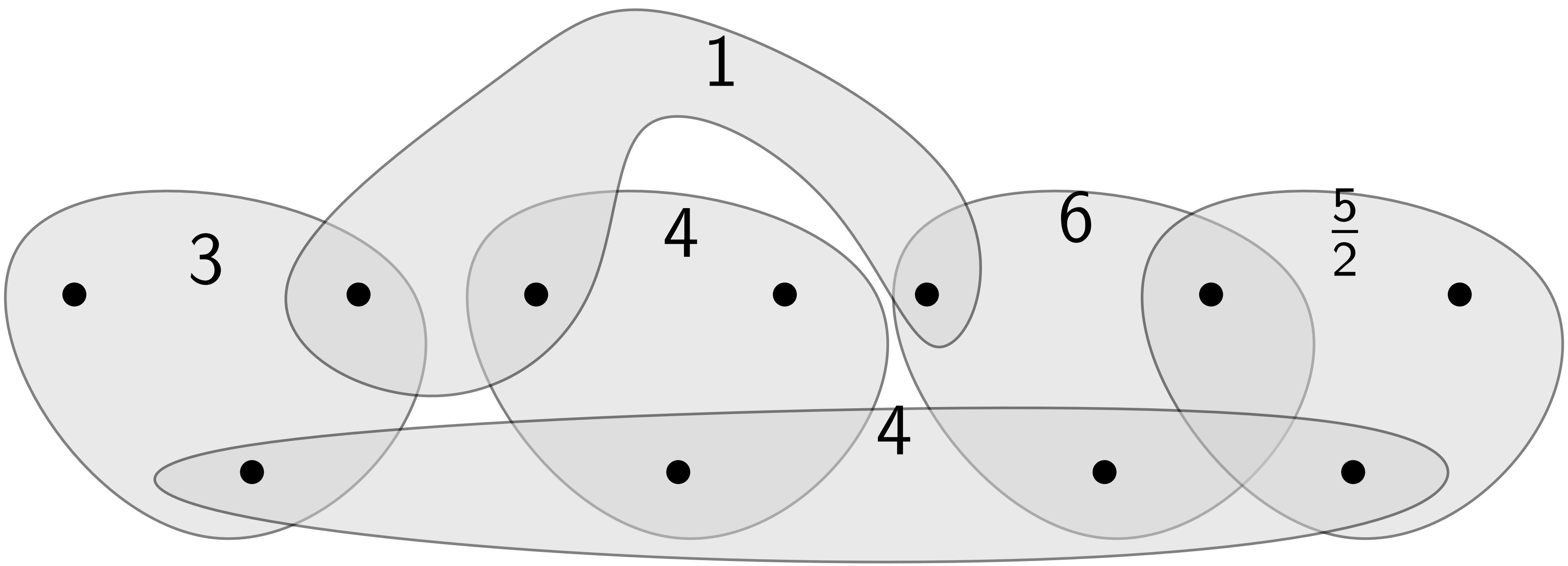
SetCover (card.)

- Given a **ground set** U and a collection $\mathcal{S}$ of **subsets** of U where $\bigcup \mathcal{S} = U$.
- Find a **cover** $\mathcal{S}' \subseteq \mathcal{S}$ of U (i.e. with $\bigcup \mathcal{S}' = U$) of minimum cardinality.



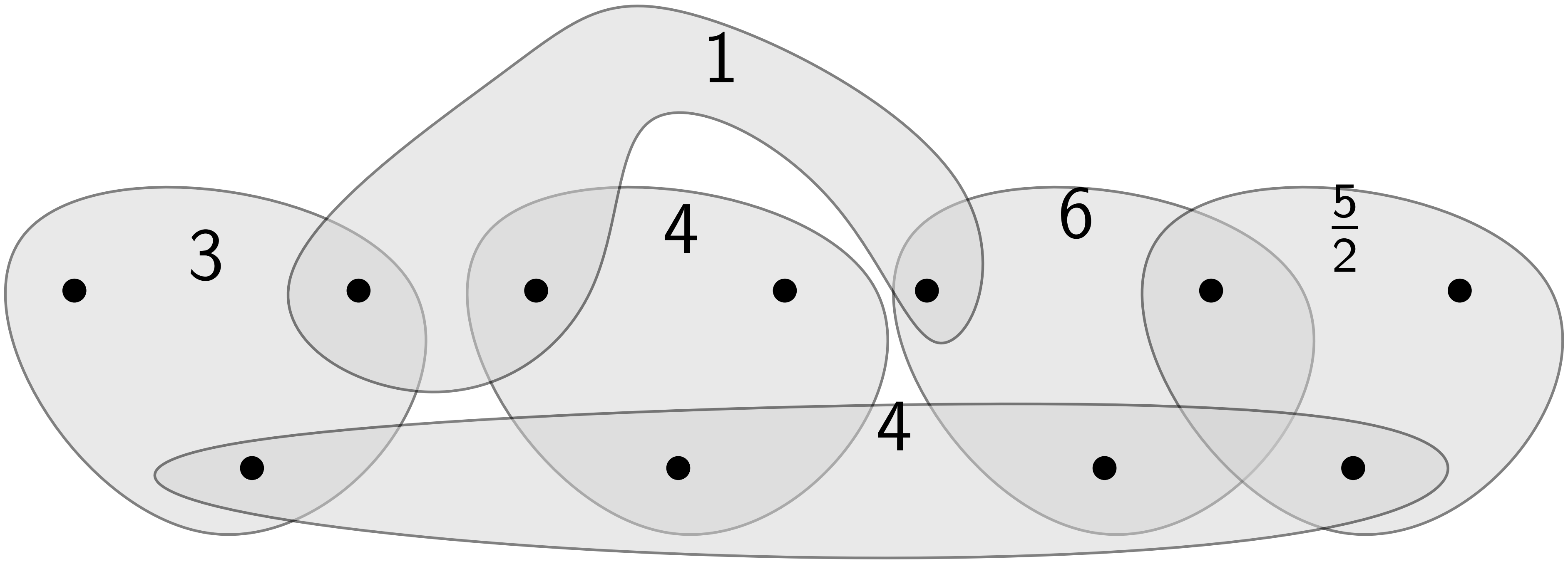
SetCover (general)

- Given a **ground set** U and a collection $\mathcal{S}$ of **subsets** of U where $\bigcup \mathcal{S} = U$. and each $S \in \mathcal{S}$ has a postive cost $c(S)$.
- Find a **cover** $\mathcal{S}' \subseteq \mathcal{S}$ of U (i.e. with $\bigcup \mathcal{S}' = U$) of minimum ~~cardinality.~~ total cost $c(\mathcal{S}') := \sum_{S \in \mathcal{S}'} c(S)$.



Iterative "re-pricing"

What is the cost of picking a set?



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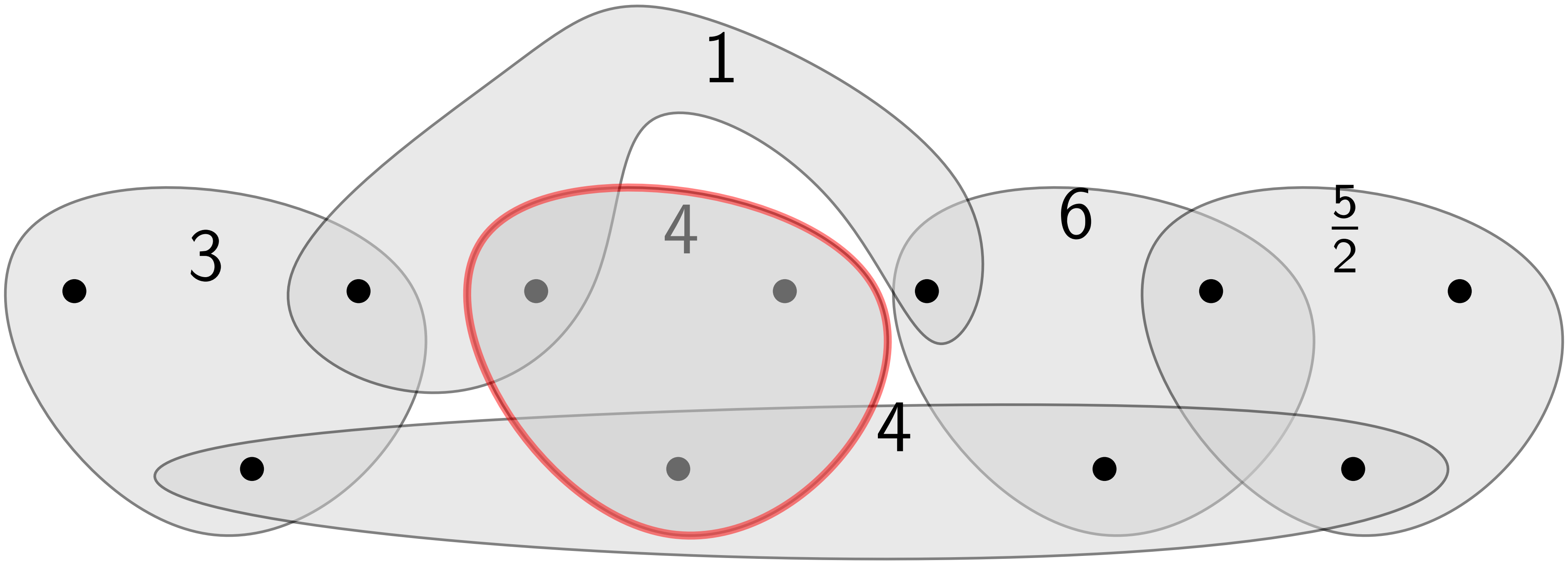

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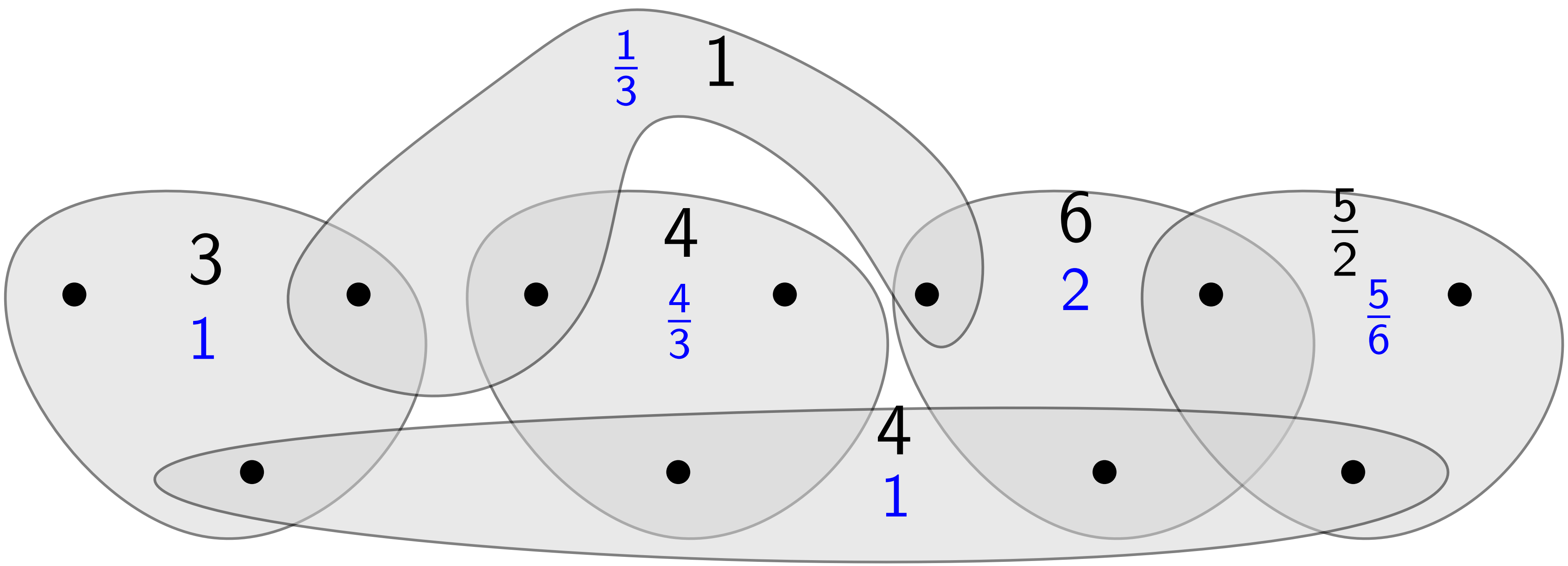

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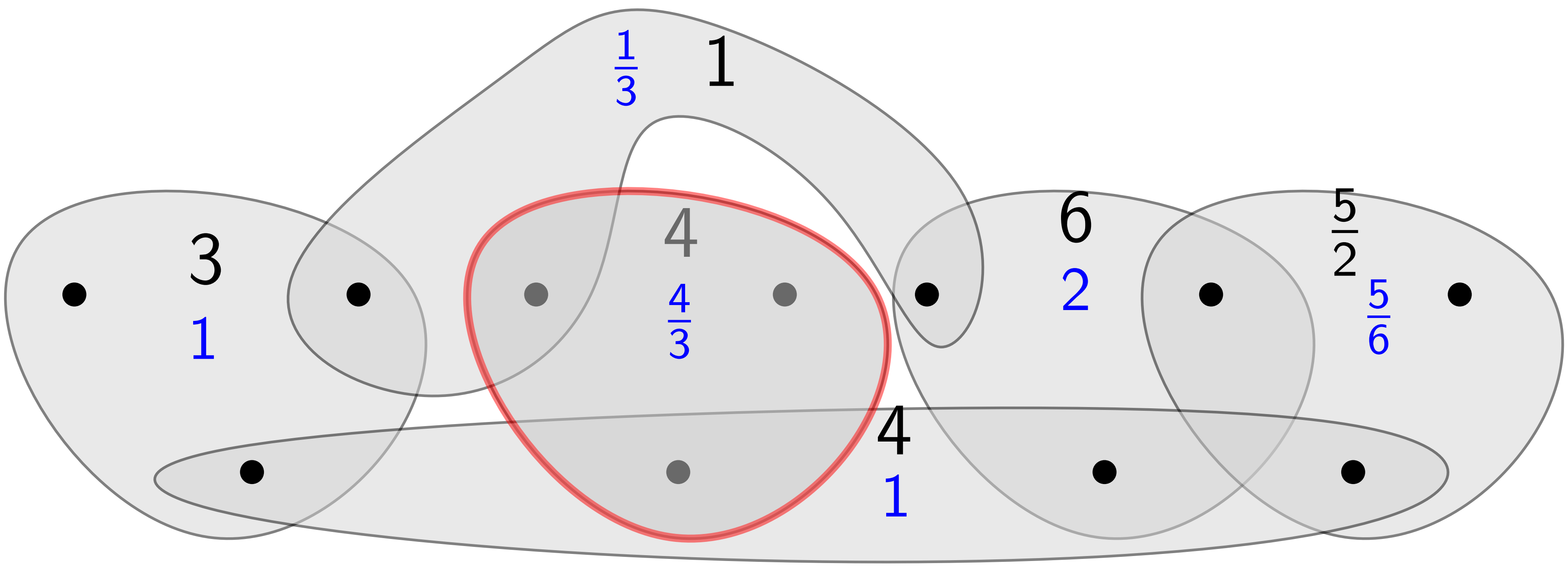

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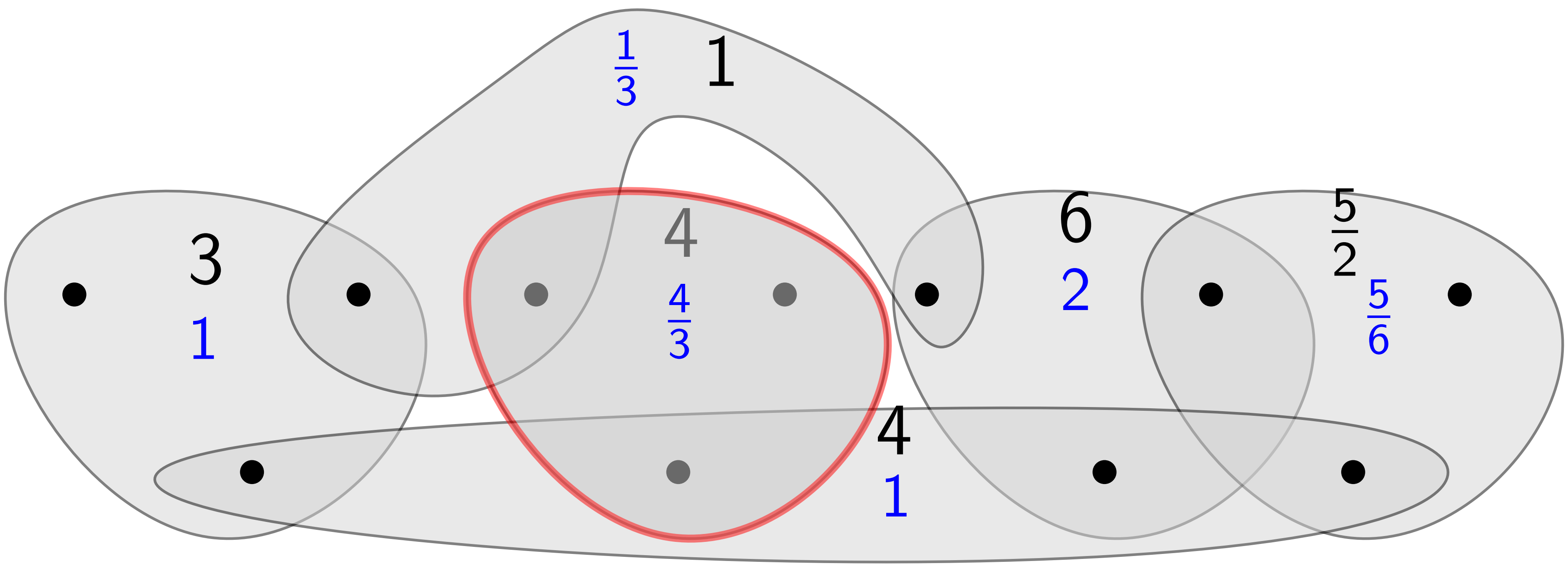
Fix price of the elements bought and re-evaluate unit prices!



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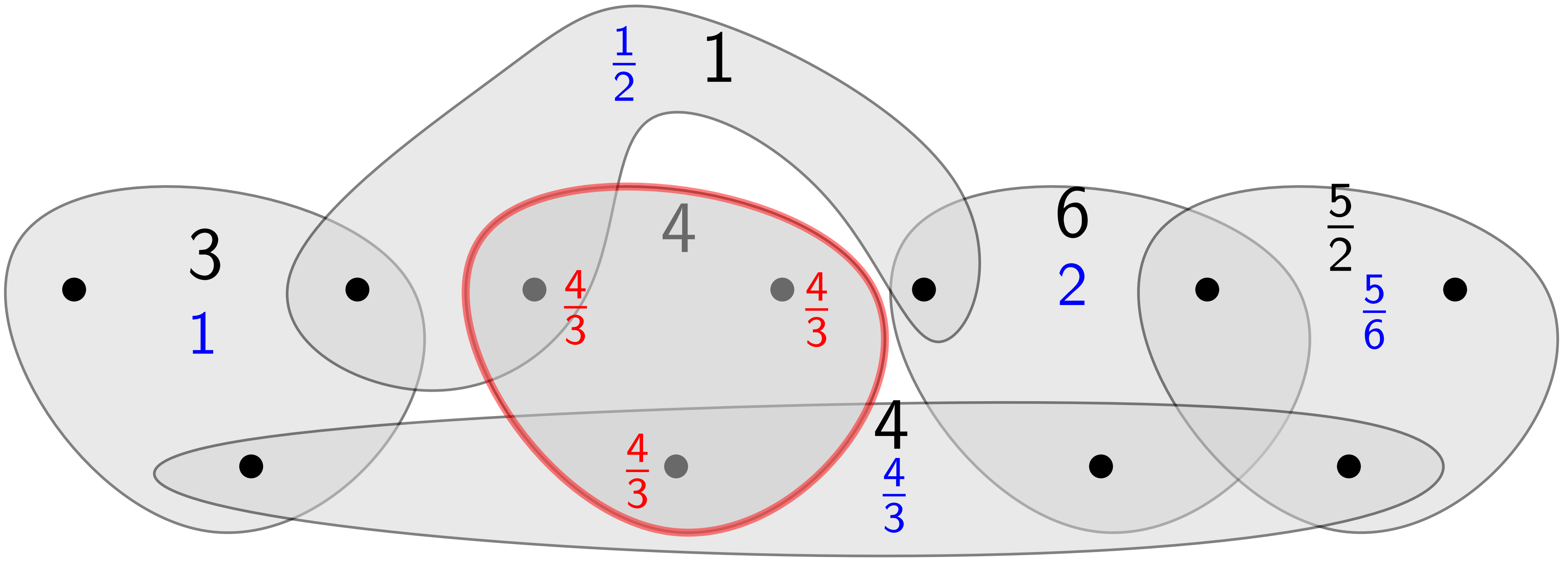

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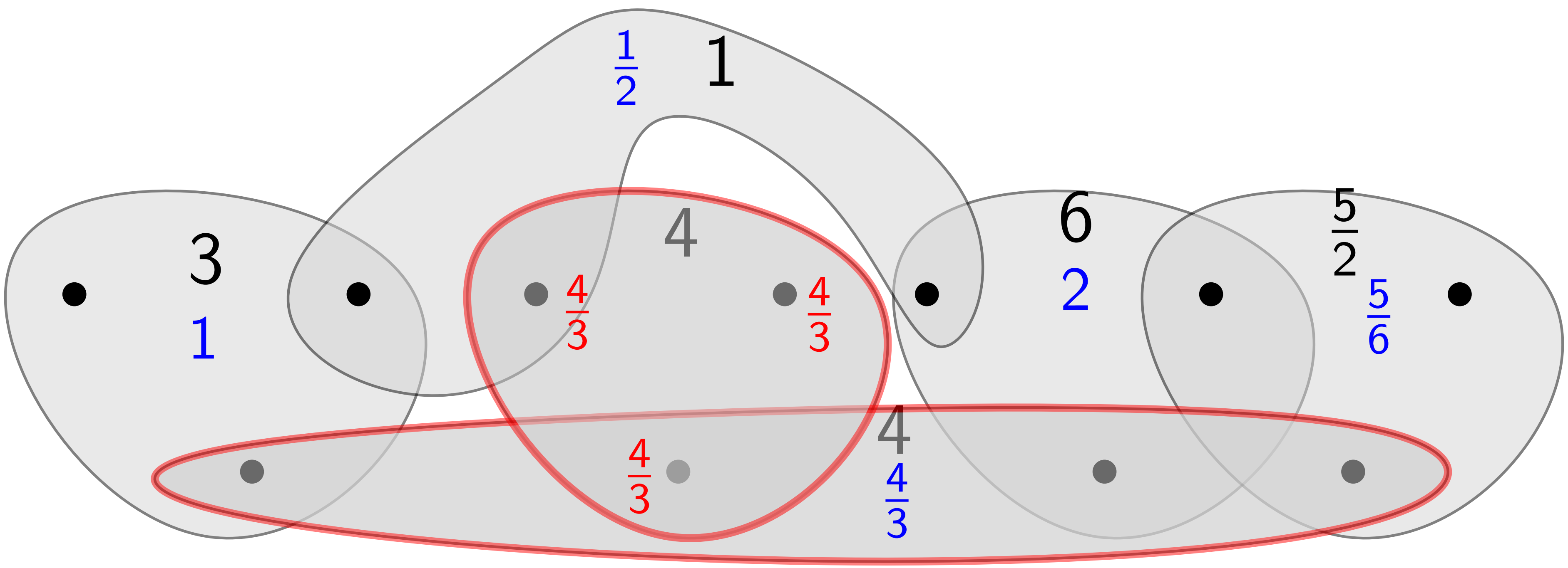

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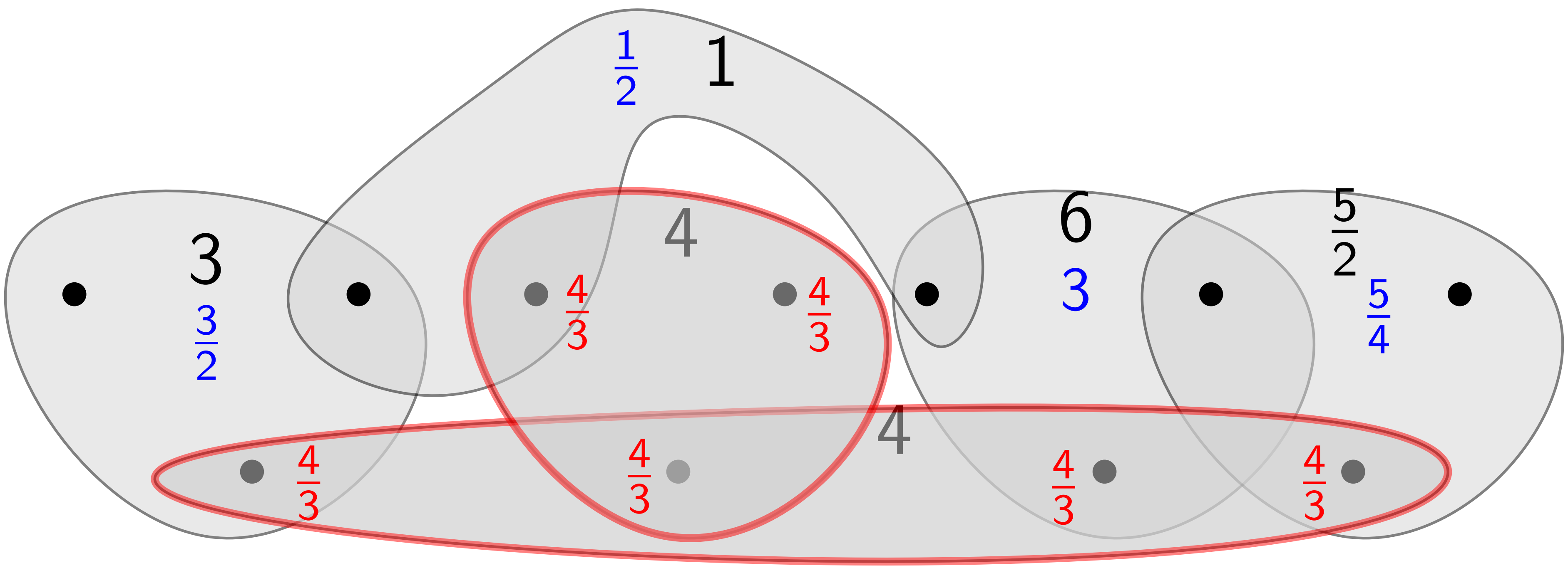

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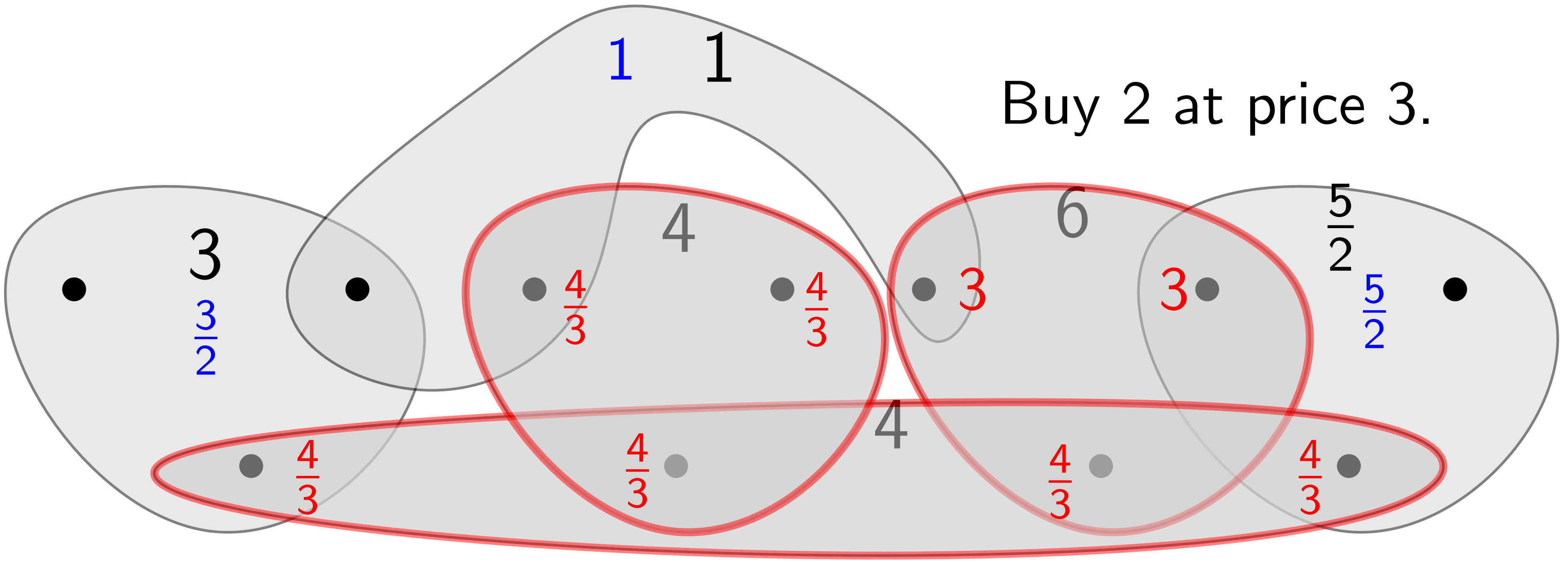

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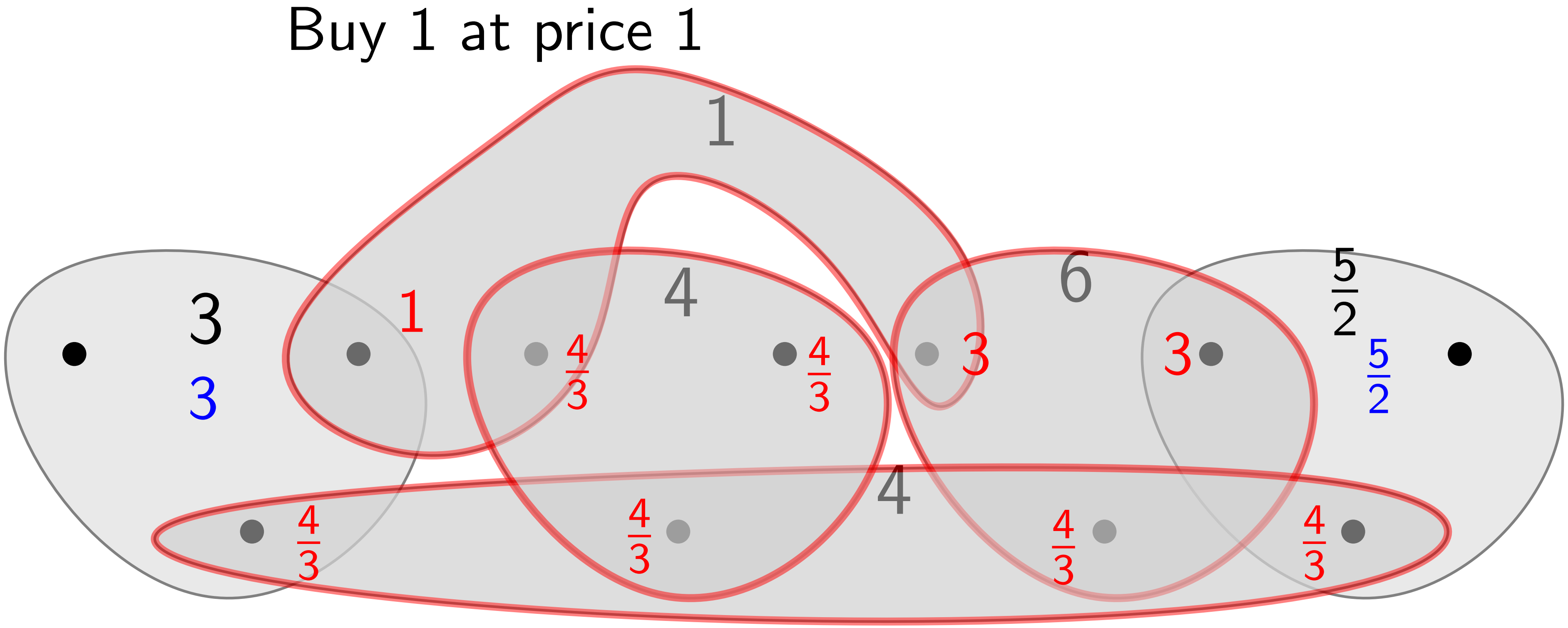

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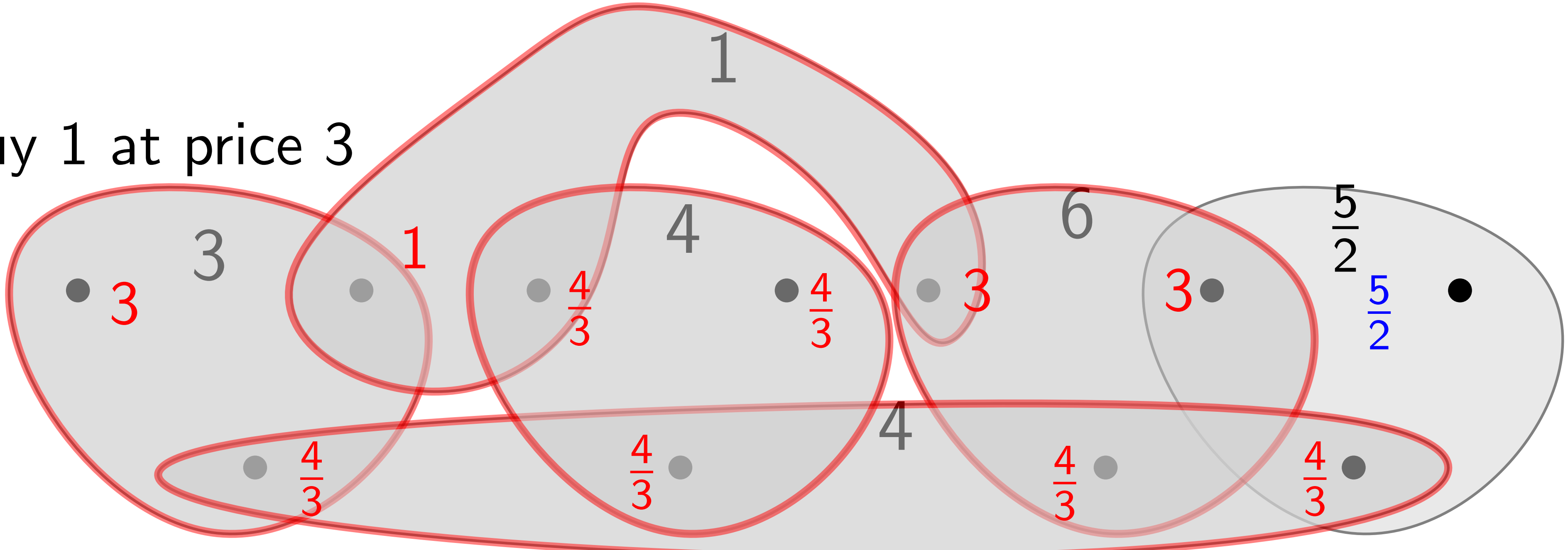
Buy 1 at price 3



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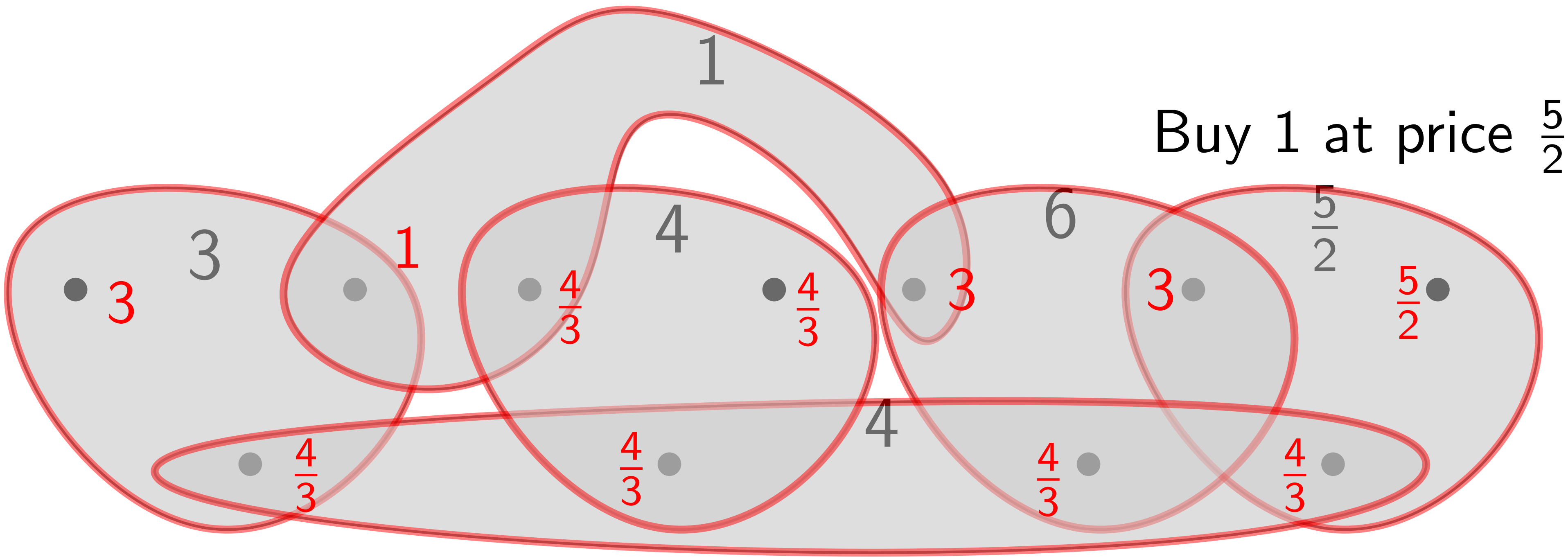

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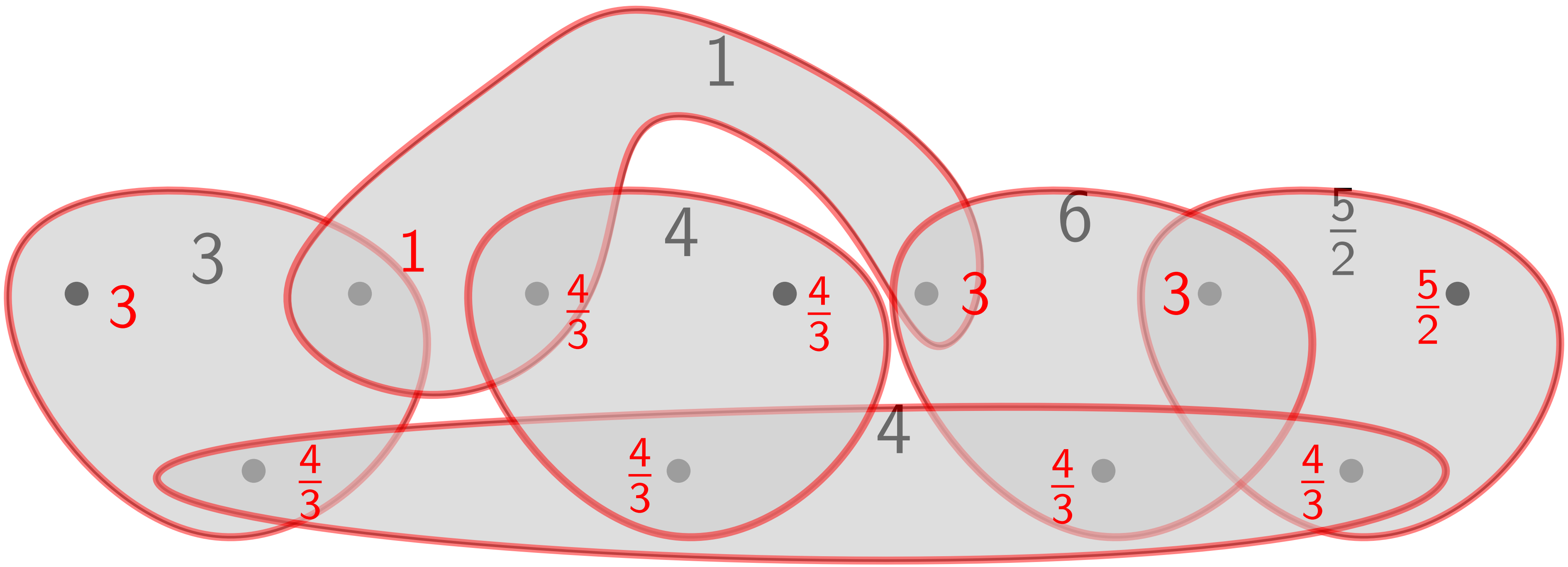
Obs: Total solution cost $= \sum_{e \in U} \text{price}(e)$



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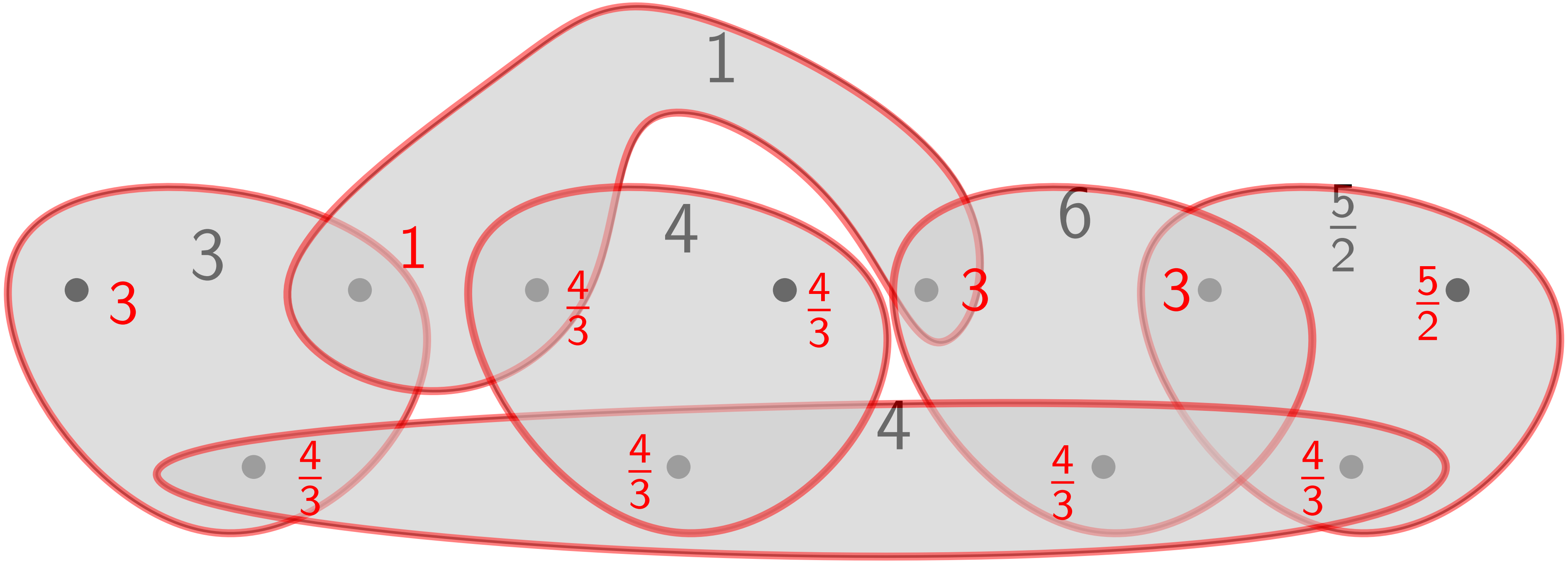
Greedy-Idea: Always choose the set with the cheapest unit price.



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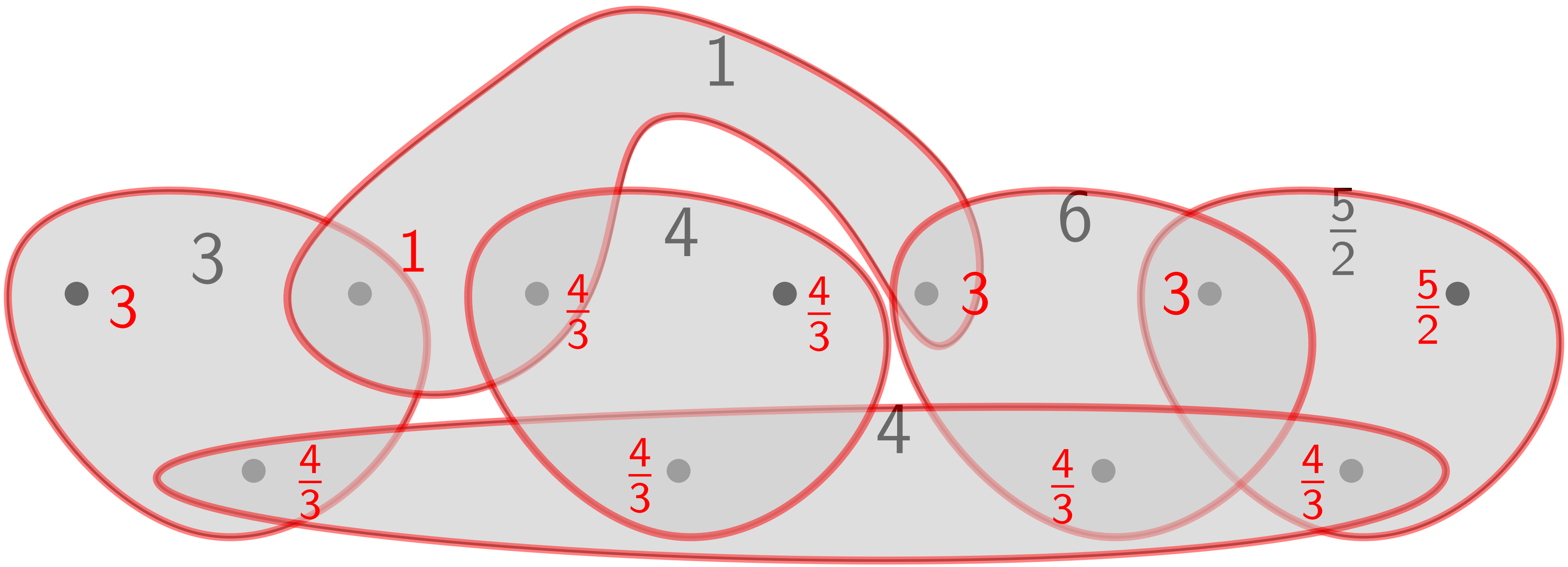
Greedy-Idea: Always choose the set with the cheapest unit price. Tie-breaking?



Greedy for SetCover

GreedySetCover($U, \mathcal{S}, c$)

```
C ← ∅
S' ← ∅
while C ≠ U do
    S ← Set S from 𝒮, with c(S)/|S\C| minimized
    foreach u ∈ S \ C do
        price(u) ← c(S)/|S\C|
    C ← C ∪ S
    S' ← S' ∪ {S}
return S'                                   // Cover of U
```

$C \leftarrow \emptyset$

$\mathcal{S}' \leftarrow \emptyset$

while $C \neq U$ **do**

- $S \leftarrow$ Set S from $\mathcal{S}$, with $\frac{c(S)}{|S \setminus C|}$ minimized
- **foreach** $u \in S \setminus C$ **do**
  - price(u) $\leftarrow \frac{c(S)}{|S \setminus C|}$
- $C \leftarrow C \cup S$
- $\mathcal{S}' \leftarrow \mathcal{S}' \cup \{S\}$

return $\mathcal{S}'$ `// Cover of` U

Analysis

Thm. GreedySetCover is a factor-$\mathcal{H}_k$ approximation alg. where k is the cardinality of the largest set in $\mathcal{S}$ and $\mathcal{H}_k := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} \in O(\log k)$.

Lemma. Let $S \in \mathcal{S}$ and $u_1, \ldots, u_l$ be the elements of S in the order they are covered ("bought") by GreedySetCover. Then

$$\text{price}(u_j) \le \frac{c(S)}{l-j+1}.$$

Lemma. $\text{price}(S) := \sum_{i=1}^{l} \text{price}(u_i) \le \mathcal{H}_l \cdot c(S)$

Is the Analysis of Greedy Tight?

Question: Does there exist a set cover instance where our greedy algorithm performs as bad (asymptotically) as our performance guarantee?

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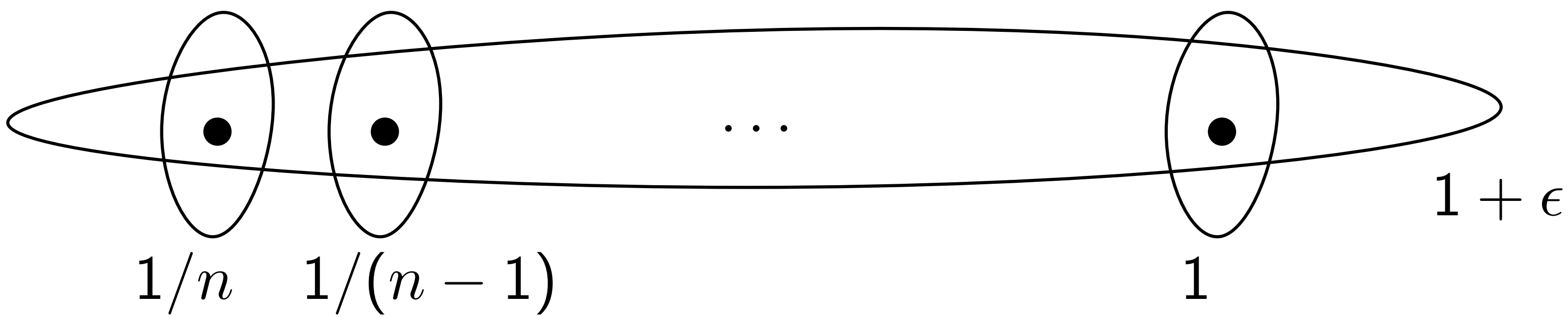
Yes :-(



ShortestSuperstring (SSS)

Given a collection $\mathcal{S} = \{s_1, \ldots, s_n\}$ of strings over a finite alphabet Σ (i.e., $\mathcal{S} \subseteq \Sigma^+$).
Find a *shortest string* s such that each s_i is a *substring* of s.

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$$s = abcbaa$$

$$\begin{array}{l} abc \\ bcb \\ cbaa \end{array}$$

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WLOG: No string s_i is a substring of any other string s_j.

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$$\begin{array}{l} s = abcbaa \\ \phantom{s = {}}abc \\ \phantom{s = {}a}bcb \\ \phantom{s = {}ab}cbaa \end{array}$$

SSS as a SetCover problem

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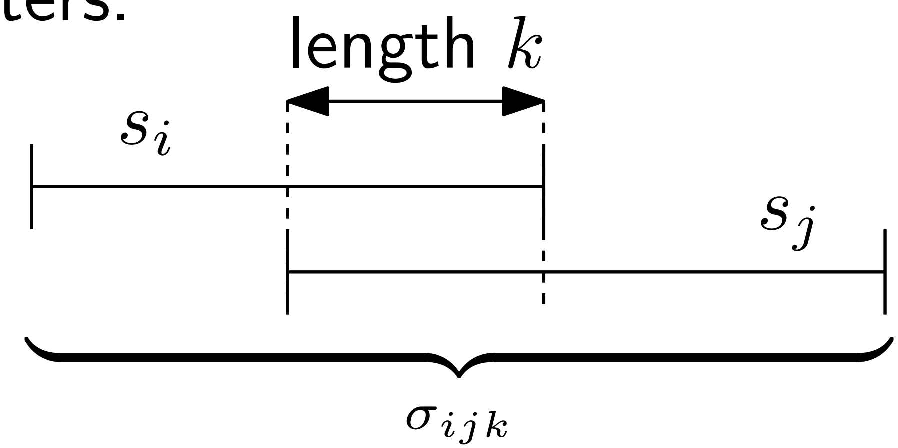
- a string σ_{ijk} has prefix s_i, suffix s_j where s_i and s_j *overlap* on k characters.



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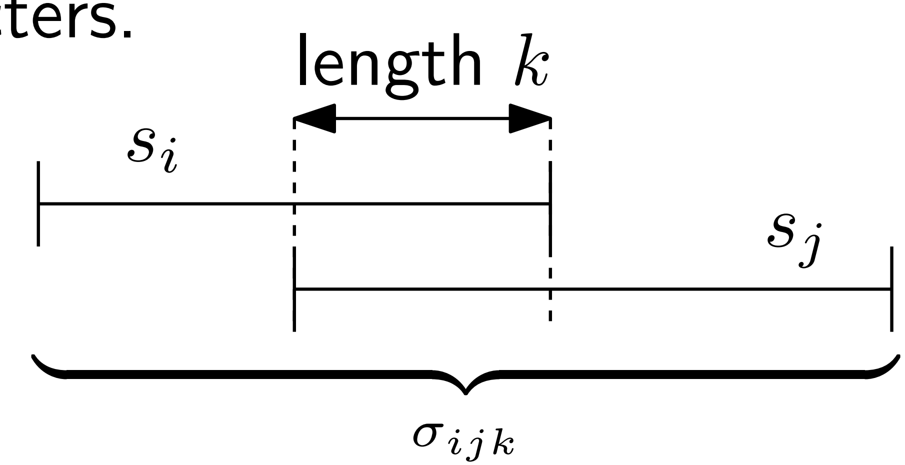

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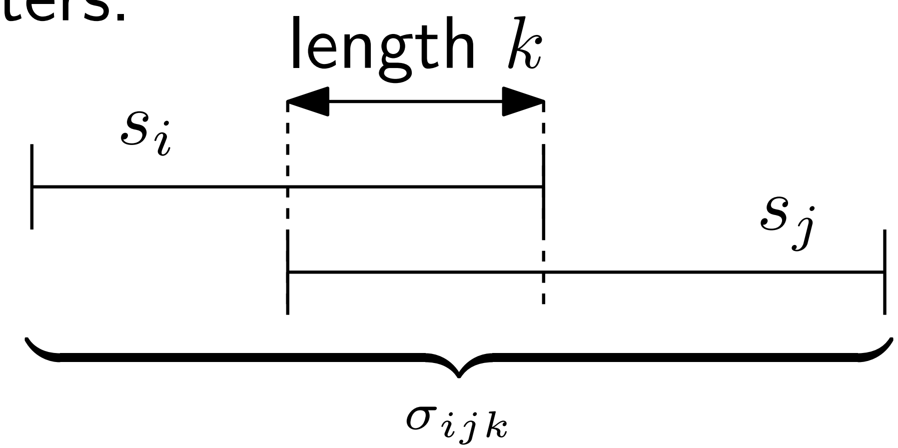

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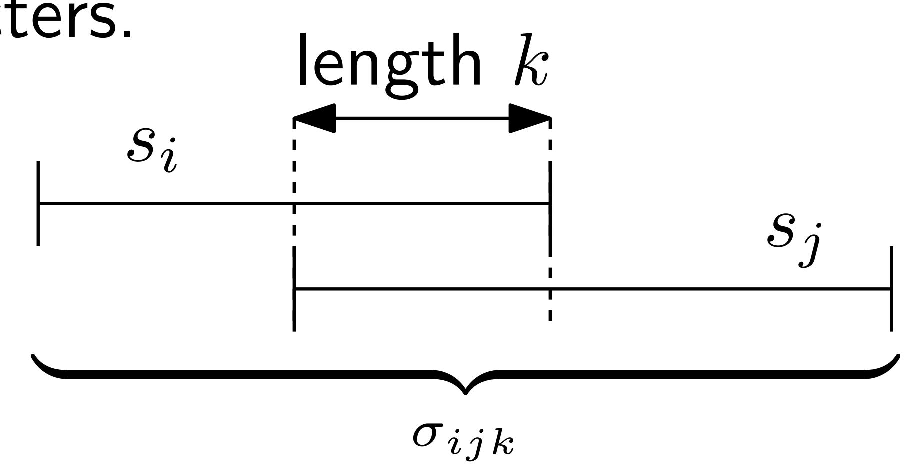

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- set family $\mathcal{S} = \{$set$(\sigma_{ijk}) \mid$ for valid choices of i, j (possibly $i = j$) and $k > 0\}$.

Relating SSS and SetCover

Lemma. Let OPT be the length of a shortest superstring of U and $\mathrm{OPT}_{\mathrm{SC}}$ be the minimum cost of the corresponding SetCover-Instance. Then:

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s_{b_1}

leftmost occurence of some string $s_{b_1} \in U$

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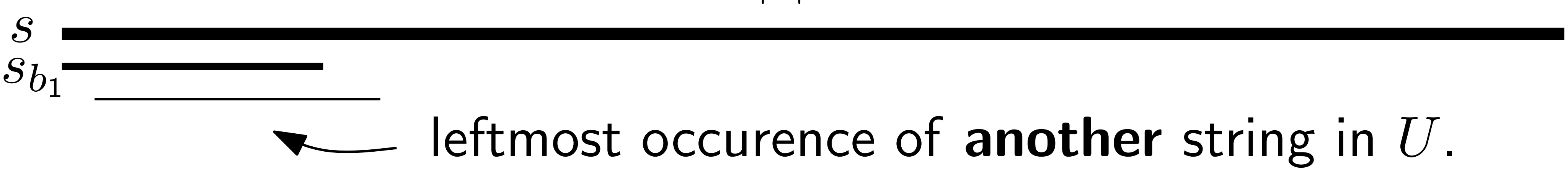

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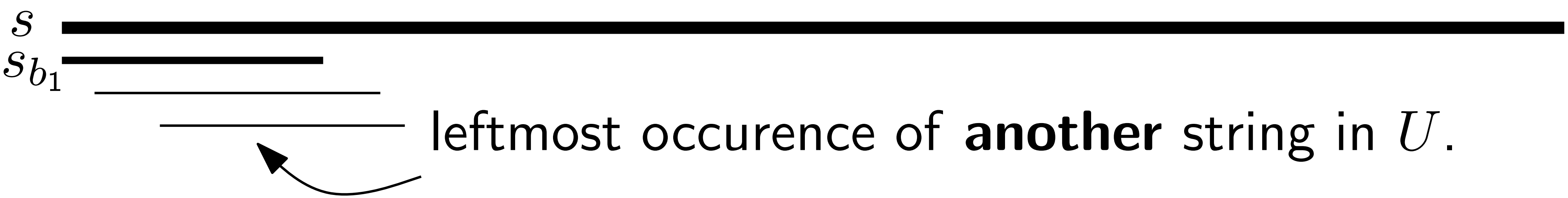

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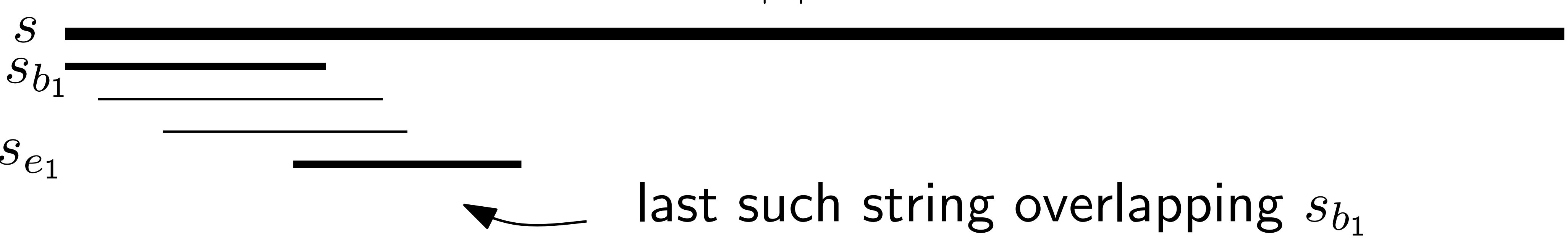

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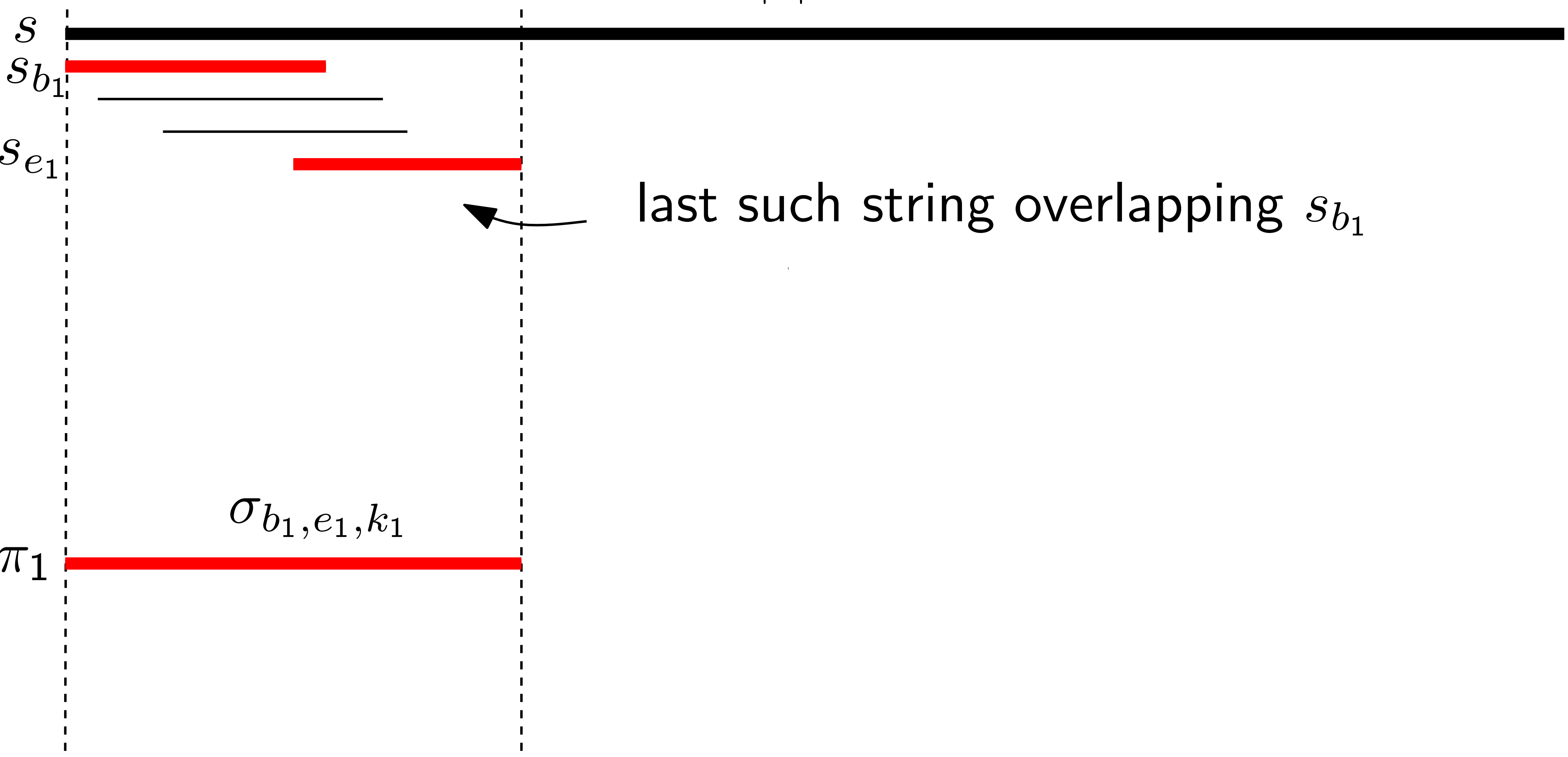

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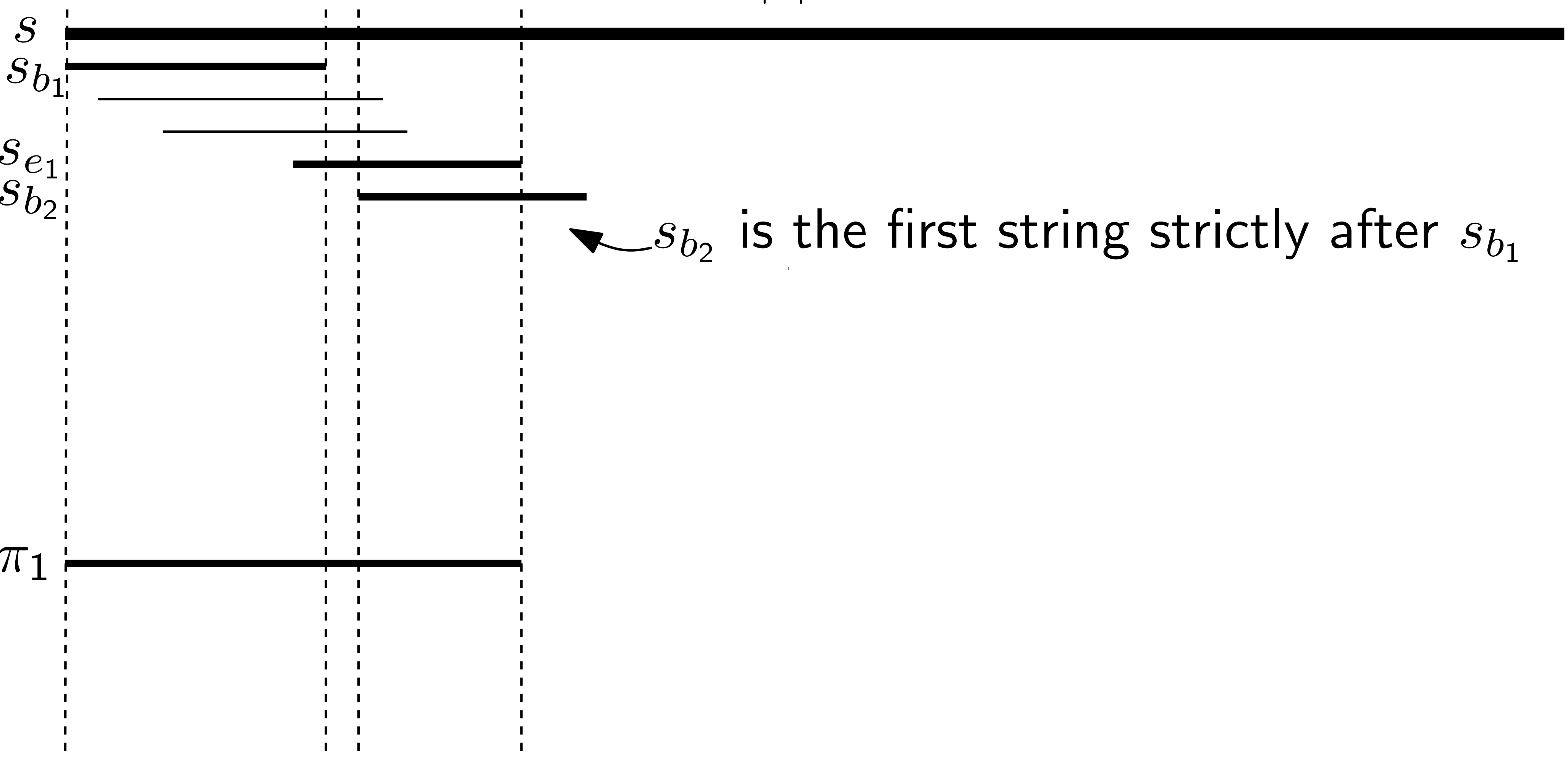

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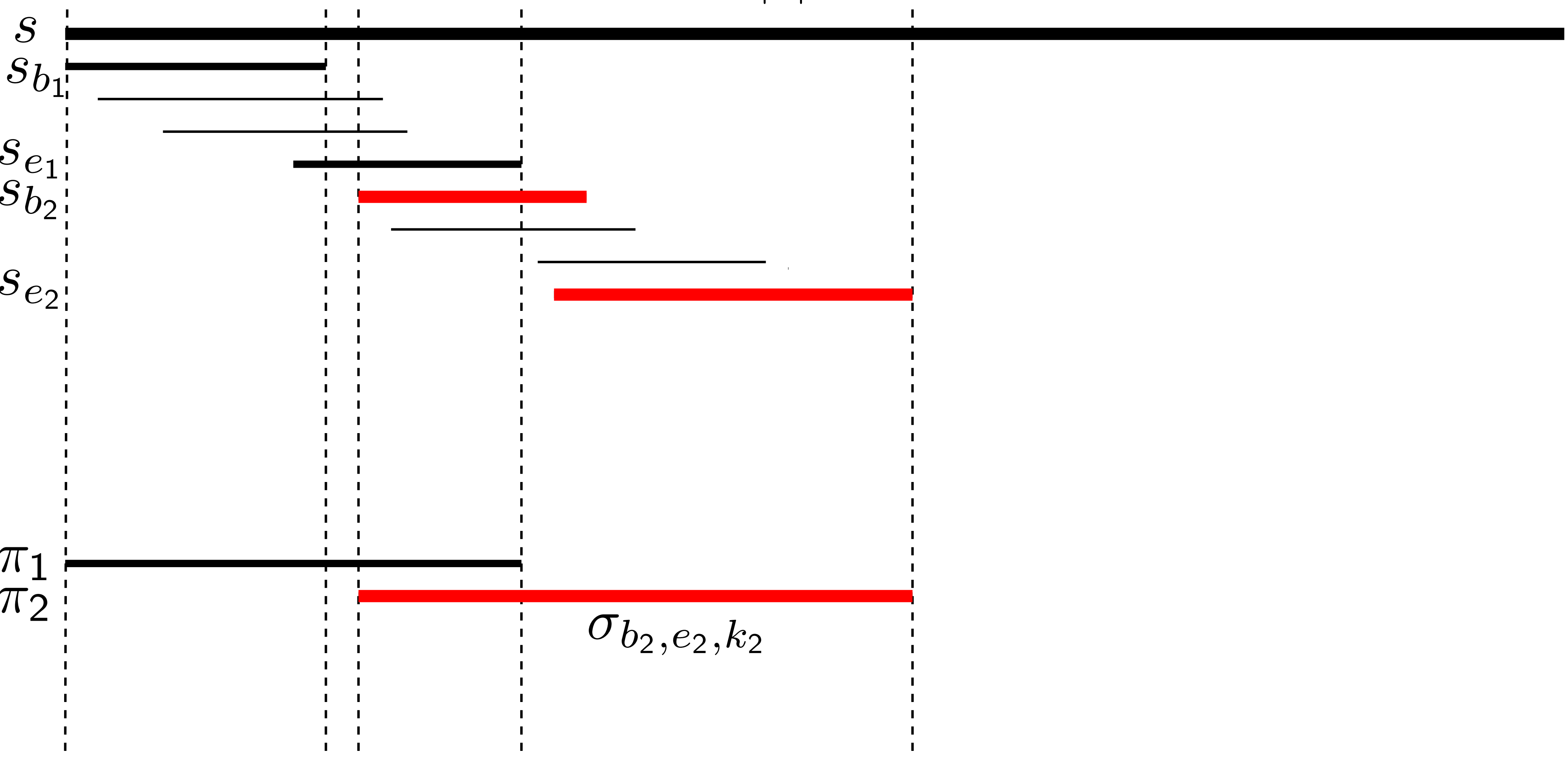

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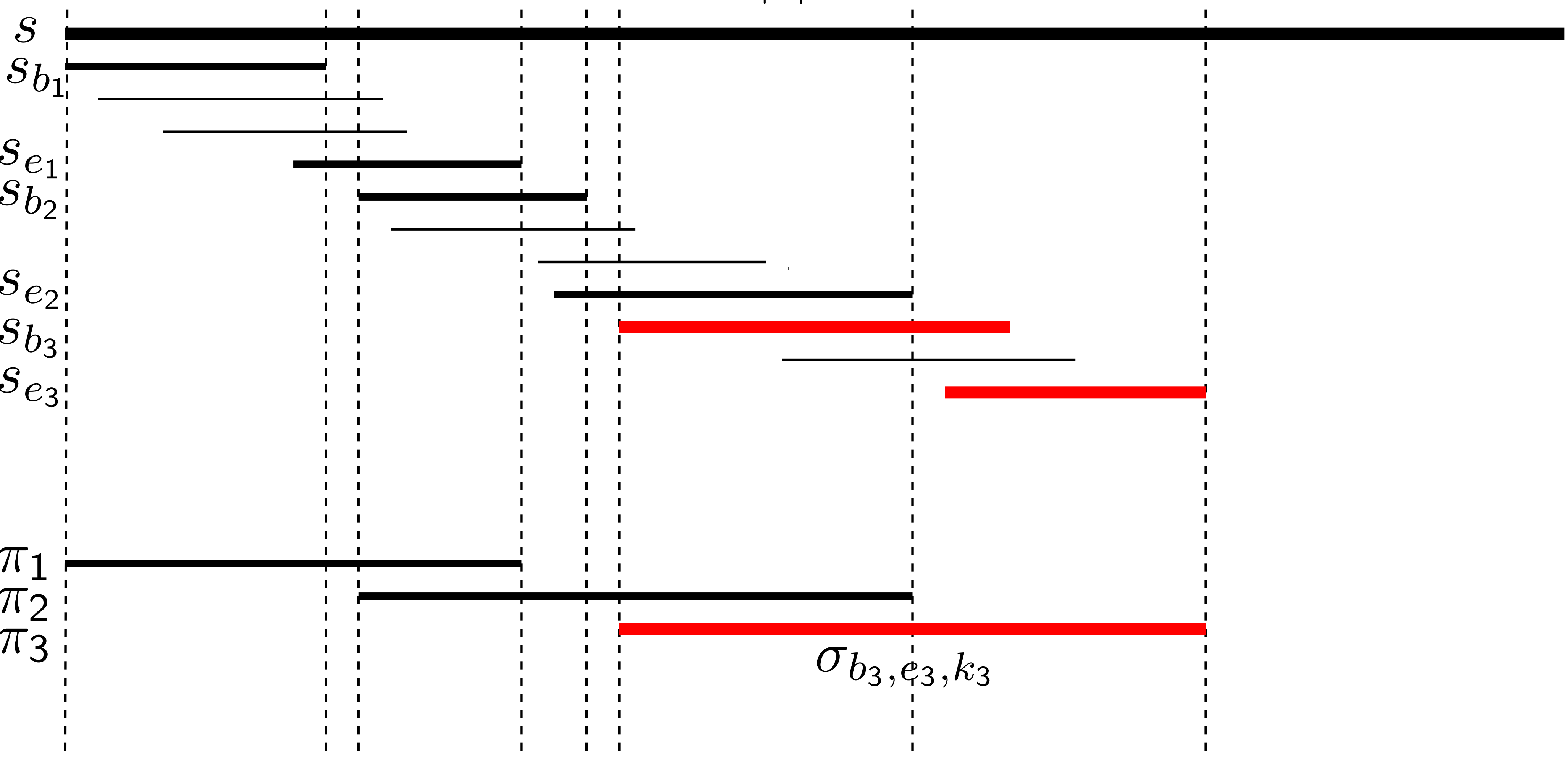

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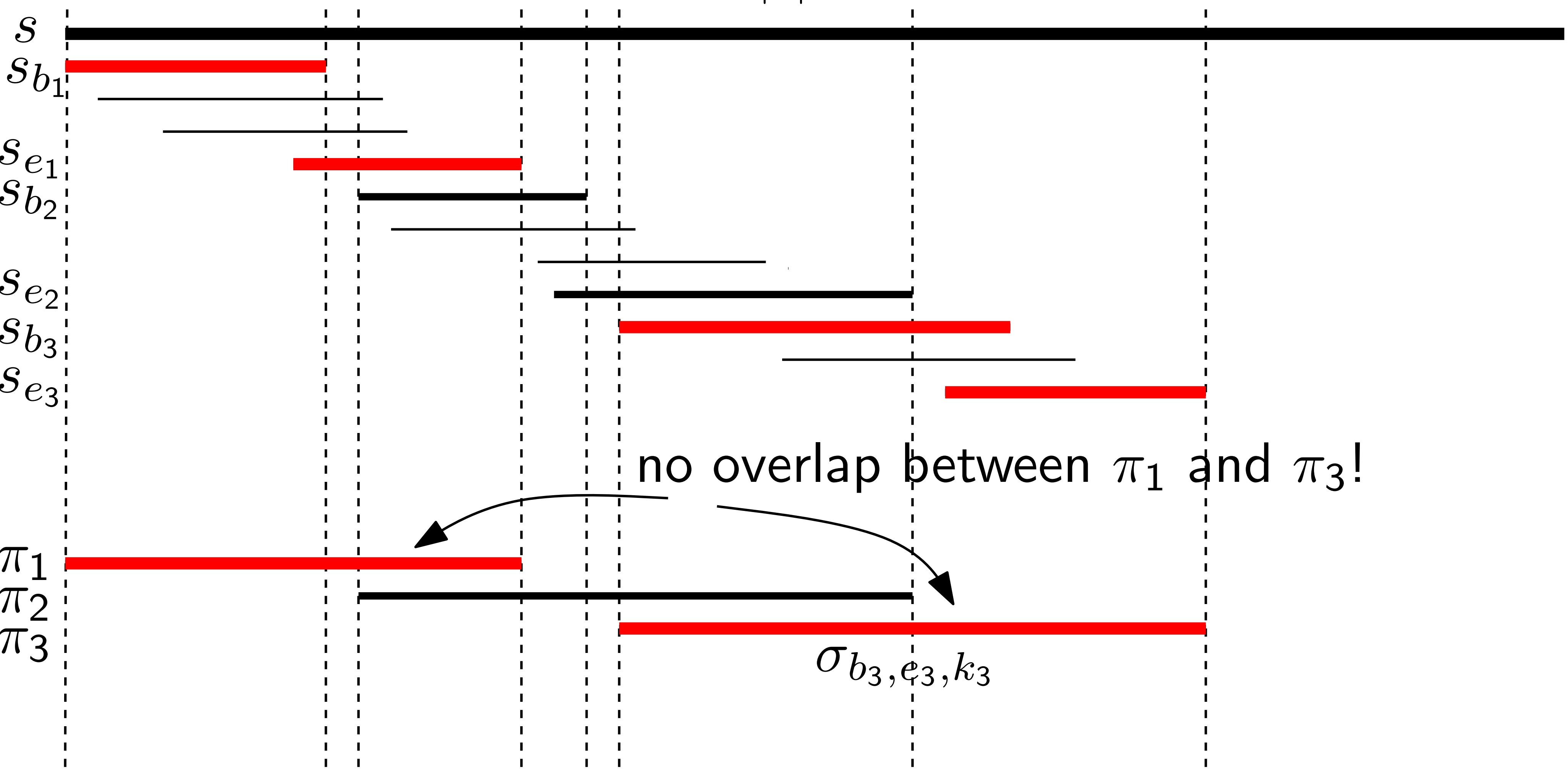

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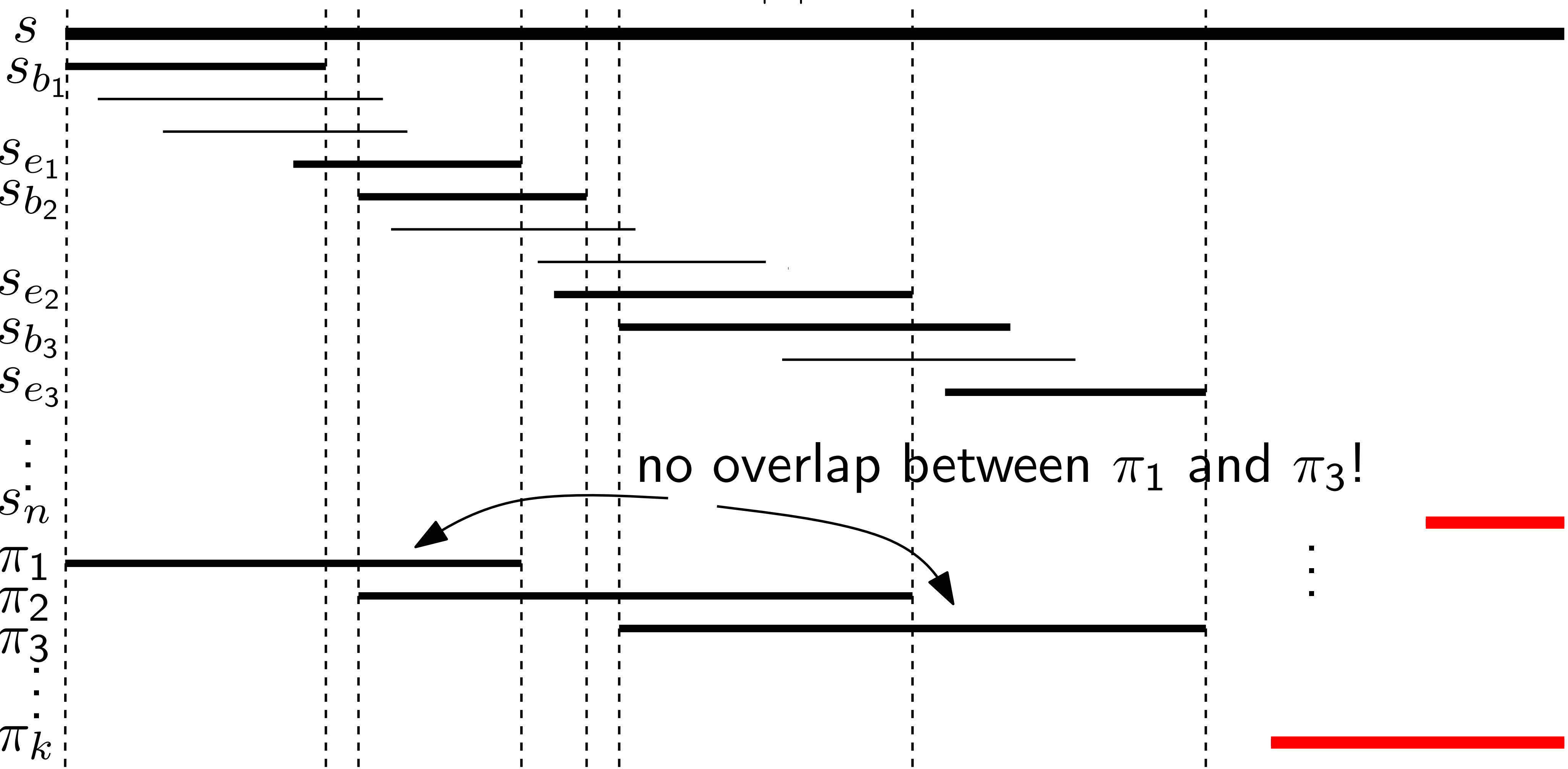

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Proof.

- each string $s_i \in U$ is a substring of some π_j
- $\mathrm{set}(\pi_1), \ldots, \mathrm{set}(\pi_k)$ is a solution for the SetCover instance with cost $\sum_i |\pi_i|$
- the substrings π_j, π_{j+2} of s do not overlap
- each character of s lies in at most two substrings (i.e., π_j and π_{j+1})
- $\sum_i |\pi_i| \leq 2|s| = 2 \cdot \mathsf{OPT}$

$\square$

Algorithm for SSS

- construct the $\textsc{SetCover}$ instance U, $\mathcal{S}$, c
- Let set$(\pi_1), \ldots,$ set(π_k) be the solution provided by the GreedySetCover algorithm.
- return $\pi_1 \circ \cdots \circ \pi_k$ as the superstring.

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Note: SSS has a factor-3 approximation alg. see [V. §7].

Algorithm for SSS

- construct the SETCOVER instance U, $\mathcal{S}$, c
- Let set$(\pi_1), \ldots,$ set(π_k) be the solution provided by the GreedySetCover algorithm.
- return $\pi_1 \circ \cdots \circ \pi_k$ as the superstring.

Thm. This algorithm for SHORTESTSUPERSTRING has approximation factor $2\mathcal{H}_n$.

Proof.

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Next Week: Steiner Trees & Multiway-Cuts

Approximation Preserving Reduction

Let Π_1, Π_2 be minimization problems. An **approximation preserving reduction** from Π_1 to Π_2 is a pair (f, g) of poly-time computable functions with the following properties.

(i) for each instance I_1 of Π_1, $I_2 := f(I_1)$ is an instance of Π_2 where $\mathsf{OPT}_{\Pi_2}(I_2) \leq \mathsf{OPT}_{\Pi_1}(I_1)$

(ii) for each feasible solution t of I_2, $s := g(I_1, t)$ is a feasible solution of I_1 where $\mathsf{obj}_{\Pi_1}(I_1, s) \leq \mathsf{obj}_{\Pi_2}(I_2, t)$

$$
\begin{tikzcd}
 & \Pi_1 & \Pi_2 \\
\text{instances} & I_1 \arrow[r, "f"] \arrow[d, dashed] & I_2 \arrow[d] \\
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Approximation Preserving Reduction

Thm. Let Π_1, Π_2 be minimization problems where there is an approximation preserving reduction from Π_1 to Π_2. Then, for each factor-α approximation algorithm of Π_2, there is a factor-α approximation algorithm of Π_1.

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- Consider a factor-α approx. alg. A of Π_2 and an instance I_1 of Π_1.
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- $\mathsf{obj}_{\Pi_1}(I_1, s) \le \mathsf{obj}_{\Pi_2}(I_2, t) \le \alpha \cdot \mathsf{OPT}_{\Pi_2}(I_2) \le \alpha \cdot \mathsf{OPT}_{\Pi_1}(I_1)$

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