Aalto University
School of Science

CS-E4530 Computational Complexity Theory

Lecture 16: Cryptography

Aalto University
School of Science
Department of Computer Science

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Agenda

- Encryption schemes
- Computational security
- One-way functions
- Public-key encryption schemes

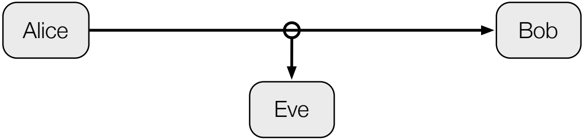
Cryptography

- ***Cryptography*** **is the study of secure communication**
  - Cryptography is *much* older than computer science
  - Traditionally, cryptography referred to the development of various ad-hoc encryption schemes
  - These schemes were usually broken sooner or later

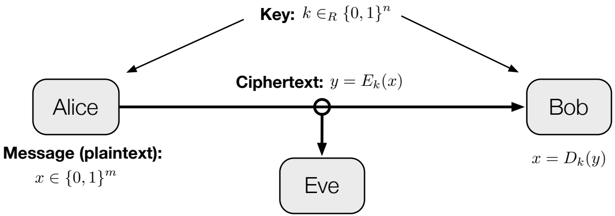
- ***Modern cryptography*** **was born in the 1970s, when computational complexity theory was applied to cryptography**
  - Modern cryptography aims to develop *provably unbreakable* encryption schemes
  - Unbreakability is conditioned on complexity assumptions

Encryption Schemes

- Two parties, *Alice* and *Bob*, wish to communicate in the presence of a malevolent eavesdropper *Eve*

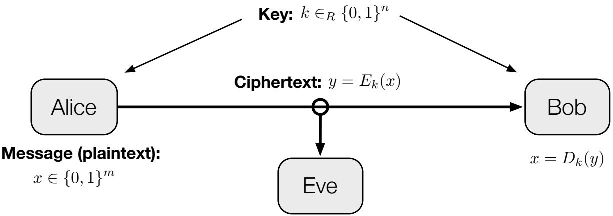


Encryption Schemes



- ***Encryption scheme* consists of two algorithms:**
  - *Encryption* algorithm E
  - *Decryption* algorithm D
- **Both algorithms are parameterised by a randomly selected *secret key* $k \in \{0,1\}^n$, known to Alice and Bob**
  - For all $k \in \{0,1\}^n$ and $x \in \{0,1\}^m$, we have $D_k(E_k(x)) = x$

Encryption Schemes



- **Transmission of a secret message x:**
  - Alice computes ciphertext $y = E_k(x)$
  - Alice sends y to Bob and Bob computes $x = D_k(y)$
- **Requirements for encryption scheme (E, D):**
  - E and D are polynomial-time
  - Eve cannot obtain *any information* about x from y, even if Eve knows E and D

Perfect Secrecy

- **What does**

  "Eve cannot obtain any information about x from y"

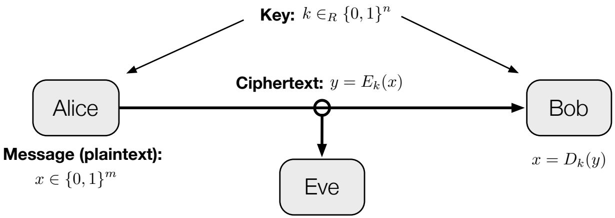
  mean?

Definition (Perfect secrecy)

Let (E,D) be a encryption scheme for messages of length m with key size n. We say that (E,D) is *perfectly secret* if for any pair of messages $x, x' \in \{0,1\}^m$, the distributions $E_{U_n}(x)$ and $E_{U_n}(x')$ are the same, where U_n denotes the uniform distribution over $\{0,1\}^n$.

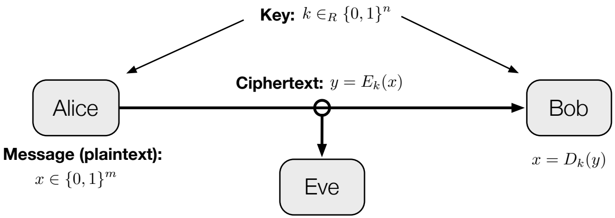
- If the key is picked at random, Eve will see the same distribution of ciphertexts regardless of the actual message

One-time Pad



- **A simple solution: *one-time pad***
  - For message $x \in \{0,1\}^n$, select key $k \in \{0,1\}^n$ uniformly at random
  - Let $E_k(x) = x \oplus k$ and $D_k(x) = x \oplus k$, where $\oplus$ is the bit-wise XOR
  - Now $D_k(E_k(x)) = (x \oplus k) \oplus k = x$

One-time Pad



- **One-time pad satisfies perfect secrecy:**
  - If k is uniformly distributed over $\{0,1\}^n$, then so is $E_k(x)$
- **One-time pads are *one-time*:**
  - If same key k is used twice, then $E_k(x) \oplus E_k(x') = x \oplus x'$, which yields nontrivial information about the messages

Computational Security

- **Perfect security means we can *fool* any adversary**
  - Perfect security requires that key length is at least message length
  - However, it is reasonable to assume that adversary has limited computational power
- ***Computational security*: aim is to fool any (randomised) polynomial-time adversary**
  - One can define various forms of computational security
  - Strength of the required security can depend on application
  - Real definitions are somewhat complicated and subtle

Computational Security: Simple Definition

- **A simple version of *computational security*:**
  - **Intuition:** adversary cannot guess any bit of the plaintext with probability significantly larger than $1/2$
  - **Formally:** we say that a scheme (E,D) for m-bit messages with n-bit keys is *computationally secure* if for any probabilistic polynomial-time algorithm A,

$$\Pr_{\substack{k \in U_n \\ x \in U_m}} \left[A(E_k(x)) = (i,b) \text{ such that } x_i = b \right] \leq 1/2 + \varepsilon(n)\,,$$

where $\varepsilon(n) = n^{-\omega(1)}$, that is, $\varepsilon(n) < n^{-c}$ for any c and for sufficiently large n

One-way Functions

- **One can show that if** $\mathrm{P} = \mathrm{NP}$**, then computationally secure encryption schemes do not exist**
  - Thus, modern cryptography requires $\mathrm{P} \neq \mathrm{NP}$
  - Assuming $\mathrm{P} \neq \mathrm{NP}$ is not quite enough, as far as we know

- **Standard assumption:** ***one-way functions*** **exist**

One-way Functions

Definition

A polynomial-time computable function $f\colon \{0,1\}^* \to \{0,1\}^*$ is a *one-way function* if for every probabilistic polynomial-time algorithm A,

$$\Pr_{\substack{x \in U_n \\ y = f(x)}} [A(y) = x' \text{ such that } f(x') = y] < \varepsilon(n)\,,$$

where $\varepsilon(n) = n^{-\omega(1)}$.

Conjecture

There exists a one-way function.

One-way Functions

- **Existence of one-way functions implies** $\mathsf{P} \neq \mathsf{NP}$
  - No one-way functions are known
- **Some candidates:**
  - *Integer multiplication:* the function $(p,q) \mapsto pq$, where p and q are prime numbers (inverse: factoring)
  - *RSA function*: $f_{\mathsf{RSA}}(x,e,p,q,) = (x^e \bmod pq, pq, e)$, where p and q are prime numbers, e is a relative prime to $\phi(pq) = (p-1)(q-1)$ and $x < pq$ is an integer

One-way Functions and Security

- **One-way functions are used as building block for encryption schemes**

Theorem

Assume one-way functions exist. Then for every $c \in \mathbb{N}$, there exists a computationally secure encryption scheme (E, D) for n^c-bit messages with n-bit keys.

Pseudorandomness

- **Another application of one-way functions: *pseudorandom generators***
  - Basic idea: turn a small number of random bits into a larger number of “random-looking” bits
  - Specifically, we require that the *pseudorandom* bits cannot be distinguished from real random bits by polynomial-time algorithms
  - Lots of practical applications

Pseudorandomness

Definition

Let $G\colon \{0,1\}^* \to \{0,1\}^*$ be a polynomial-time computable function, and let $\ell\colon \mathbb{N} \to \mathbb{N}$ be a polynomial-time computable function with $\ell(n) > n$. We say that G is a *secure pseudorandom generator of stretch $\ell(n)$* if $|G(x)| = \ell(|x|)$ for every $x \in \{0,1\}^*$ and for every probabilistic polynomial-time algorithm A, we have that

$$\left| \Pr_{x \in U_n} [A(G(x)) = 1] - \Pr_{z \in U_{\ell(n)}} [A(z) = 1] \right| < \varepsilon(n)\,,$$

where $\varepsilon(n) = n^{-\omega(1)}$.

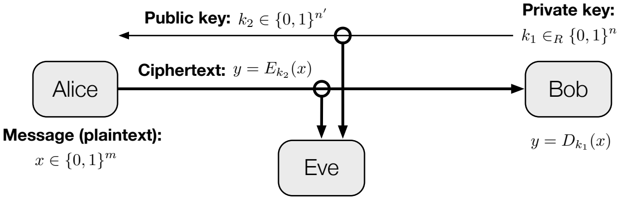
Definition

If one-way functions exist, then there is a secure pseudorandom generator with stretch n^c for every $c \in \mathbb{N}$.

Public-key Encryption

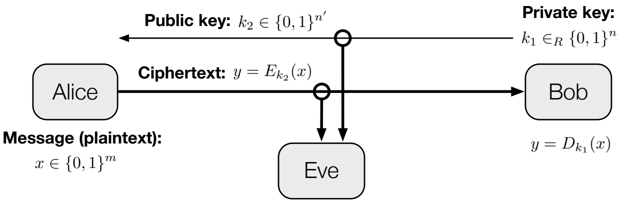
- **The notion of encryption schemes we have been discussing so far is *private-key encryption***
  - Alice and Bob need to share a secret key k
  - Impractical: this key needs to be shared somehow!
- **Modern cryptography is based on *public-key encryption***
  - Both the algorithms (E, D) and the encryption key are known publicly

Public-key Encryption



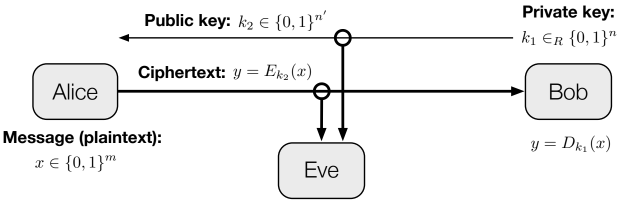
- **In public-key encryption, Bob generates two keys:**
  - A *private* key k_1 and a *public* key k_2
  - Bob sends the public key to Alice unencrypted
- **Alice can now send an encrypted message to Bob**
  - Alice encrypts the message as $y = E_{k_2}(x)$
  - Bob decrypts the message as $x = D_{k_1}(y)$

Public-key Encryption



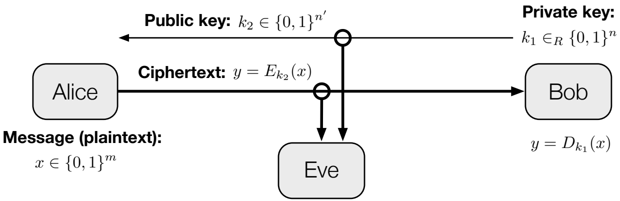
- **The security requirement is now:**
  - Public key should not reveal information about private key
  - Knowing public key should not reveal information about the message
- **This can be achieved using one-way functions**

RSA Encryption



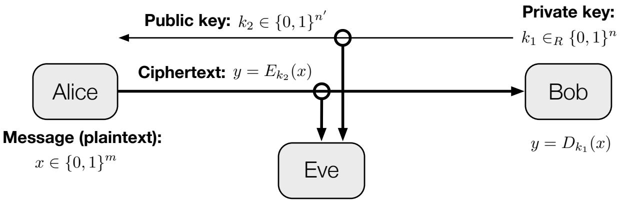
- **Recall the definition of RSA function:**
  - $f_{\mathrm{RSA}}(x,e,p,q) = (x^e \bmod pq, pq, e)$, where p and q are prime numbers, e is a relative prime to $\phi(pq) = (p-1)(q-1)$ and $x < pq$ is an integer
  - e can be selected to be a fixed prime number

RSA Encryption



- **The keys for the RSA system are now as follows:**
  - Bob generates two large primes p and q
  - Bob's private key is $k_1 = (p,q,d)$, where $d = e^{-1} \bmod \phi(pq)$, that is, $ed = 1 + k\phi(pq)$ for some k (given p, q and e, one can compute d by extended Euclid's algorithm)
  - Bob's public key is $k_2 = (pq, e)$

RSA Encryption



- Alice encrypts a message x as

$$y = x^e \bmod pq$$

- Bob decrypts a message y by

$$y^d = x^{ed} = x^{1+k\phi(pq)} = x(x^{\phi(pq)})^k = x \bmod pq\,,$$

where $x^{\phi(pq)} = 1 \bmod pq$ by an extension of Fermat's theorem

Lecture 16: Summary

- Encryption schemes
- Basic idea of computational security
- One-way functions
- Pseudorandom generators
- Public-key encryption