Aalto University
School of Science

Department of Computer Science

Combinatorics of Efficient Computations

Approximation Algorithms

Lecture 3: Steiner Tree & Multiway Cut

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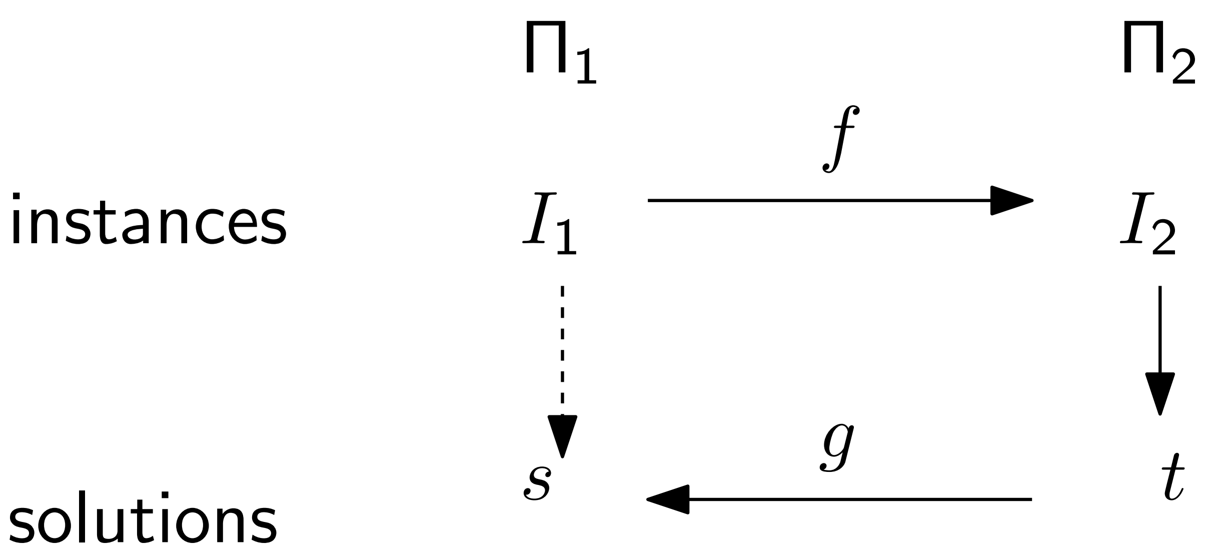
Approximation Preserving Reduction

Let Π_1, Π_2 be minimization problems. An **approximation preserving reduction** from Π_1 to Π_2 is a pair (f, g) of poly-time computable functions with the following properties.

(i) for each instance I_1 of Π_1, $I_2 := f(I_1)$ is an instance of Π_2 where $\mathsf{OPT}_{\Pi_2}(I_2) \leq \mathsf{OPT}_{\Pi_1}(I_1)$

(ii) for each feasible solution t of I_2, $s := g(I_1, t)$ is a feasible solution of I_1 where $\mathsf{obj}_{\Pi_1}(I_1, s) \leq \mathsf{obj}_{\Pi_2}(I_2, t)$



Approximation Preserving Reduction

Thm. Let Π_1, Π_2 be minimization problems where there is an approximation preserving reduction from Π_1 to Π_2. Then, for each factor-α approximation algorithm of Π_2, there is a factor-α approximation algorithm of Π_1.

Proof.

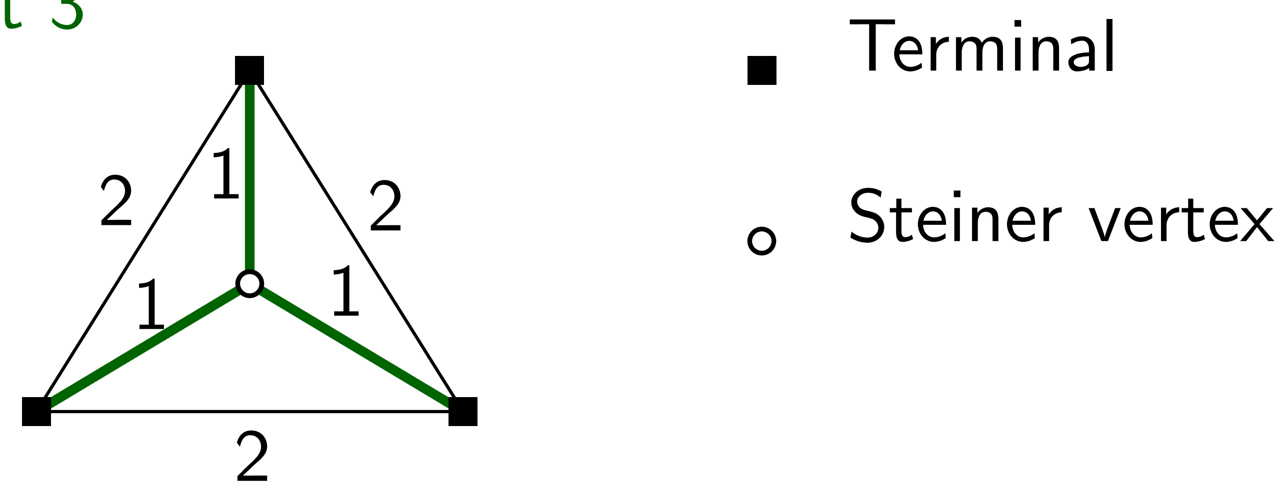
- Consider a factor-α approx. alg. A of Π_2 and an instance I_1 of Π_1.
- Let $I_2 := f(I_1)$, $t := A(I_2)$ and $s := g(I_1, t)$
- $\mathsf{obj}_{\Pi_1}(I_1, s) \leq \mathsf{obj}_{\Pi_2}(I_2, t) \leq \alpha \cdot \mathsf{OPT}_{\Pi_2}(I_2) \leq \alpha \cdot \mathsf{OPT}_{\Pi_1}(I_1)$

□

SteinerTree

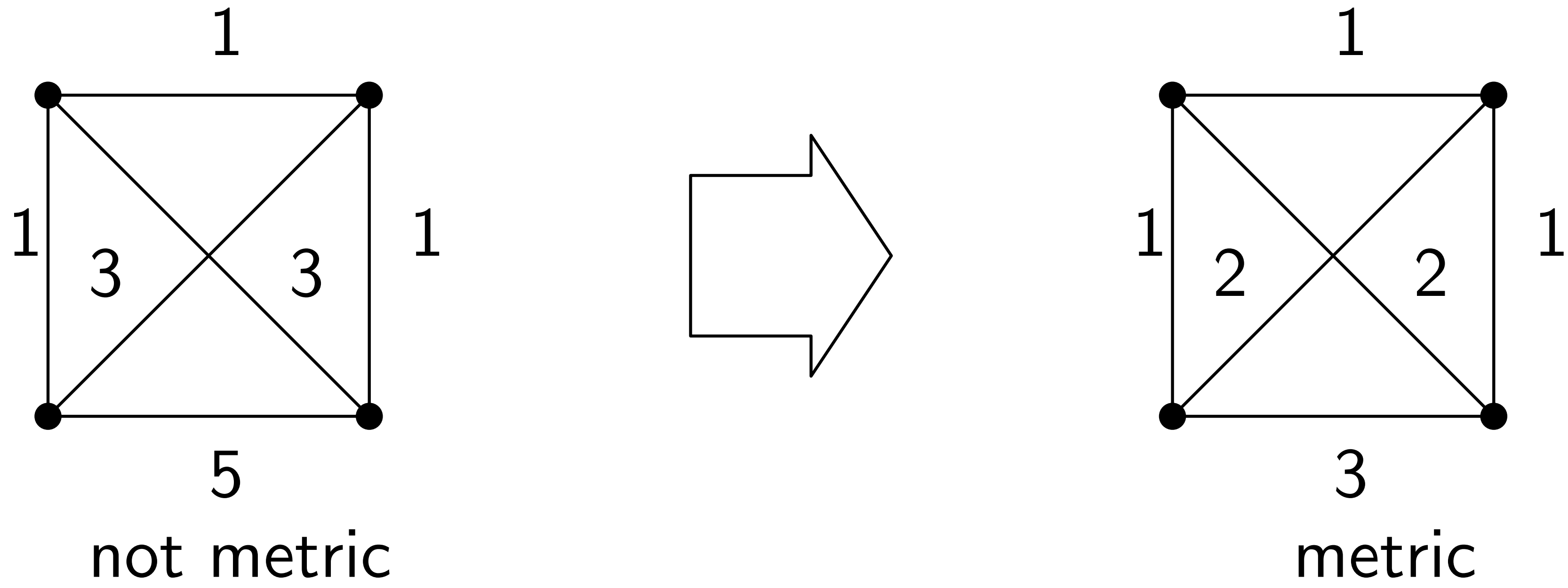
Given: a graph $G = (V, E)$ with edge weights $c\colon E \to \mathbb{Q}^+$ and a partition (T, S) of V into a set T of **Terminals** and a set S of **Steiner vertices**.
Find: a subtree $B = (V', E')$ of G of minimum cost $(c(E') := \sum_{e \in E'} c(e))$ containing all terminals, i.e., $T \subseteq V'$.

optimal solution: cost 3



MetricSteinerTree

Restriction of SteinerTree where the cost function is **metric**, i.e., graph G is complete (i.e., a clique) and for every triple (u, v, w) of vertices, we have $c(u, w) \leq c(u, v) + c(v, w)$.

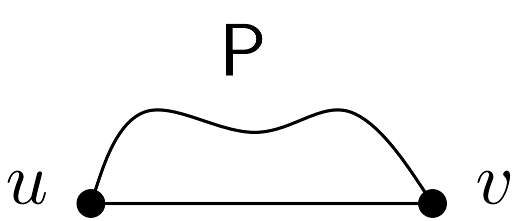


MetricSteinerTree

Thm. There is an approximation preserving reduction from SteinerTree to MetricSteinerTree.

Proof. **Part 1:** The mapping f.

- Instance I_1 of SteinerTree: Graph $G_1 = (V, E_1)$, edge weights c_1, partition $V = S \dot{\cup} T$
- Metric Instance $I_2 := f(I_1)$: complete graph $G_2 = (V, E_2)$, partition (T, S) as in I_1, and cost funtion c_2 as below.
- $c_2(u, v) :=$ length of a shortest u–v-path in G_1, for every $u, v \in V$
- $c_2(u, v) \leq c_1(u, v)$ for every $(u, v) \in E$



$c_2(u, v) := c_1(P)$

MetricSteinerTree

Thm. There is an approximation preserving reduction from SteinerTree to MetricSteinerTree.

Proof. **Part 2:** $\mathsf{OPT}(I_2) \leq \mathsf{OPT}(I_1)$.

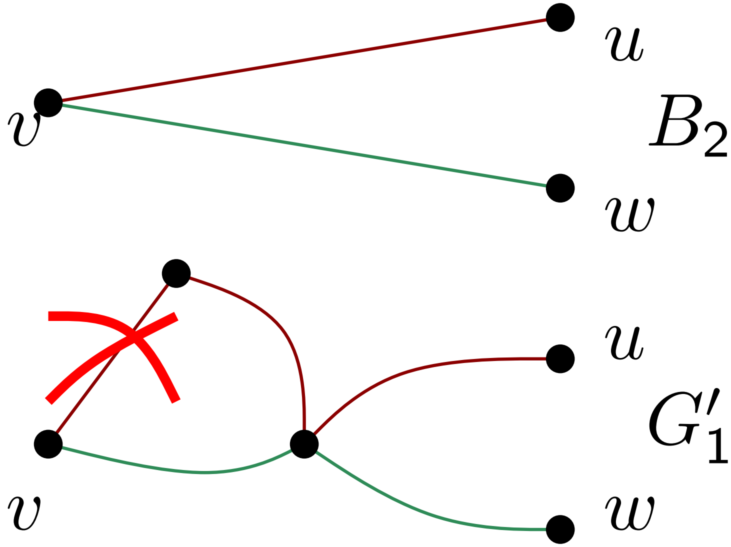
- Let B^* be an optimal subtree of I_1
- B^* is also feasible for I_2, since $E_1 \subseteq E_2$ and the vertex sets V, S, T are the same
- Also, recall $c_2(u, v) \leq c_1(u, v)$ for every edge uv of G.
- Thus, $\mathsf{OPT}(I_2) \leq c_2(B^*) \leq c_1(B^*) = \mathsf{OPT}(I_1)$

MetricSteinerTree

Thm. There is an approximation preserving reduction from SteinerTree to MetricSteinerTree.

Proof. **Part 3:** The mapping g.

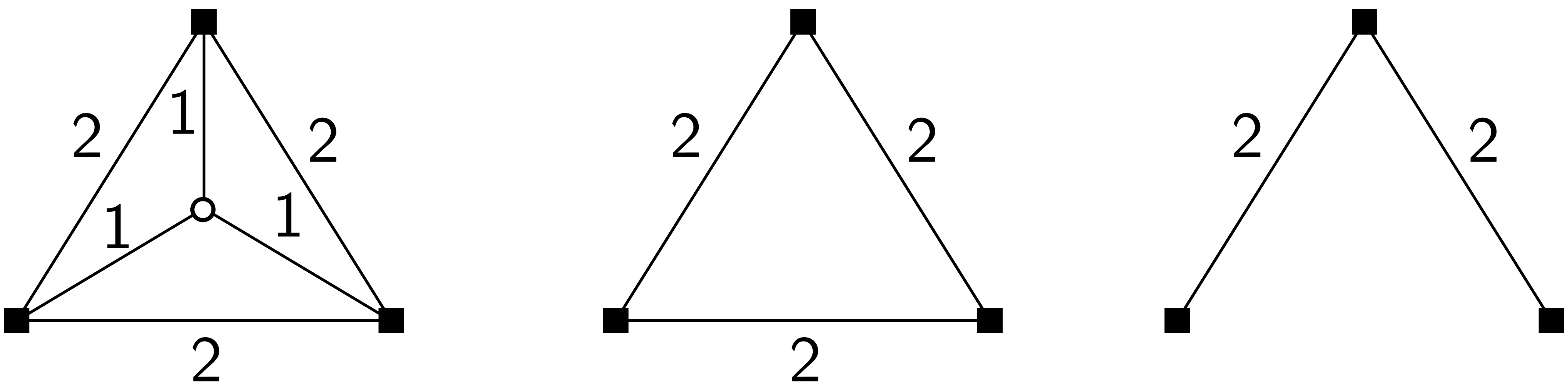
- Let B_2 be a steiner tree for G_2
- Construct $G_1' \subseteq G_1$ from B_2 by replacing each edge (u, v) of B_2 by a shortest u–v-path in G_1.
- Note: $c_1(G_1') \leq c_2(B_2)$
- G_1' connects all terminals
- G_1' not necessarily a tree
- Pick B_1 as a spanning tree of G_1'
- $\rightsquigarrow$ B_1 is a Steiner tree of G_1
- $c_1(B_1) \leq c_1(G_1') \leq c_2(B_2)$ $\square$



2-Approximation for SteinerTree

Thm. For an instance of MetricSteinerTree, let B be a minimum spanning tree (MST) of the subgraph $G[T] := (T, \{ uv \mid u, v \in T \})$ induced by the terminal set T. We have:

$$c(B) \le 2 \cdot \mathsf{OPT}$$

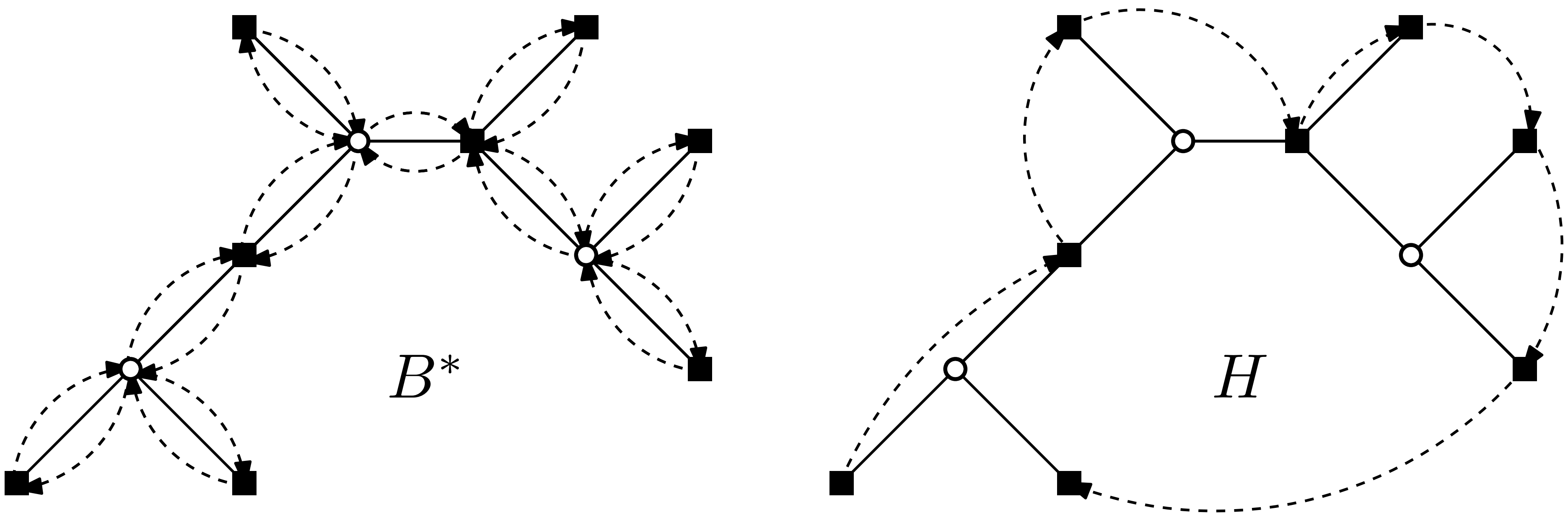


G $G[T]$ B

Proof

*see glossary document

- Consider an optimal steiner tree B^*
- duplicate all edges in B^* $\rightsquigarrow$ Eulerian (Multi-)Graph B' with cost $c(B') = 2 \cdot \mathrm{OPT}$
- Find an Eulerian tour T' in B' $\rightsquigarrow$ $c(T') = c(B') = 2 \cdot \mathrm{OPT}$
- Find a Hamiltonian path H in $G[T]$ by “short-cutting” Steiner vertices and previously visited terminals $\rightsquigarrow$ $c(H) \leq c(T) = 2 \cdot \mathrm{OPT}$, since G is metric.
- Note: any MST B of $G[T]$ has $c(B) \leq c(H) \leq 2 \cdot \mathrm{OPT}$, since H is a spanning tree of $G[T]$ $\square$



Is there a tight example?

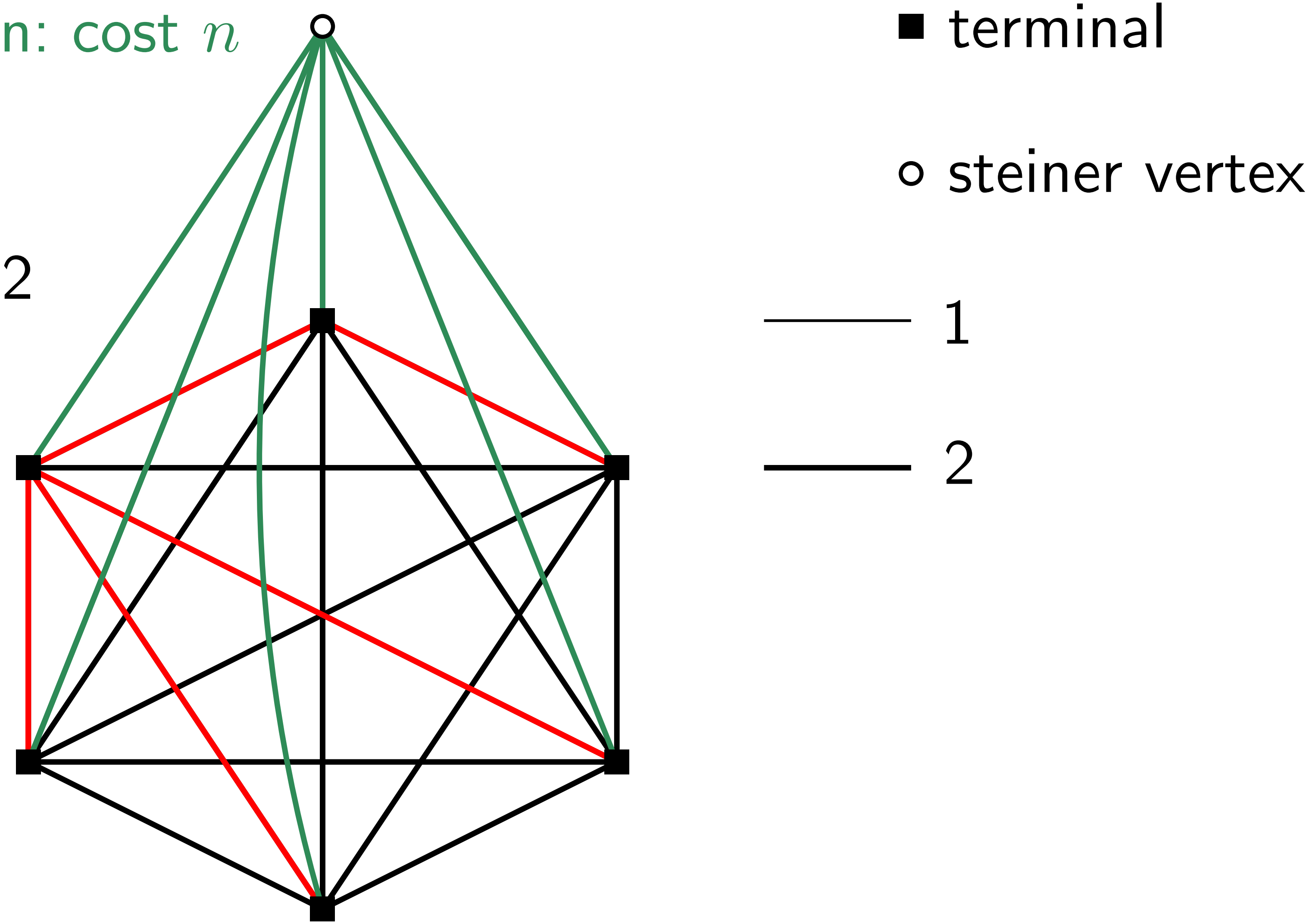
i.e., is there a graph where our algorithmic solution is $2 \cdot \text{OPT}$?

MST of $G[T]$: cost $2(n-1)$

Optimal solution: cost n

$$\frac{2(n-1)}{n} \rightarrow 2$$

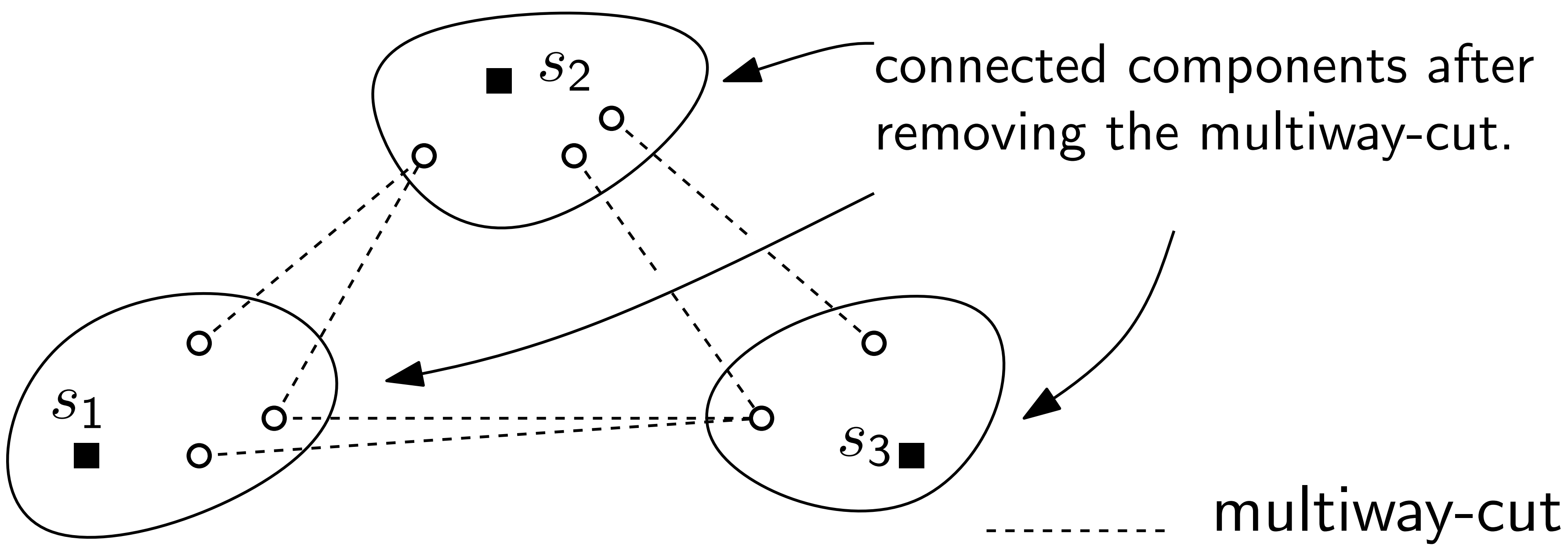
K_n



MultiwayCut

Given: a connected graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{Q}_+$ and a set $S = \{s_1, \ldots, s_k\} \subseteq V$ of terminals.
Find: a minimum cost **multiway-cut**, where a subset E' of E is a multiway-cut when no path in the graph $(V, E \setminus E')$ connects two terminals.

NP-hard for each **fixed** $k \geq 3$. What about $k = 2$?

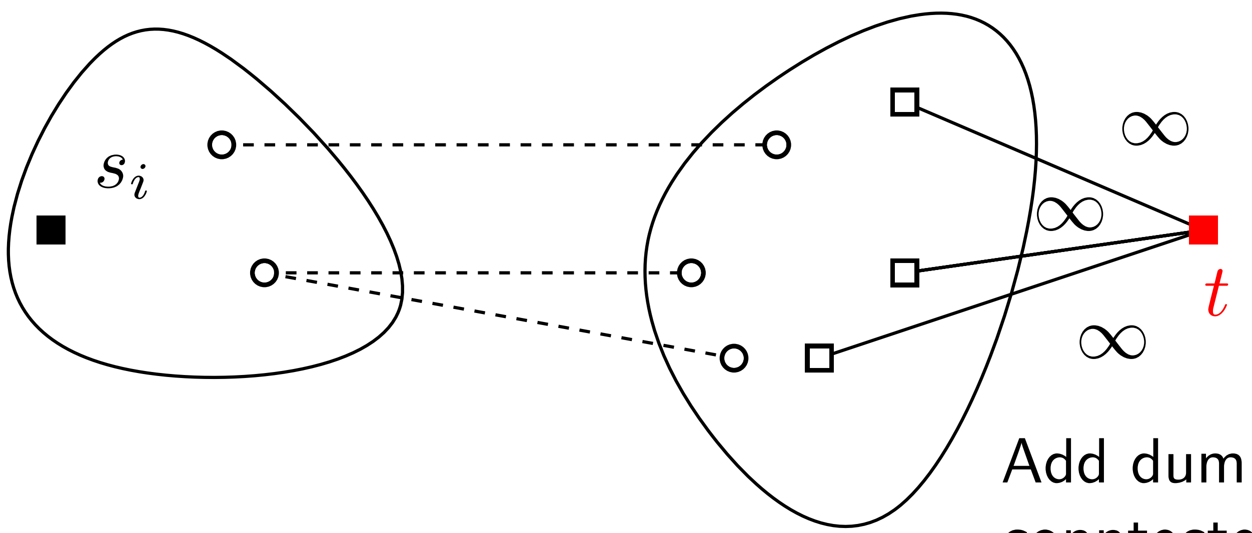


Isolating Cuts

An **isolating cut** for a terminal s_i is a set of edges separating s_i from all other terminals.

Can we compute a minimum isolating cut efficiently?

Yes :-)

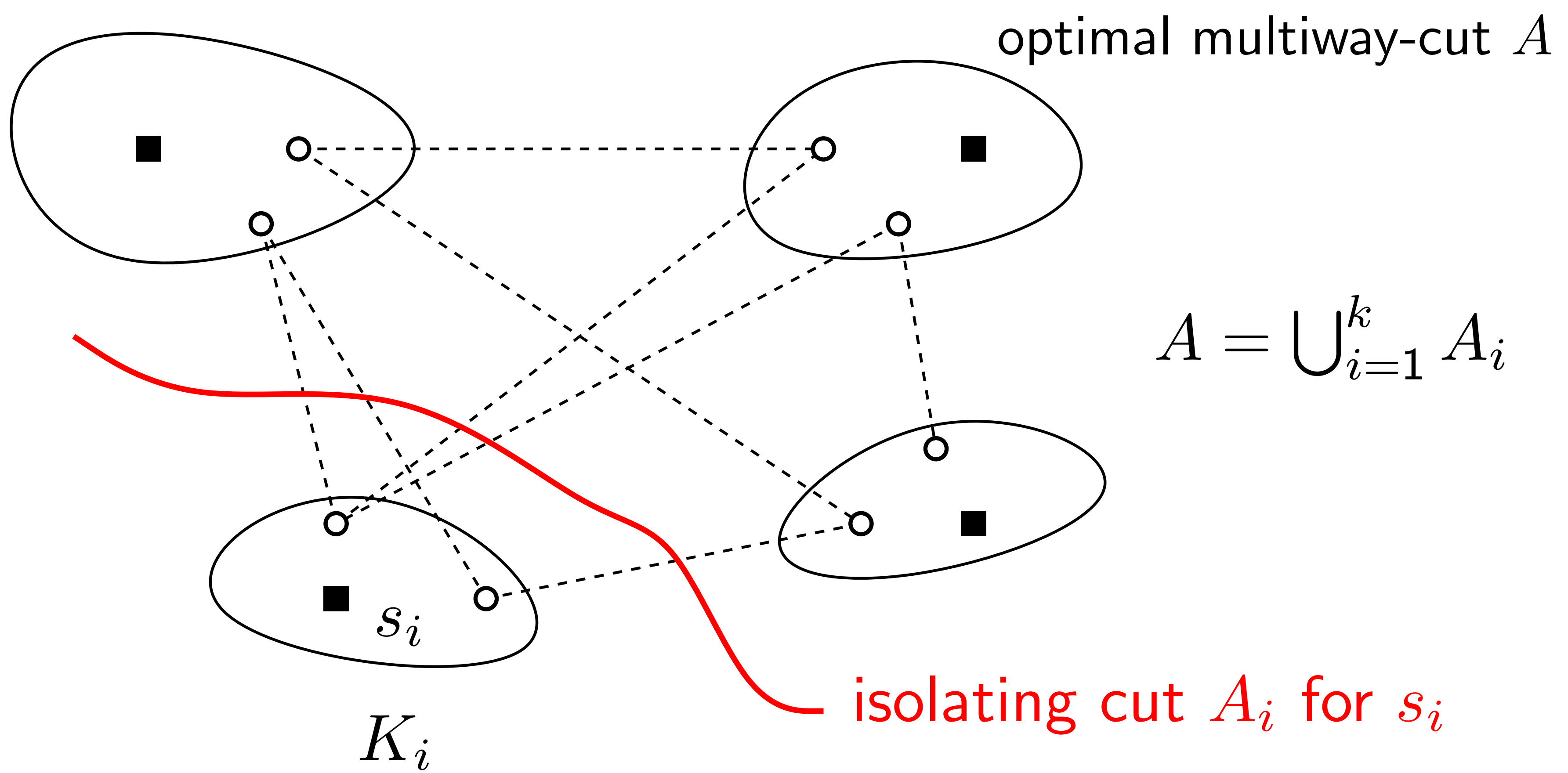


Add dummy terminal t conntected to each $s \in S \setminus \{s_i\}$, and compute minimum s_i–t cut.

Algorithm for MultiwayCut

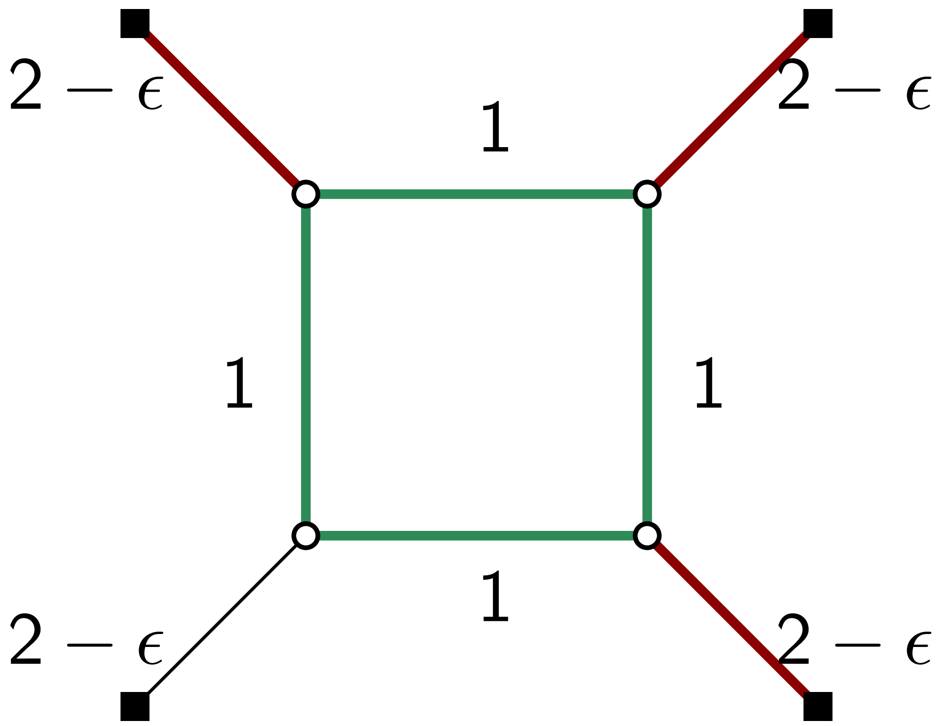
- For each terminal s_i, compute a minimum isolating cut C_i.
- Return the union of the $k-1$ cheapest such isolating cuts.

Thm. The above is a factor-$(2-\frac{2}{k})$ approx. alg.



Is our approximation factor tight?

i.e., is there an example where our algorithm produces a multiway-cut whose cost is $(2 - \frac{2}{k}) \cdot OPT$?



isolating cuts: $(k-1)(2-\epsilon)$

OPT $= k$

$(k-1)(2-\epsilon)/k \approx 2 - 2/k$

Next Week: Linear Programming