Aalto University
School of Science

Department of Computer Science

Combinatorics of Efficient Computations

Approximation Algorithms

Lecture 3: Steiner Tree & Multiway Cut

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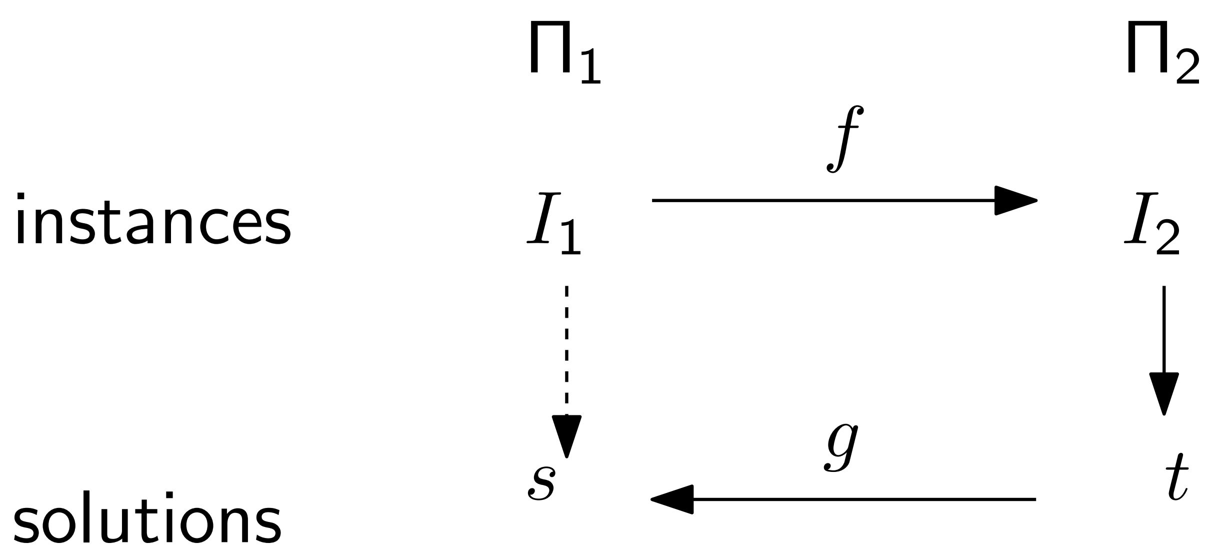
2019

Approximation Preserving Reduction

Let Π_1, Π_2 be minimization problems. An **approximation preserving reduction** from Π_1 to Π_2 is a pair (f, g) of poly-time computable functions with the following properties.

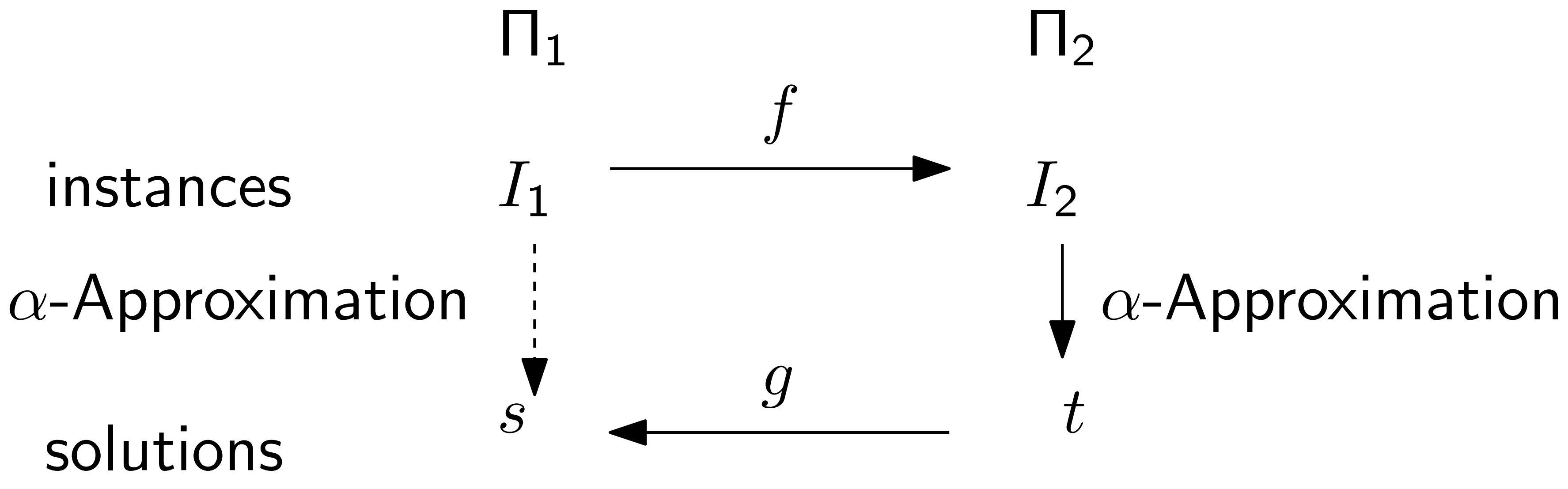
(i) for each instance I_1 of Π_1, $I_2 := f(I_1)$ is an instance of Π_2 where $\mathsf{OPT}_{\Pi_2}(I_2) \leq \mathsf{OPT}_{\Pi_1}(I_1)$

(ii) for each feasible solution t of I_2, $s := g(I_1, t)$ is a feasible solution of I_1 where $\mathsf{obj}_{\Pi_1}(I_1, s) \leq \mathsf{obj}_{\Pi_2}(I_2, t)$



Approximation Preserving Reduction

Thm. Let Π_1, Π_2 be minimization problems where there is an approximation preserving reduction from Π_1 to Π_2. Then, for each factor-α approximation algorithm of Π_2, there is a factor-α approximation algorithm of Π_1.



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Proof.

- Consider a factor-α approx. alg. A of Π_2 and an instance I_1 of Π_1.
- Let $I_2 := f(I_1)$, $t := A(I_2)$ and $s := g(I_1, t)$
- $\mathsf{obj}_{\Pi_1}(I_1, s) \leq \mathsf{obj}_{\Pi_2}(I_2, t) \leq \alpha \cdot \mathsf{OPT}_{\Pi_2}(I_2) \leq \alpha \cdot \mathsf{OPT}_{\Pi_1}(I_1)$

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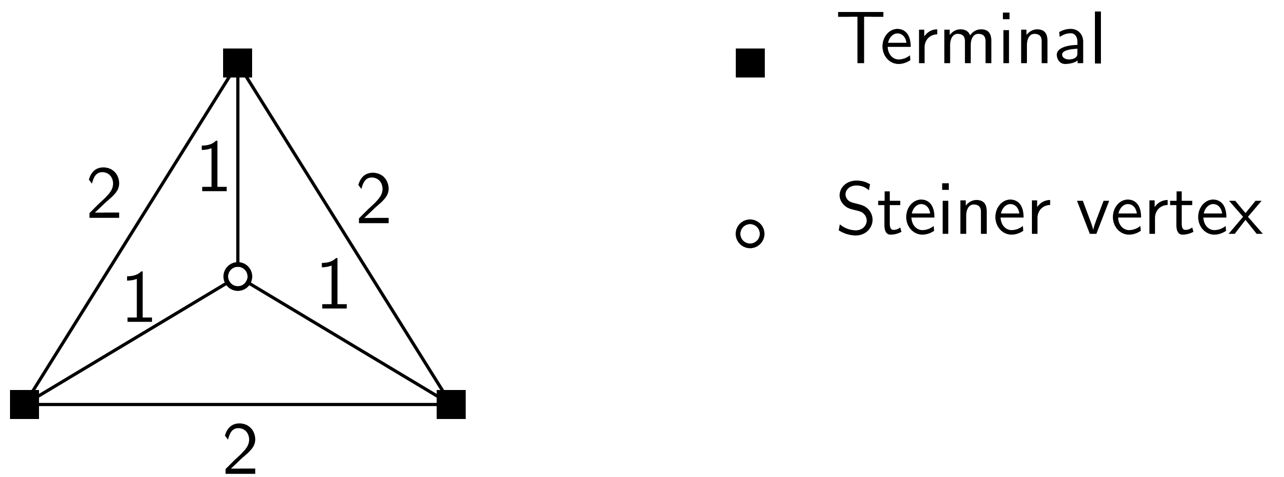
$\square$

SteinerTree

Given: a graph $G = (V, E)$ with edge weights $c\colon E \to \mathbb{Q}^+$ and a partition (T, S) of V into a set T of **Terminals** and a set S of **Steiner vertices**.
Find: a subtree $B = (V', E')$ of G of minimum cost $(c(E') := \sum_{e \in E'} c(e))$ containing all terminals, i.e., $T \subseteq V'$.



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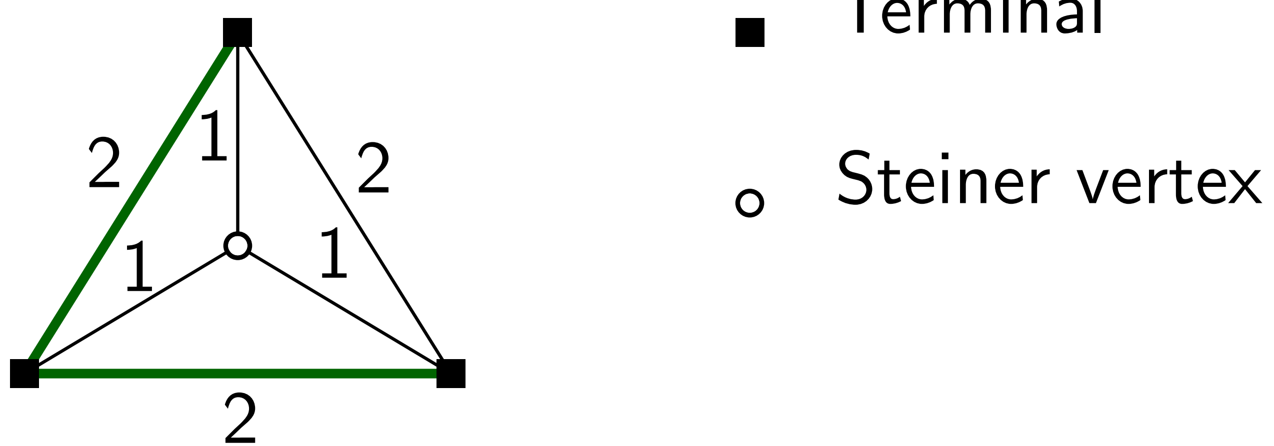
feas. solution: cost 4



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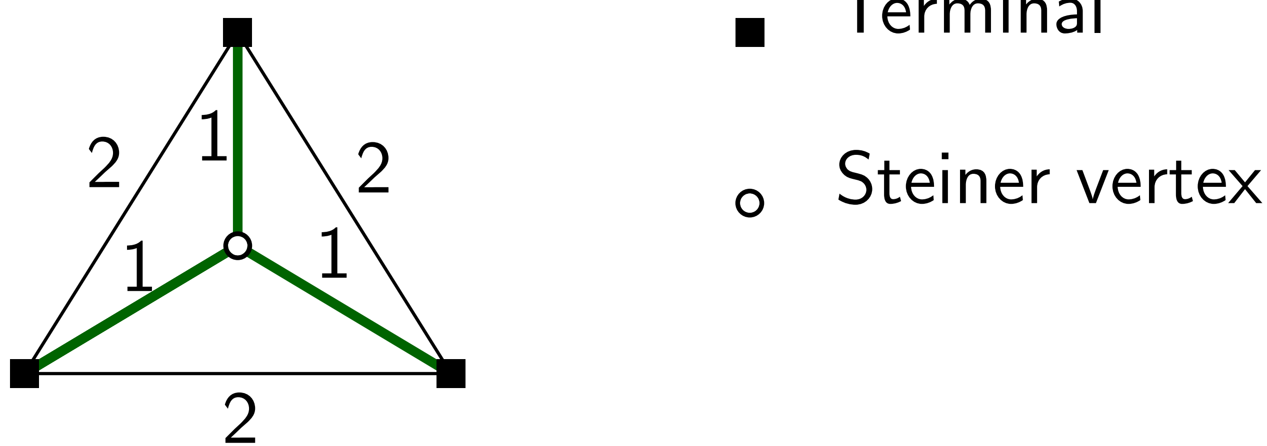
optimal solution: cost 3



MetricSteinerTree

Restriction of SteinerTree where the cost function is **metric**, i.e., graph G is complete (i.e., a clique) and for every triple (u, v, w) of vertices, we have $c(u, w) \leq c(u, v) + c(v, w)$.

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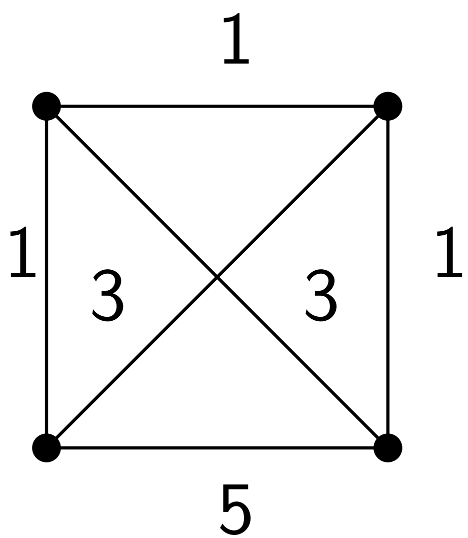

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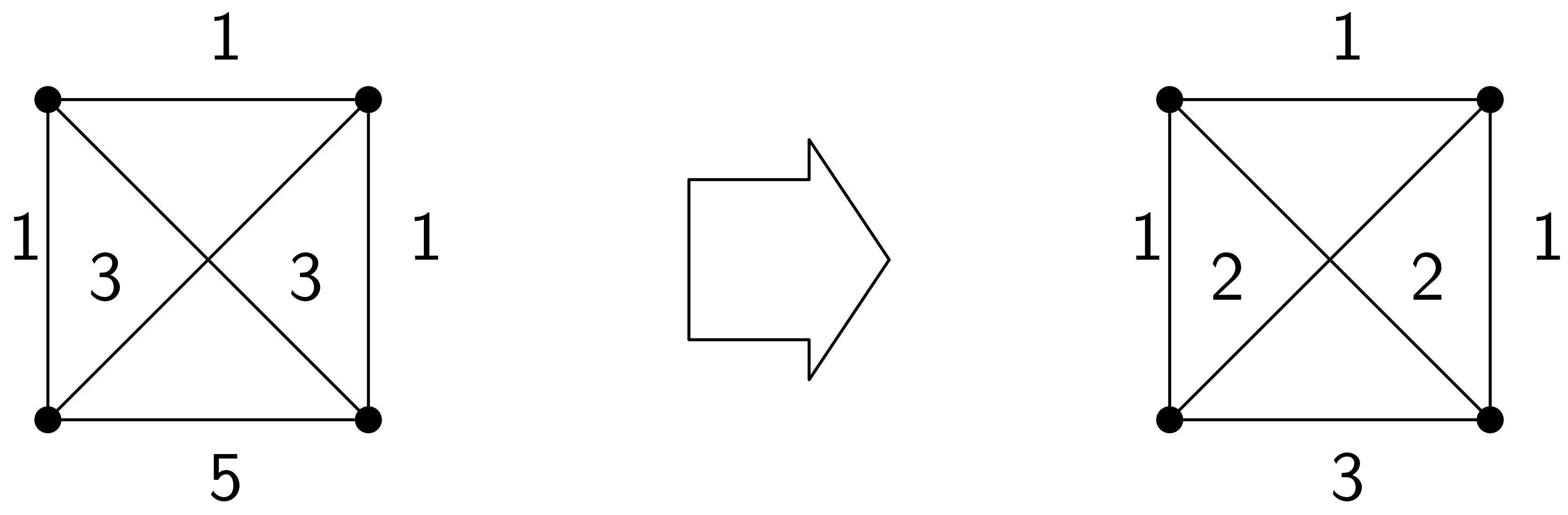

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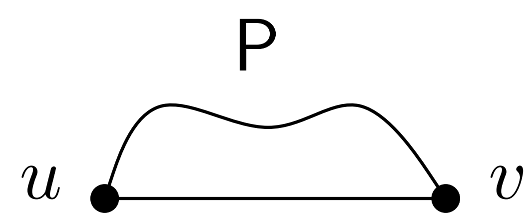

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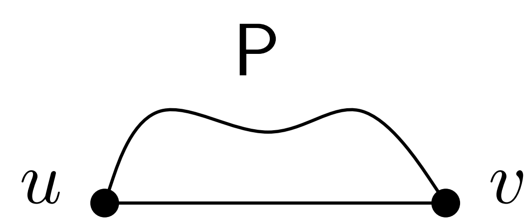

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- Also, recall $c_2(u,v) \leq c_1(u,v)$ for every edge uv of G.
- Thus, $\mathrm{OPT}(I_2) \leq c_2(B^*) \leq c_1(B^*) = \mathrm{OPT}(I_1)$

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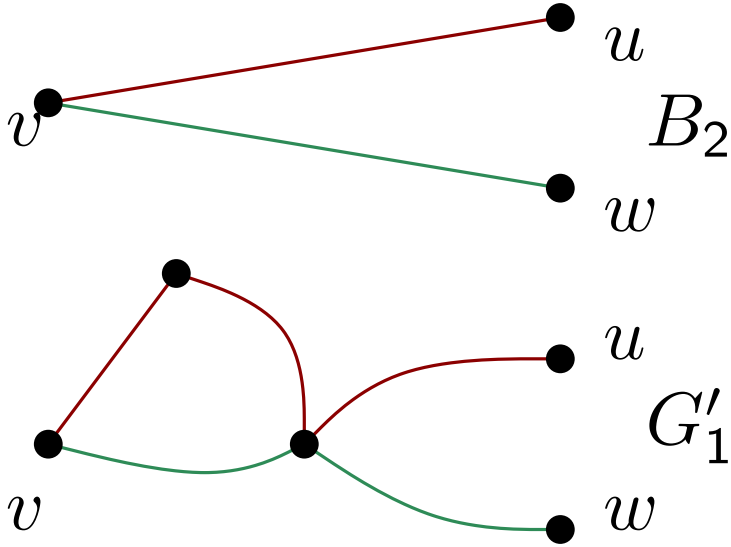

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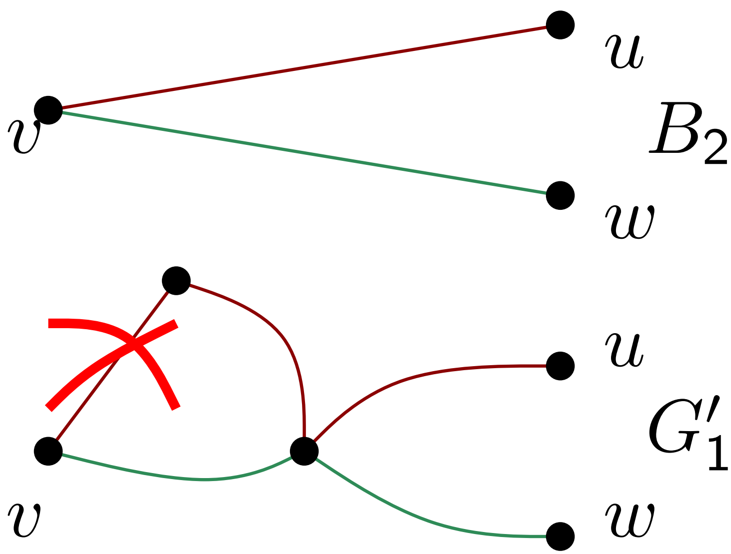

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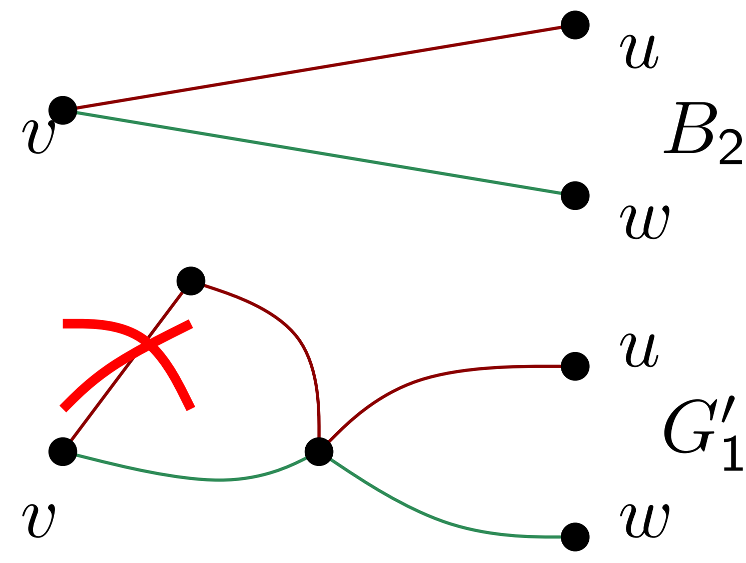

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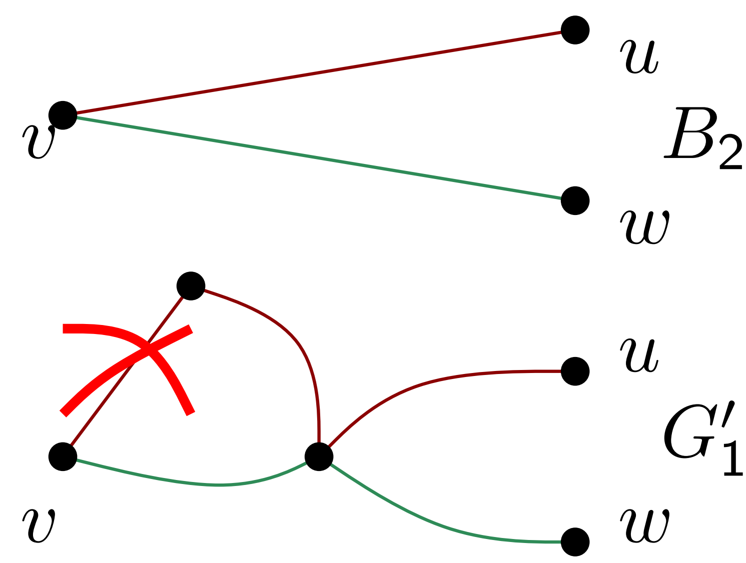

2-Approximation for SteinerTree

Thm. For an instance of MetricSteinerTree, let B be a minimum spanning tree (MST) of the subgraph $G[T] := (T, \{\, uv \mid u, v \in T \,\})$ induced by the terminal set T. We have:

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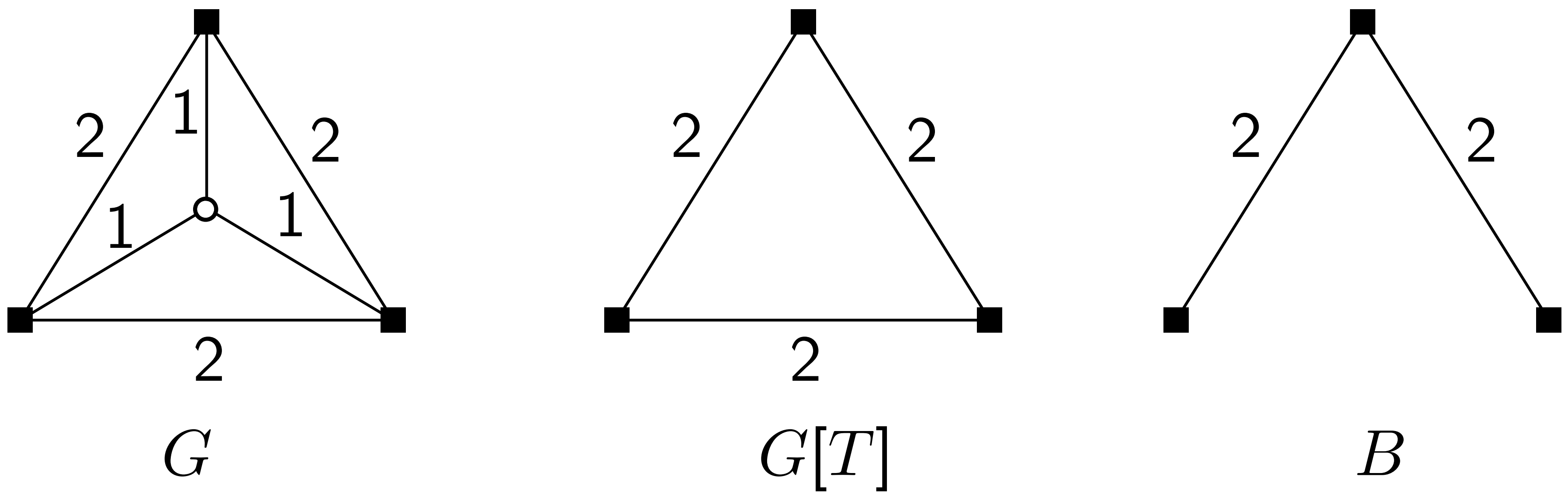

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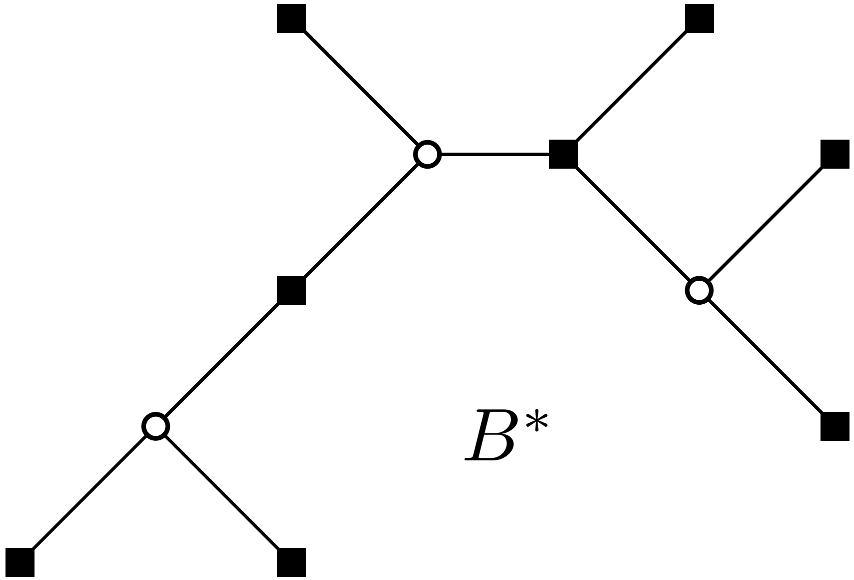

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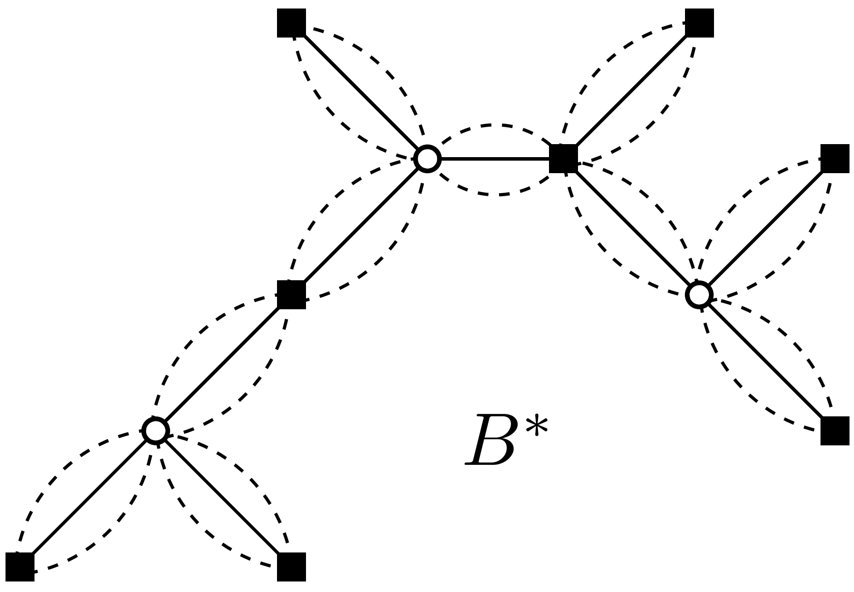

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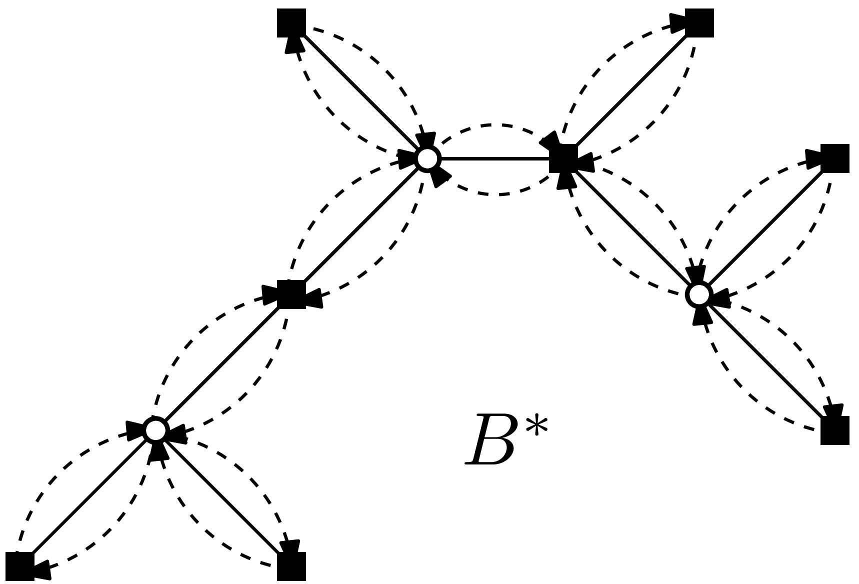

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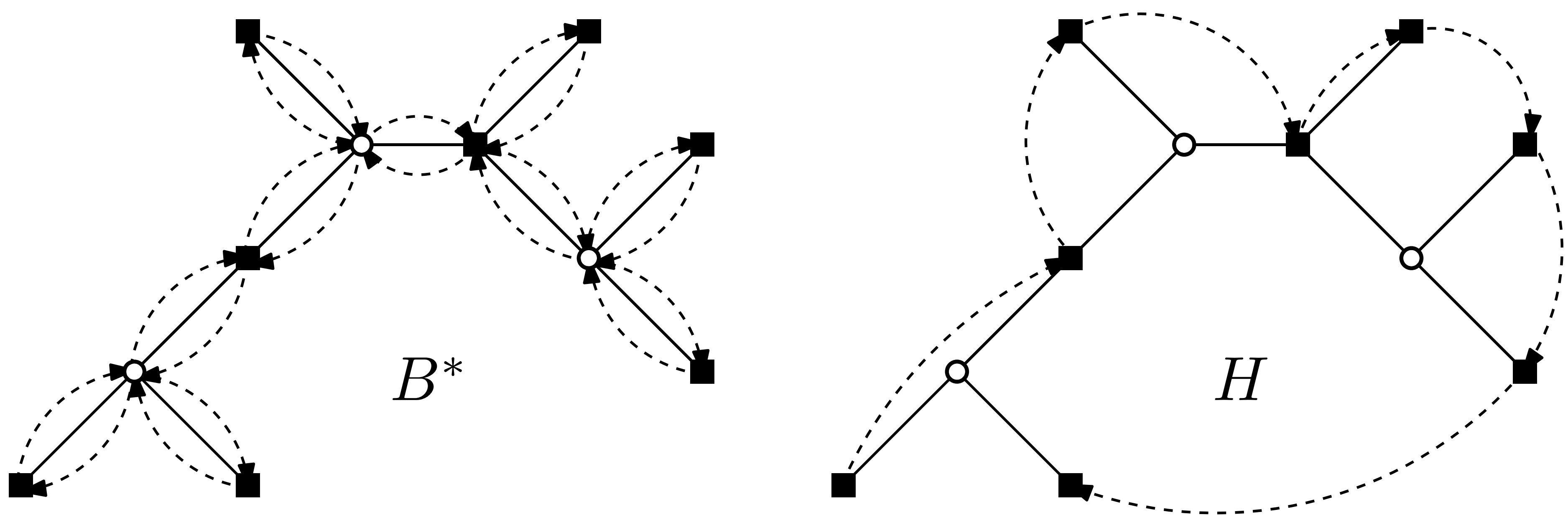

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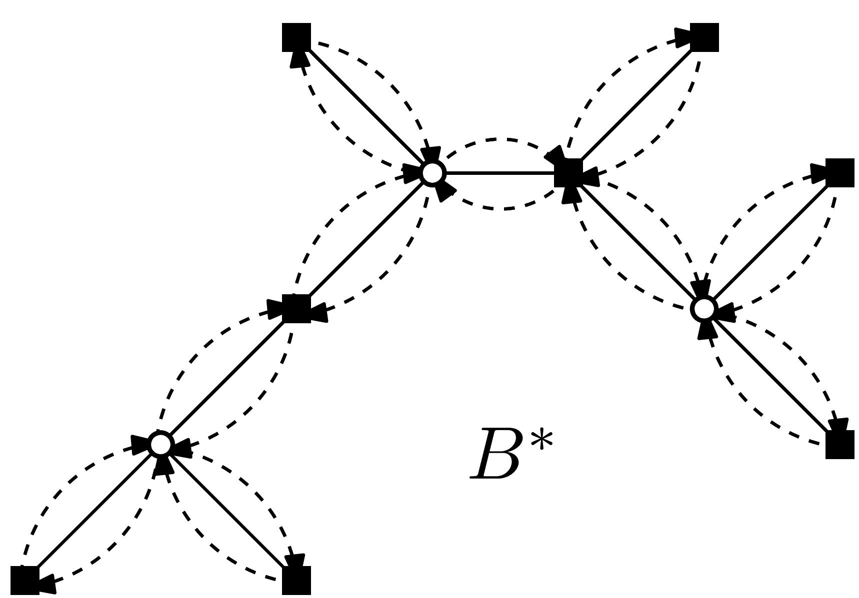

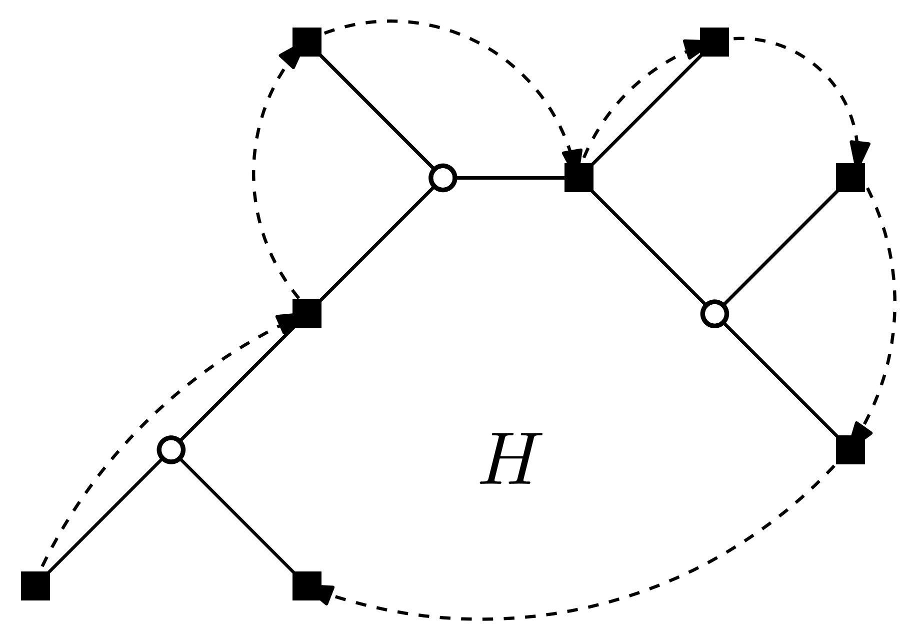

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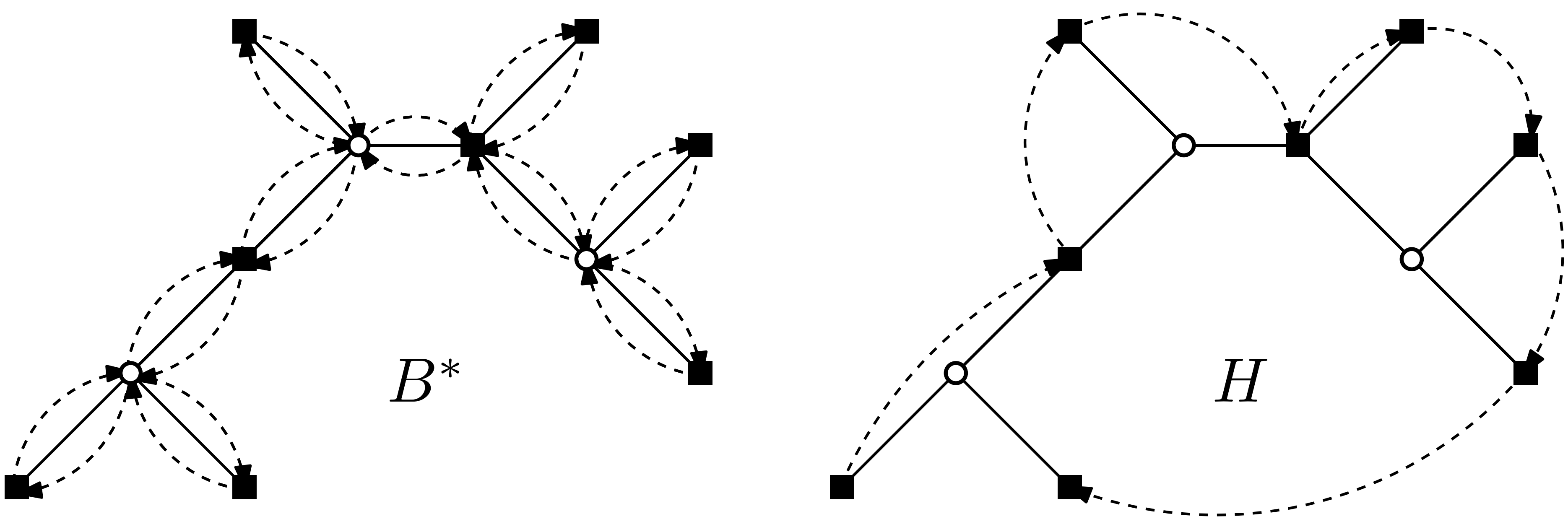

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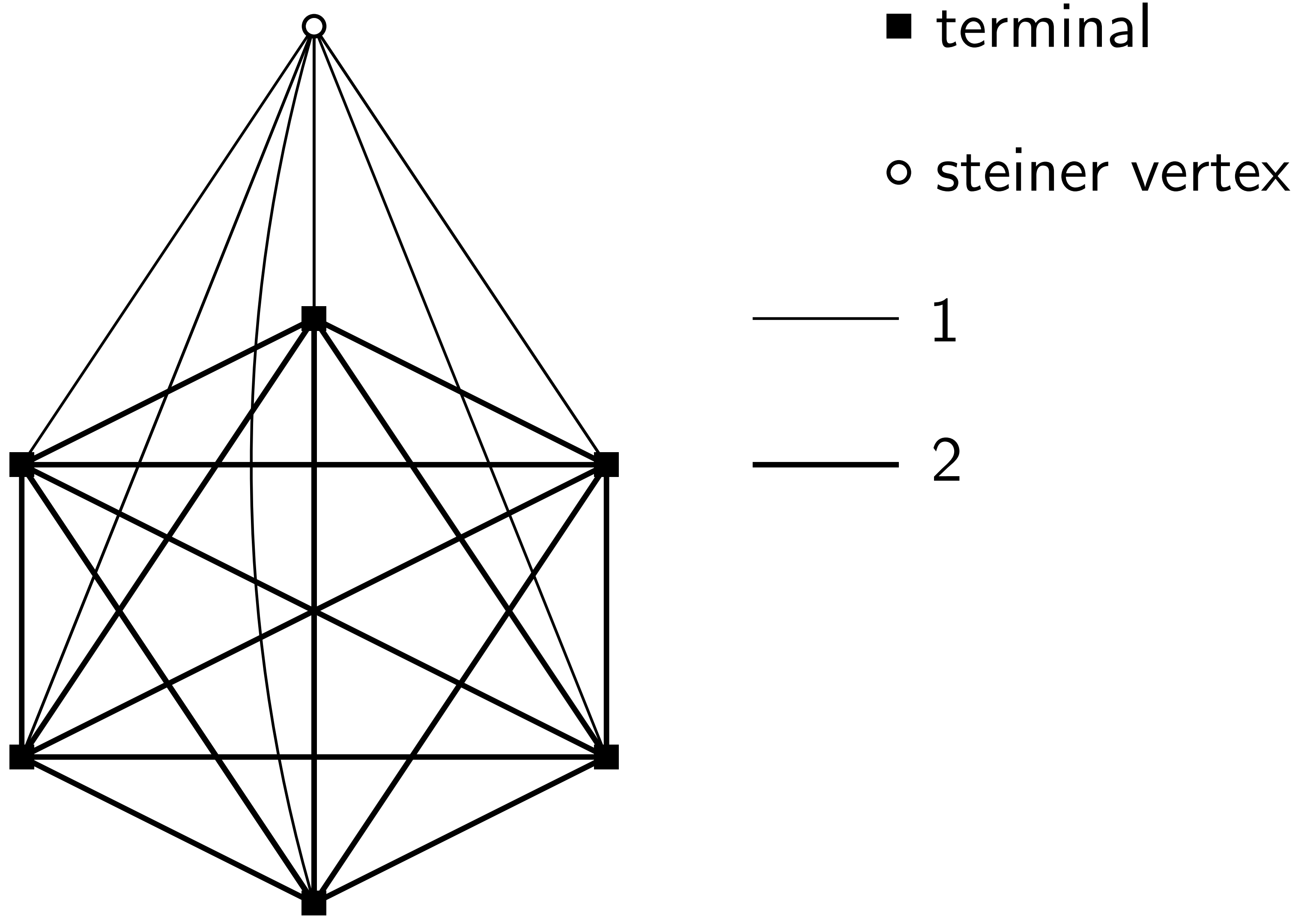
K_n



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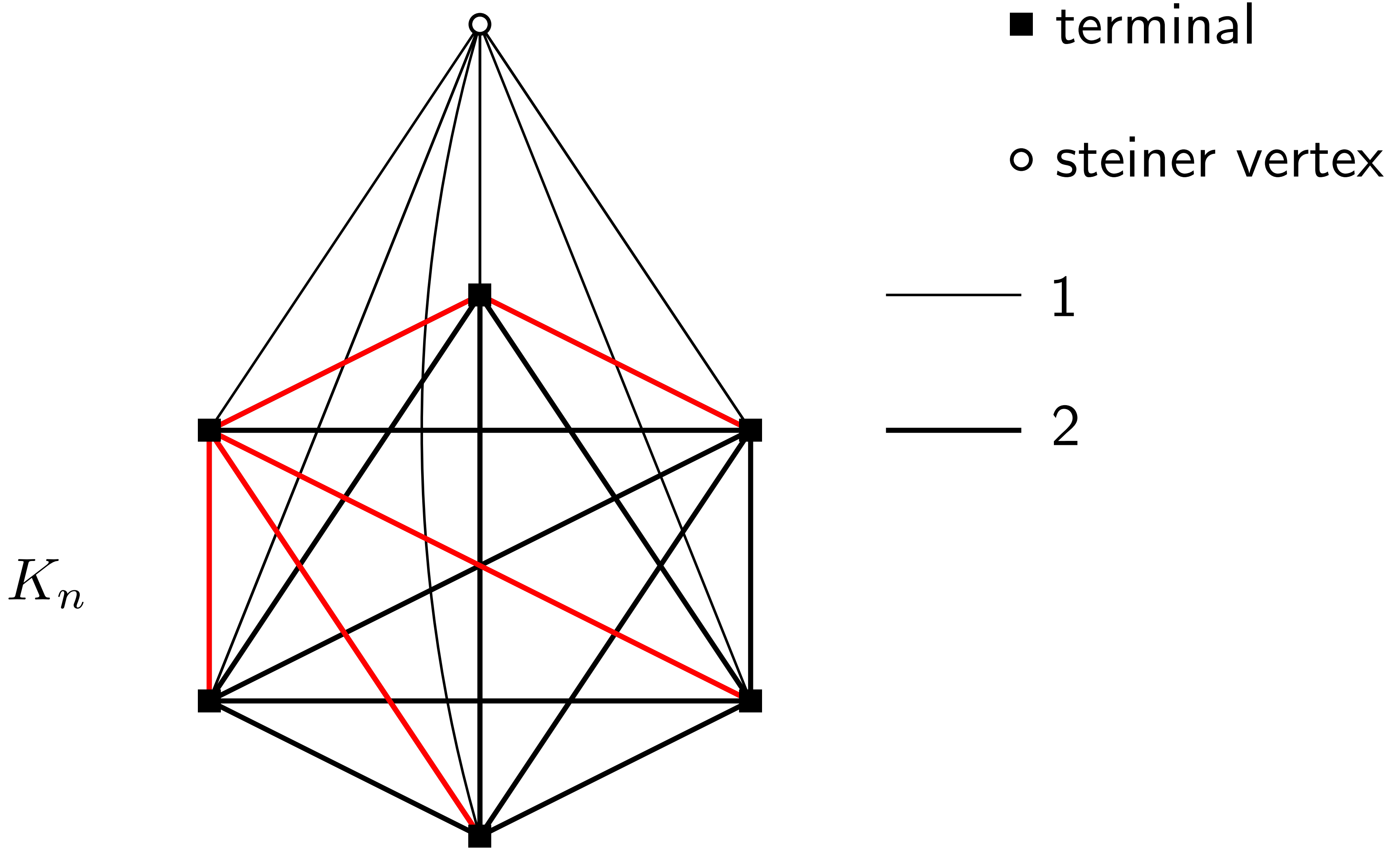
MST of $G[T]$: cost $2(n-1)$



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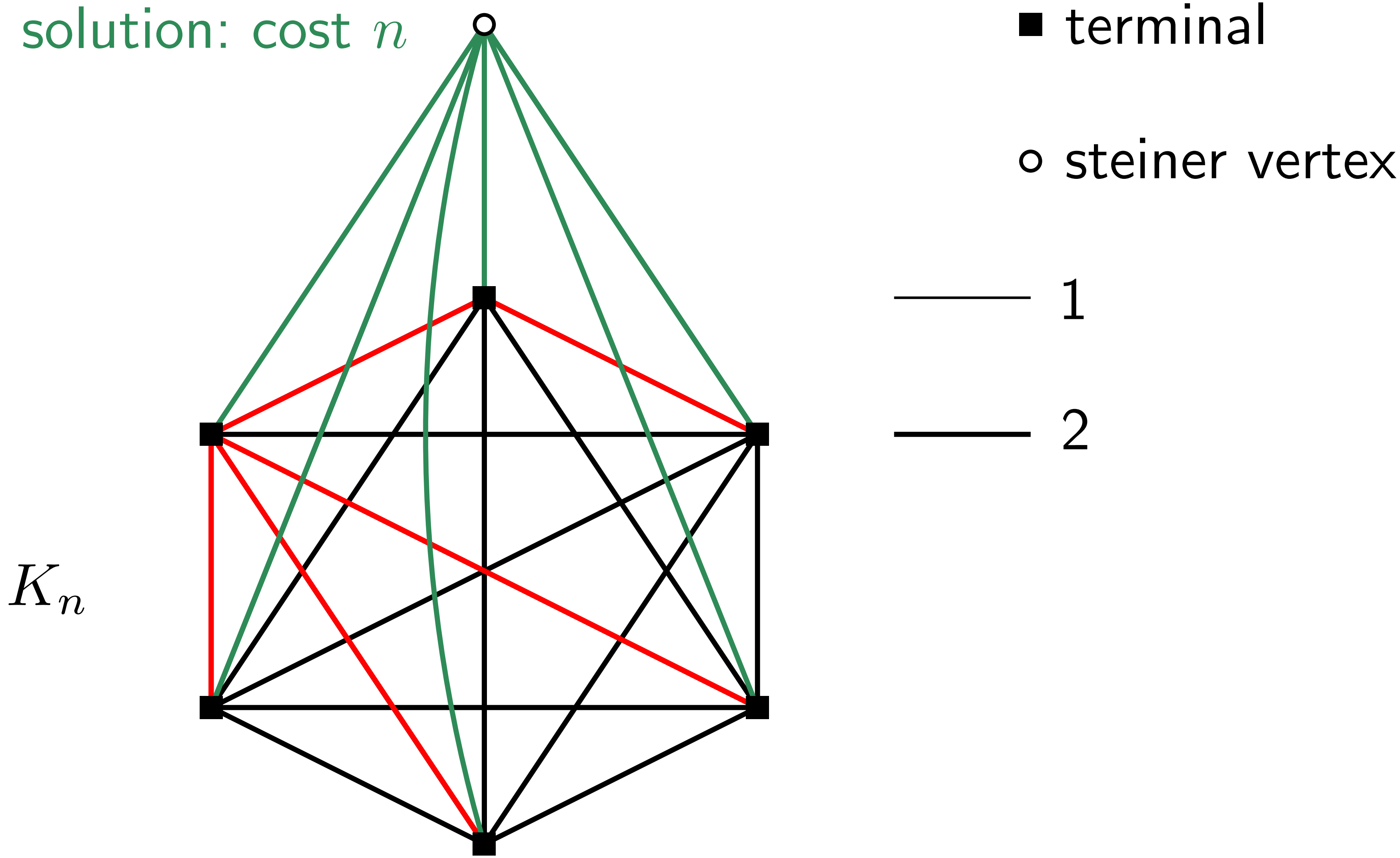

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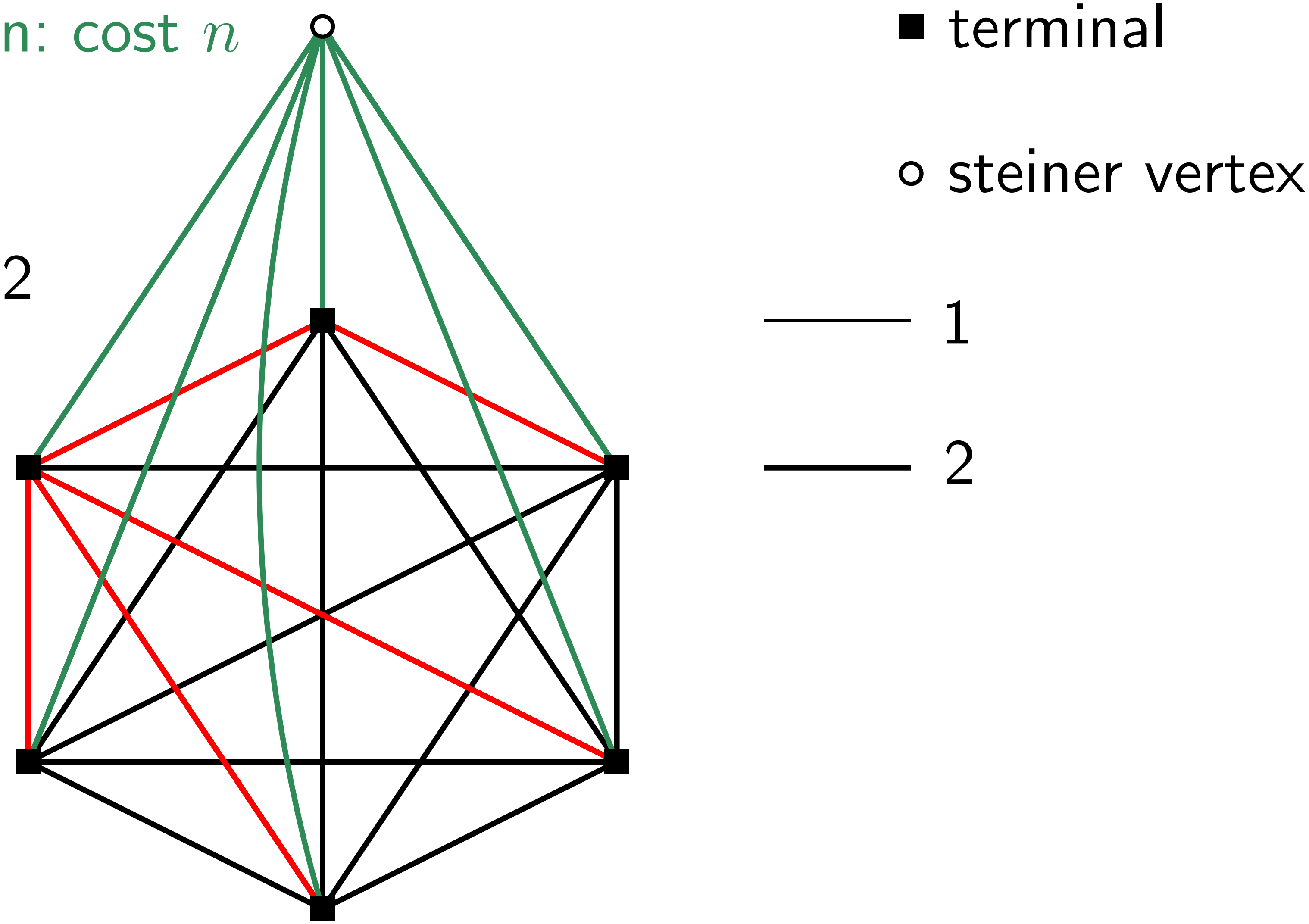
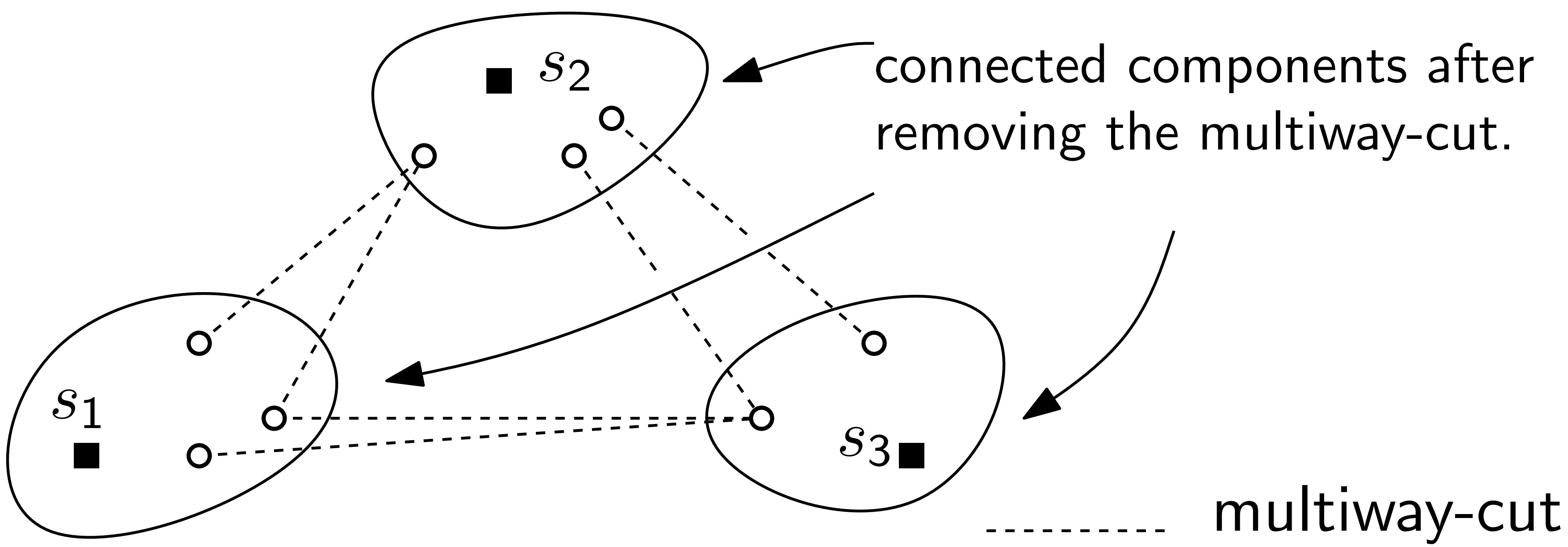
$$\frac{2(n-1)}{n} \to 2$$

K_n



MultiwayCut

Given: a connected graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{Q}_+$ and a set $S = \{s_1, \ldots, s_k\} \subseteq V$ of terminals.
Find: a minimum cost **multiway-cut**, where a subset E' of E is a multiway-cut when no path in the graph $(V, E \setminus E')$ connects two terminals.



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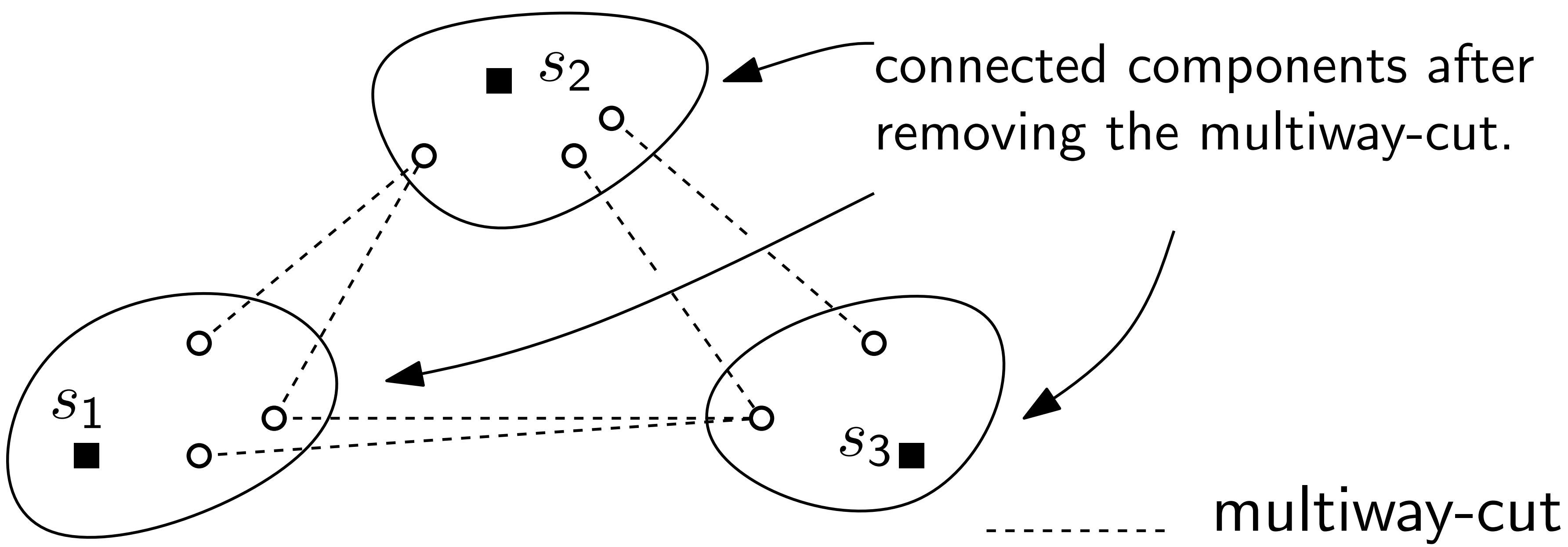

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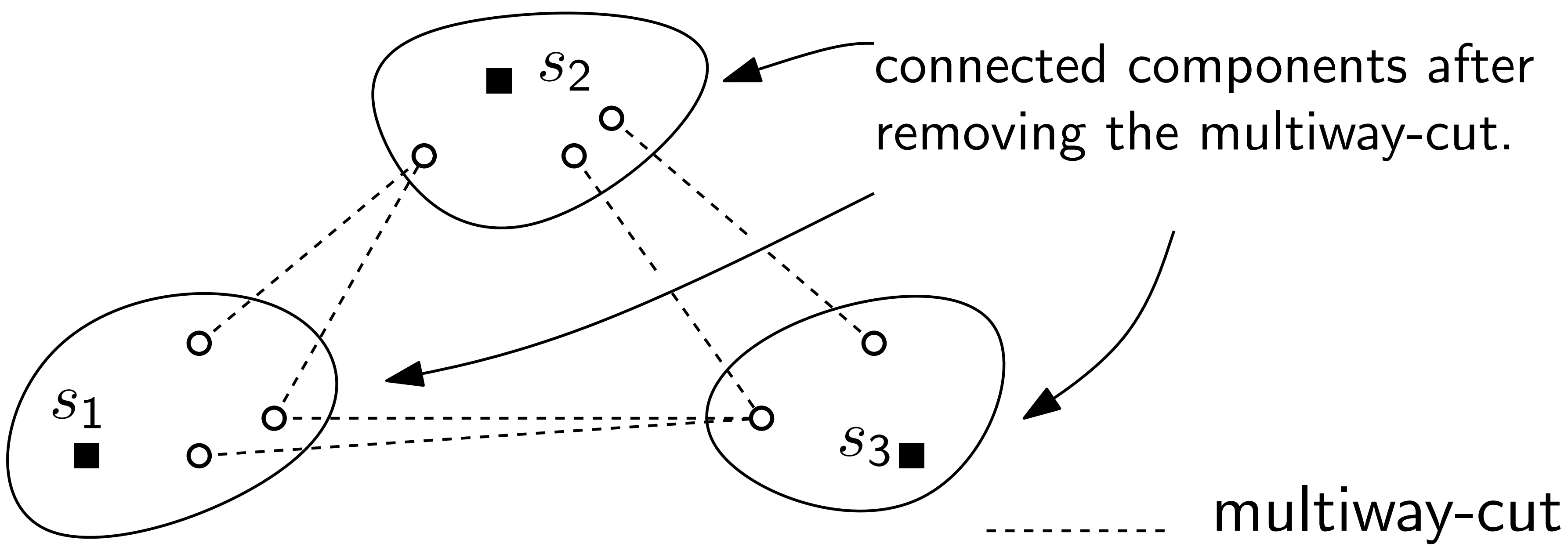
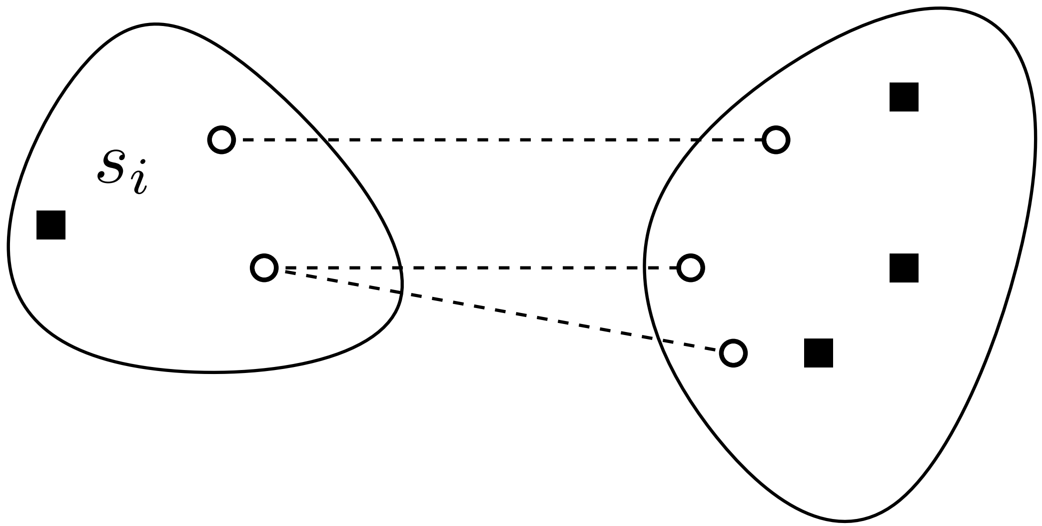
NP-hard for each **fixed** $k \geq 3$. What about $k = 2$?



Isolating Cuts

An **isolating cut** for a terminal s_i is a set of edges separating s_i from all other terminals.



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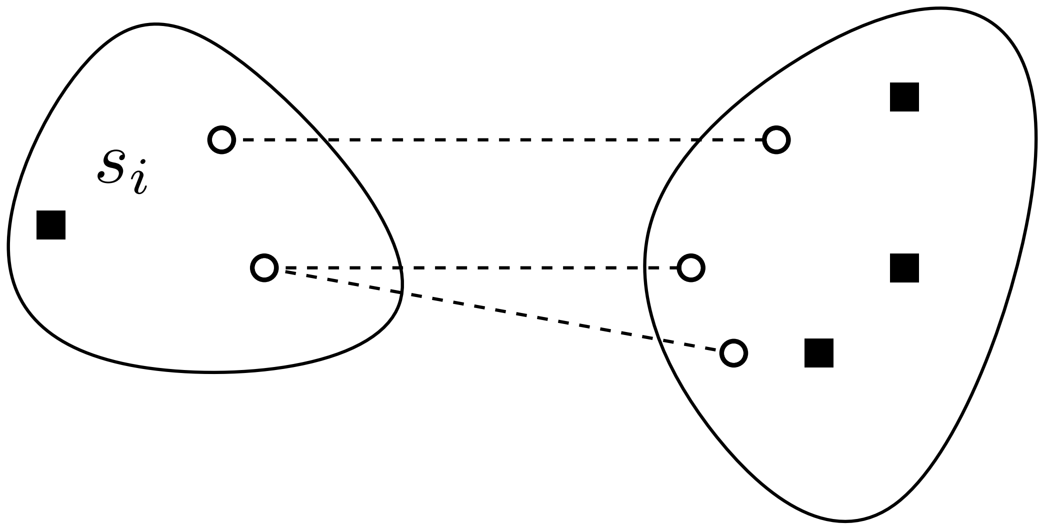

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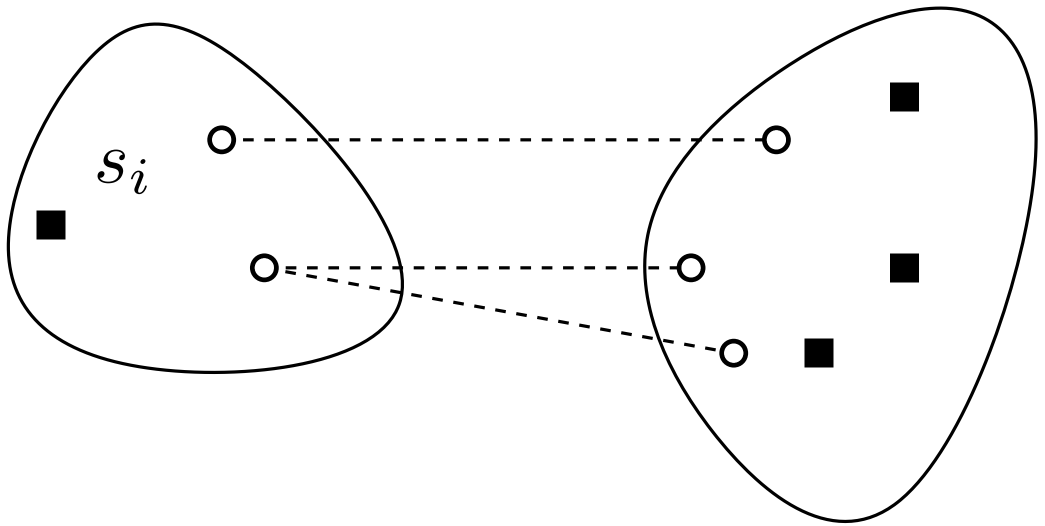

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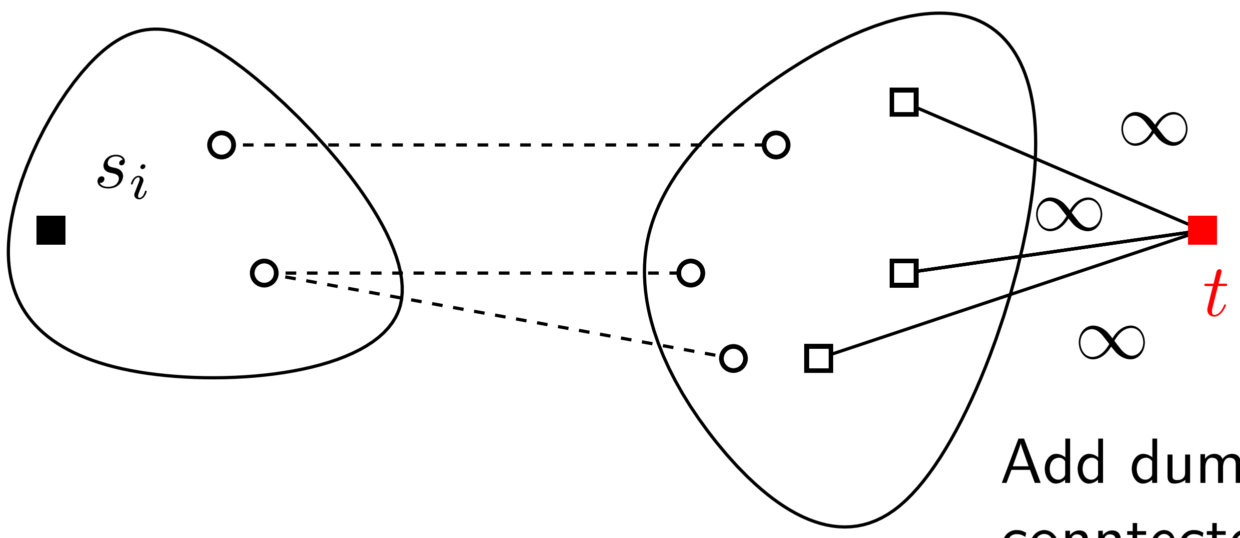

Add dummy terminal t conntected to each $s \in S \setminus \{s_i\}$, and compute minimum s_i–t cut.

Algorithm for MultiwayCut

- For each terminal s_i, compute a minimum isolating cut C_i.
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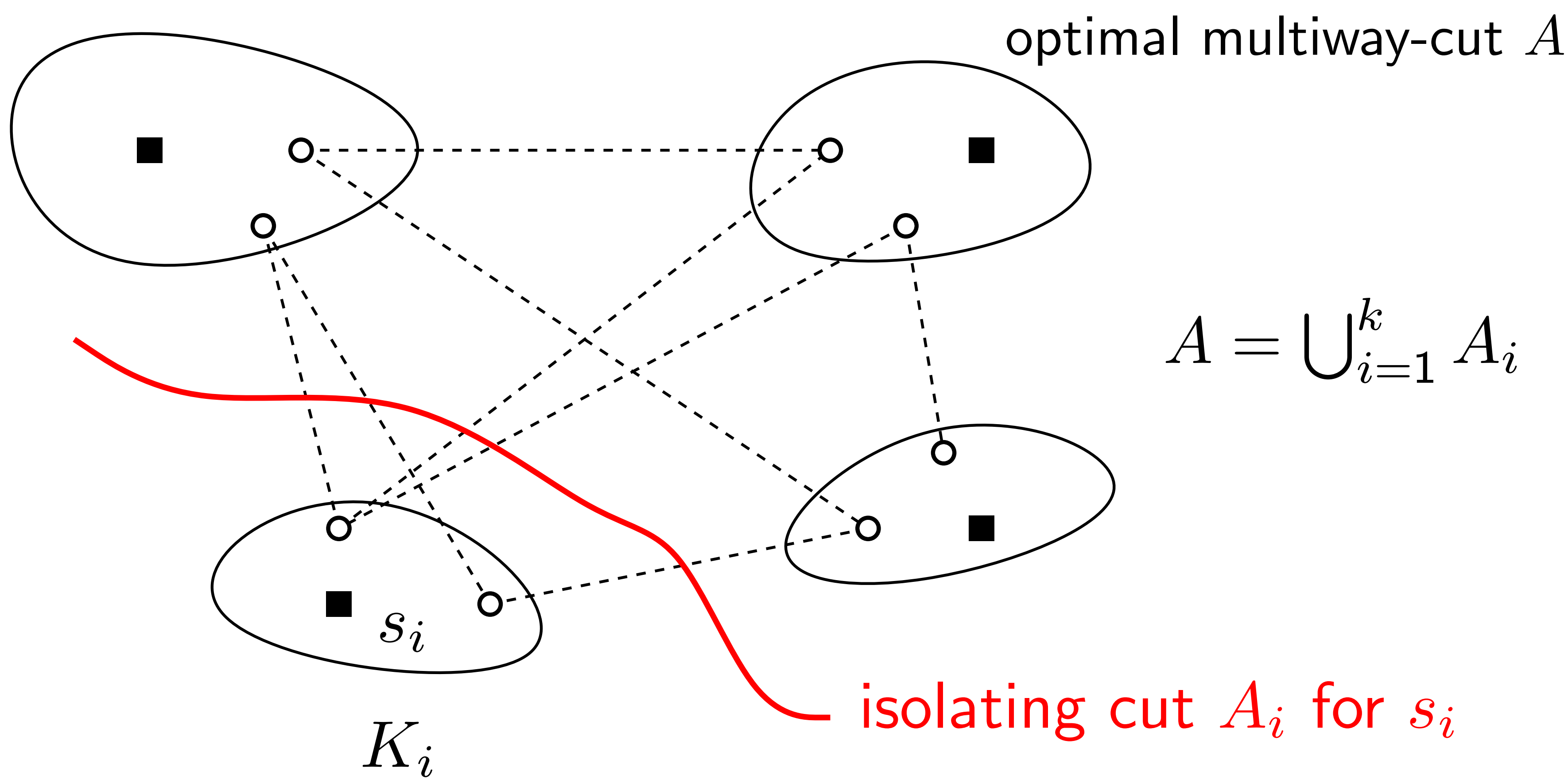
Thm. The above is a factor-$(2 - \frac{2}{k})$ approx. alg.



Is our approximation factor tight?

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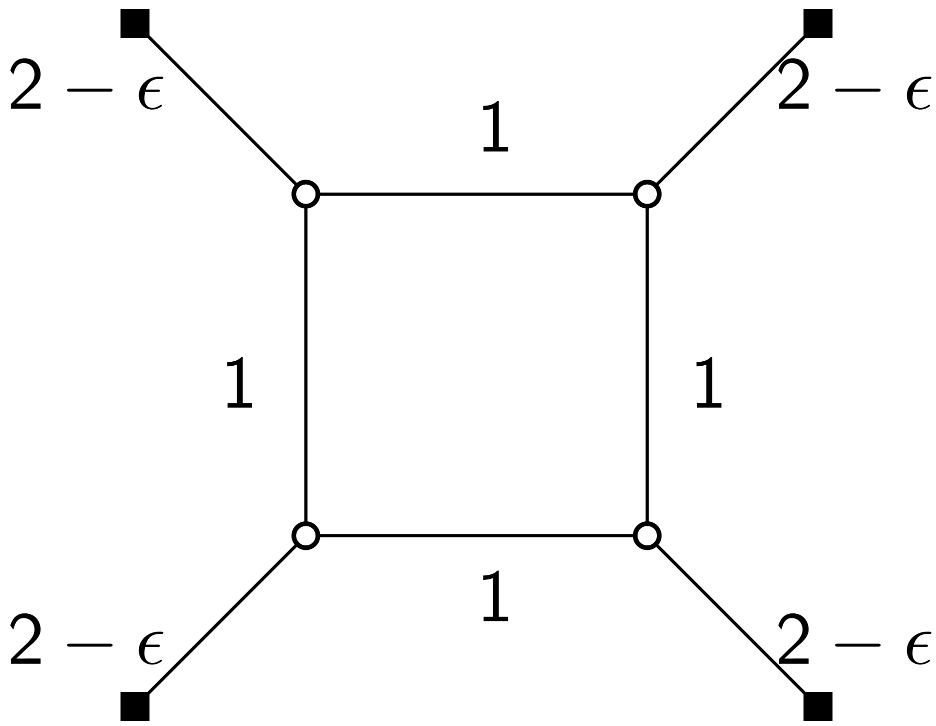

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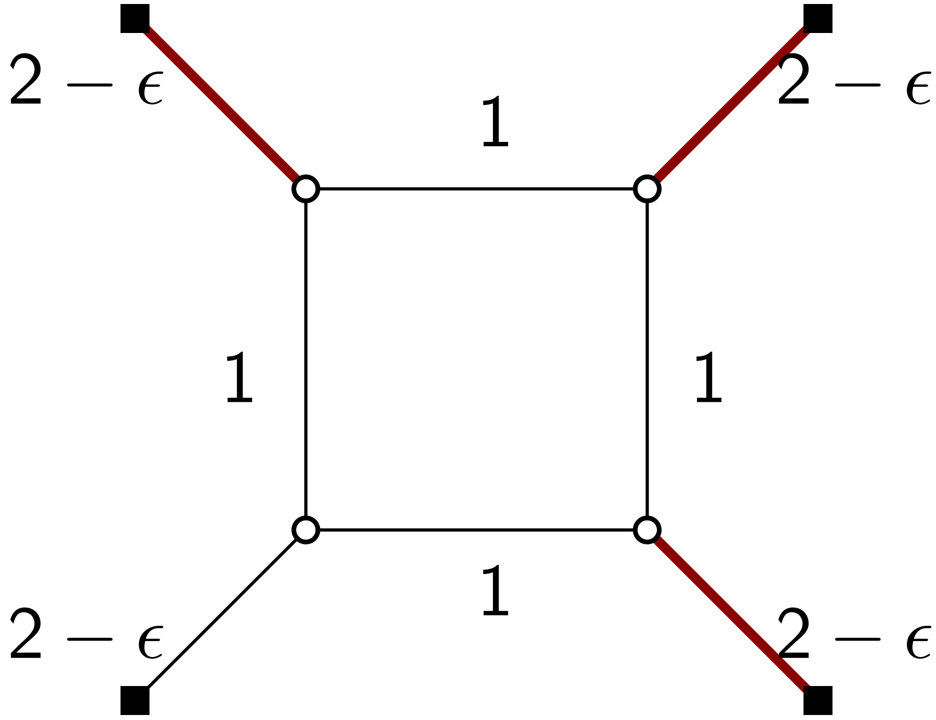

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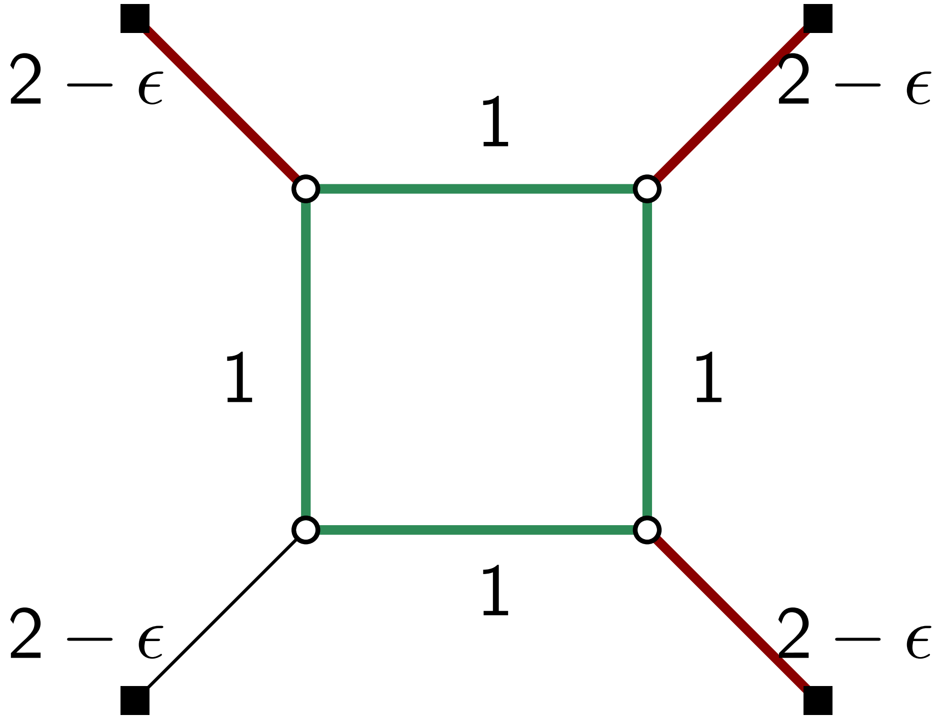

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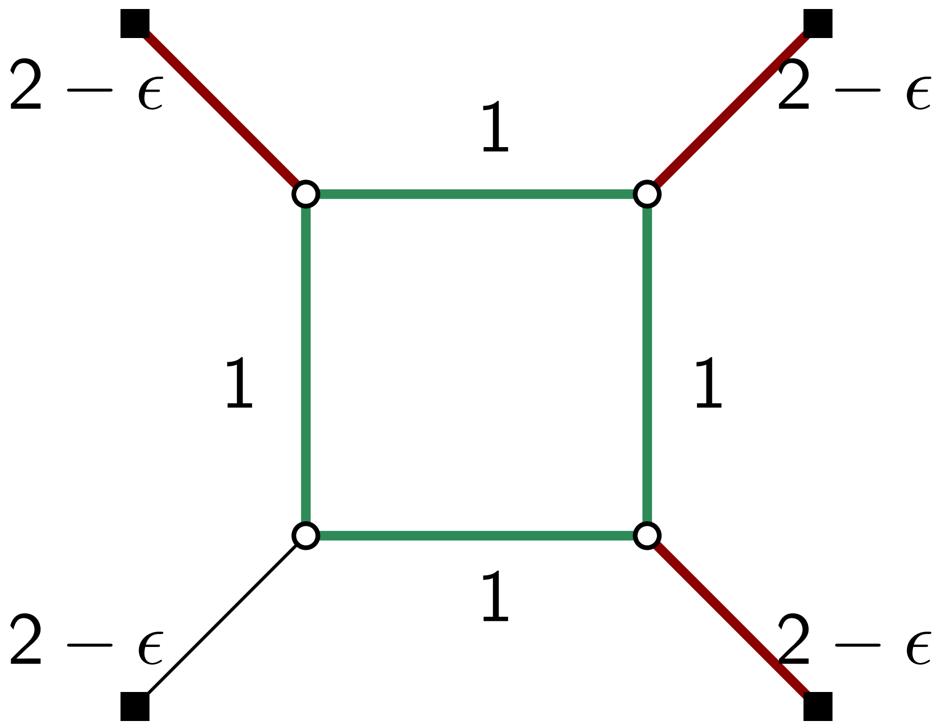

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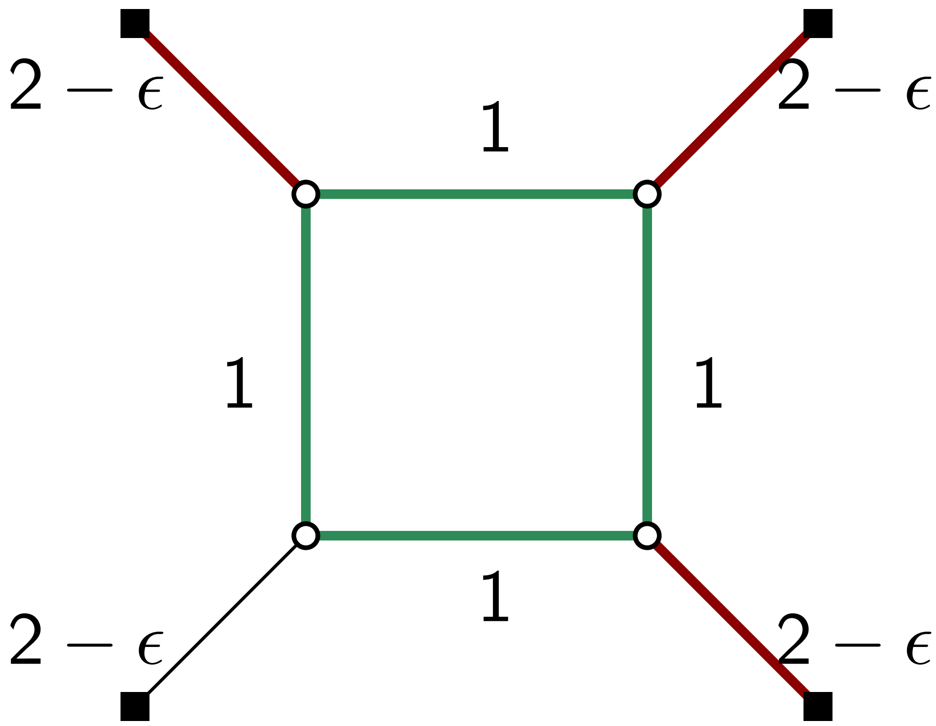

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Next Week: Linear Programming