Aalto University
School of Electrical
Engineering

Brief Recap of Chapter 9 of Brown et al. (2014)

Simo Särkkä

Contents of Chapter 9 ”One-Dimensional Fourier Imaging, k-Space, and Gradient Echoes”

Introduction

- We can use gradients to vary the Larmor frequencies spatially
- Sampled signal then becomes Fourier transform of the effective spin density
- We can recover the image with inverse Fourier transform
- Gradient echoes becomes useful in that
- Spin-echoes can be combined with gradient echoes

Magnetization and Effective Spin Density

- **The demodulated signal is given as**

$$s(t) = \omega_0 \Lambda \mathcal{B}_\perp \int d^3 r M_\perp(\vec{r}, 0) e^{i(\Omega t + \phi(\vec{r}, t))}$$

- **The phase accumulates as**

$$\phi(\vec{r}, t) = -\int_0^t dt' \omega(\vec{r}, t')$$

- **The magnetization in terms of proton density:**

$$M_\perp(\vec{r}, 0) = M_0(\vec{r}) = \frac{1}{4} \rho_0(\vec{r}) \frac{\gamma^2 \hbar^2}{kT} B_0$$

- **If we define effective spin density**

$$\rho(\vec{r}) \equiv \omega_0 \Lambda \mathcal{B}_\perp M_0(\vec{r}) = \frac{1}{4} \omega_0 \Lambda \mathcal{B}_\perp \rho_0(\vec{r}) \frac{\gamma^2 \hbar^2}{kT} B_0$$

- **then we have**

$$s(t) = \int d^3 r \rho(\vec{r}) e^{i(\Omega t + \phi(\vec{r}, t))}$$

Frequency Encoding of the Spin Position

- **The 1d version of the signal expression is**

$$s(t) = \int dz \rho(z) e^{i(\Omega t + \phi(z,t))}$$

- **Assume that we apply a field gradient:**

$$B_z(z,t) = B_0 + zG(t)$$

- **The accumulated phase now takes the form**

$$\phi_G(z,t) = -\gamma z \int_0^t dt' G(t')$$

- **We can define**

$$k(t) = \gamma \int_0^t dt' G(t')$$

- **Which allows us to write**

$$s(k) = \int dz \rho(z) e^{-i2\pi kz}$$

- **But this is the Fourier transform of *ρ*(z)!**

Frequency Encoding of the Spin Position

- **Thus by a suitable control of G(t) we can evaluate**

$$s(k) = \int dz \rho(z) e^{-i2\pi kz}$$

- **The inverse Fourier transform then gives**

$$\rho(z) = \int dk\ s(k) e^{+i2\pi kz}$$

- **The signal s(k) is the k-space (Fourier-space) signal**
- **We can, for example, keep the gradient constant:**

$$k = \gamma G t$$

Problem

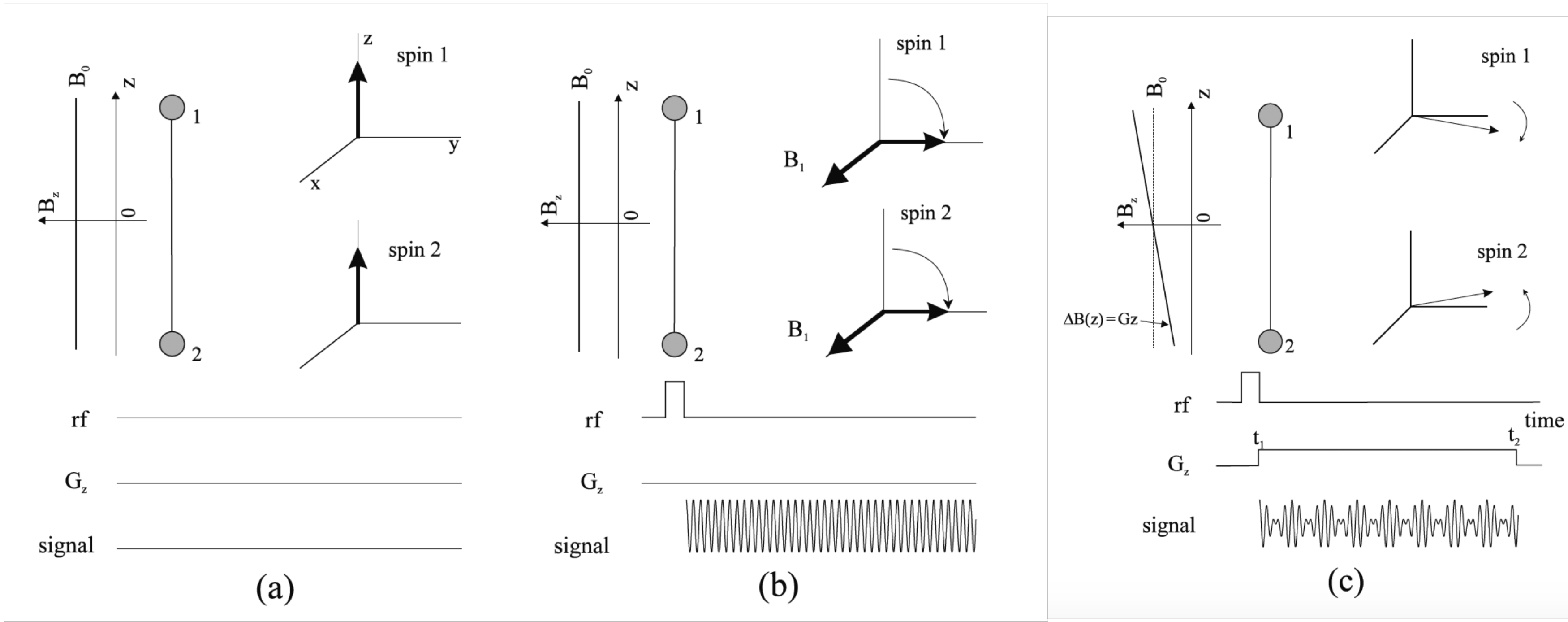
Problem 7.1

Consider a boxcar spin-density distribution with width z_0, centered at $z = 0$, and given by $\rho(z) = \rho_0 \text{rect}(z/z_0)$. Find the signal $s(k)$ for this spin density from (9.15). The answer will involve the sinc function, $\text{sinc}(\pi k z_0)$. Then check, using integral tables for example, that the answer gives back the correct spin density through the inverse transform (9.17).

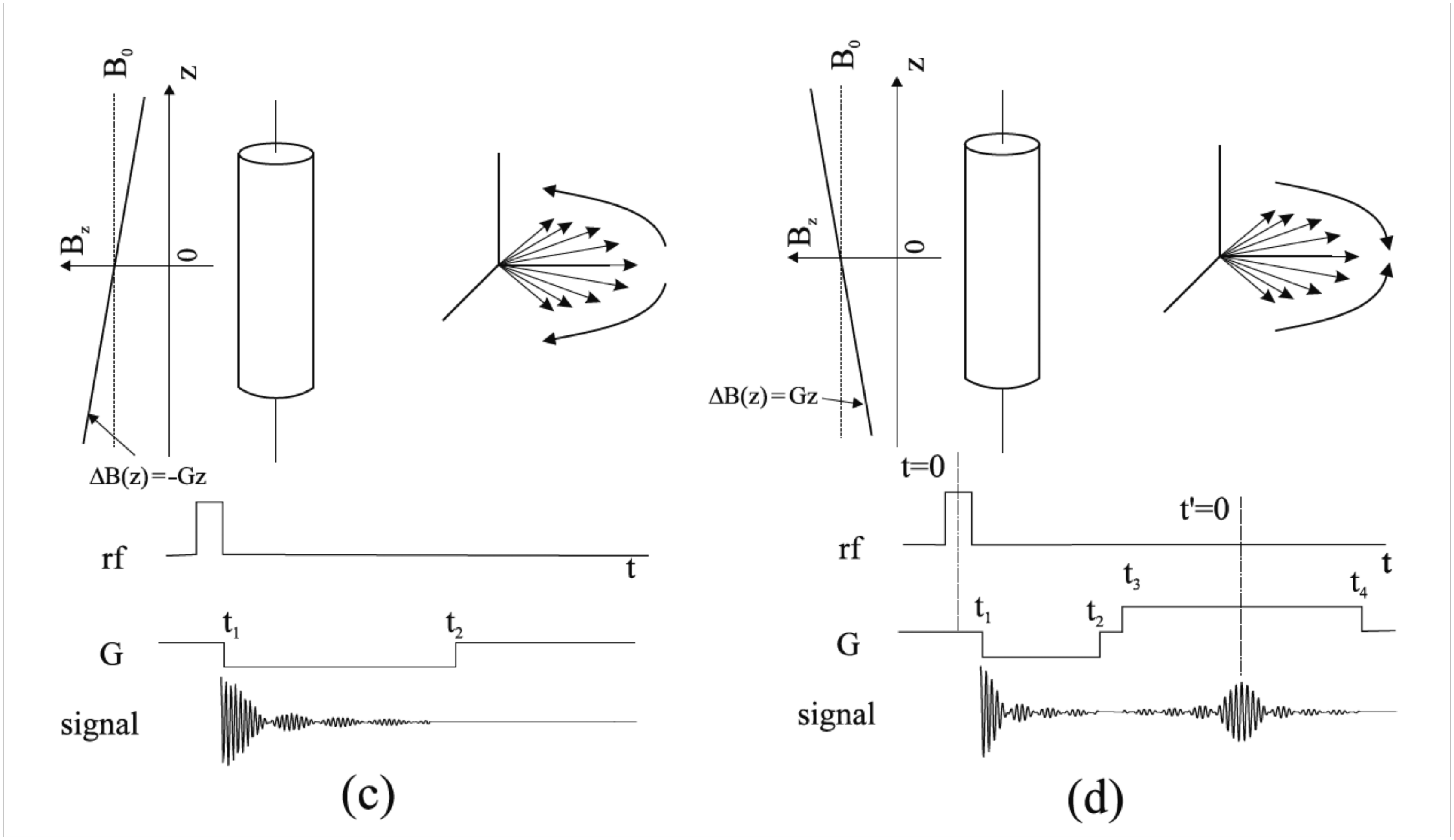
$$\text{rect}(z) \equiv \begin{cases} 0 & z < -1/2 \\ 1 & -1/2 \le z \le 1/2 \\ 0 & z > 1/2 \end{cases}$$

$$\text{sinc}(z) = \frac{\sin z}{z}$$

Simple Two-Spin Example



Gradient Echo and k-Space Diagrams



Analysis of gradient echo

- **On the first part we have:**

$$\phi_G(z,t) = +\gamma G z(t - t_1) \qquad t_1 < t < t_2$$

- **On the second part:**

$$\phi_G(z,t) = +\gamma G z(t_2 - t_1) - \gamma G z(t - t_3) \qquad t_3 < t < t_4$$

- **The gradient echo occurs at:**

$$t = t_3 + t_2 - t_1 \equiv T_E$$

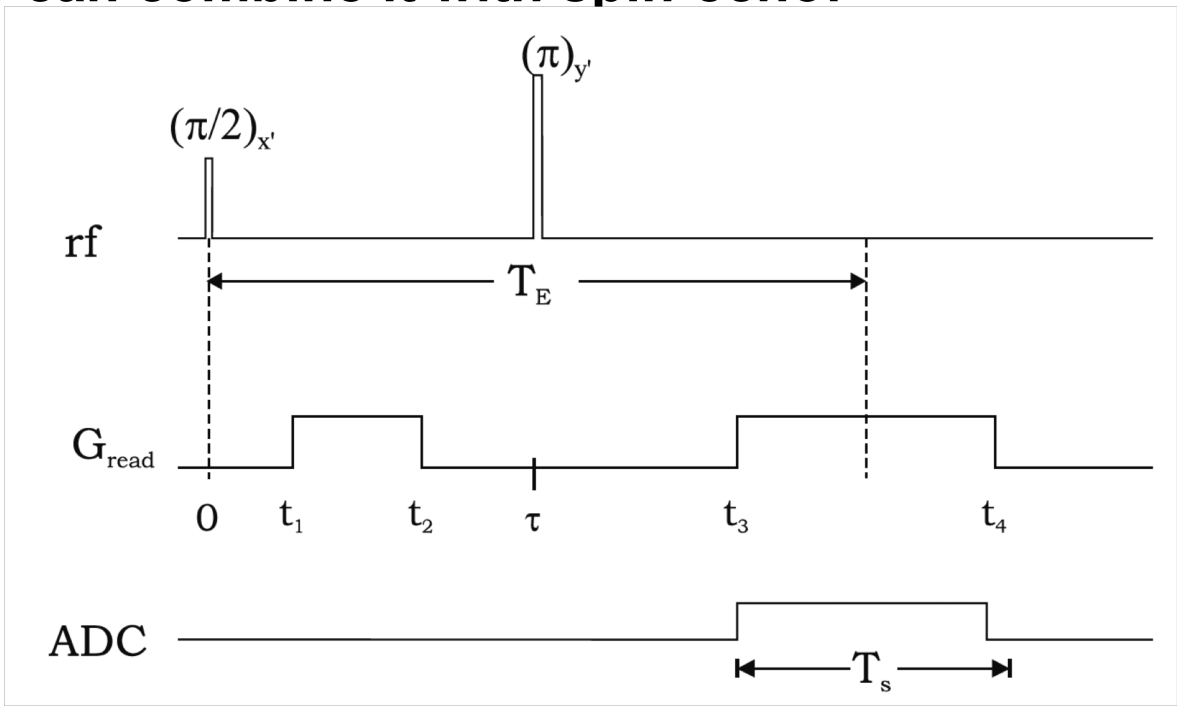
- **General echo condition:**

$$\int G(t)dt = 0$$

- **If we define** $t' \equiv t - t_3 - (t_2 - t_1) = t - T_E$ **then around echo we have**
- $$\phi_G(z,t) = -\gamma G z t'$$

General Spin Echo Imaging

- **Gradient echo does not eliminate T2' decay**
- **But we can combine it with spin-echo:**



Analysis of spin-echo

- **During the first gradient we have**

$$\phi(z,t) = -\gamma \Delta B(z) t - \gamma G z (t - t_1) \qquad t_1 < t < t_2$$

- **The π-pulse then inverts the phase**
- **During the second gradient we have**

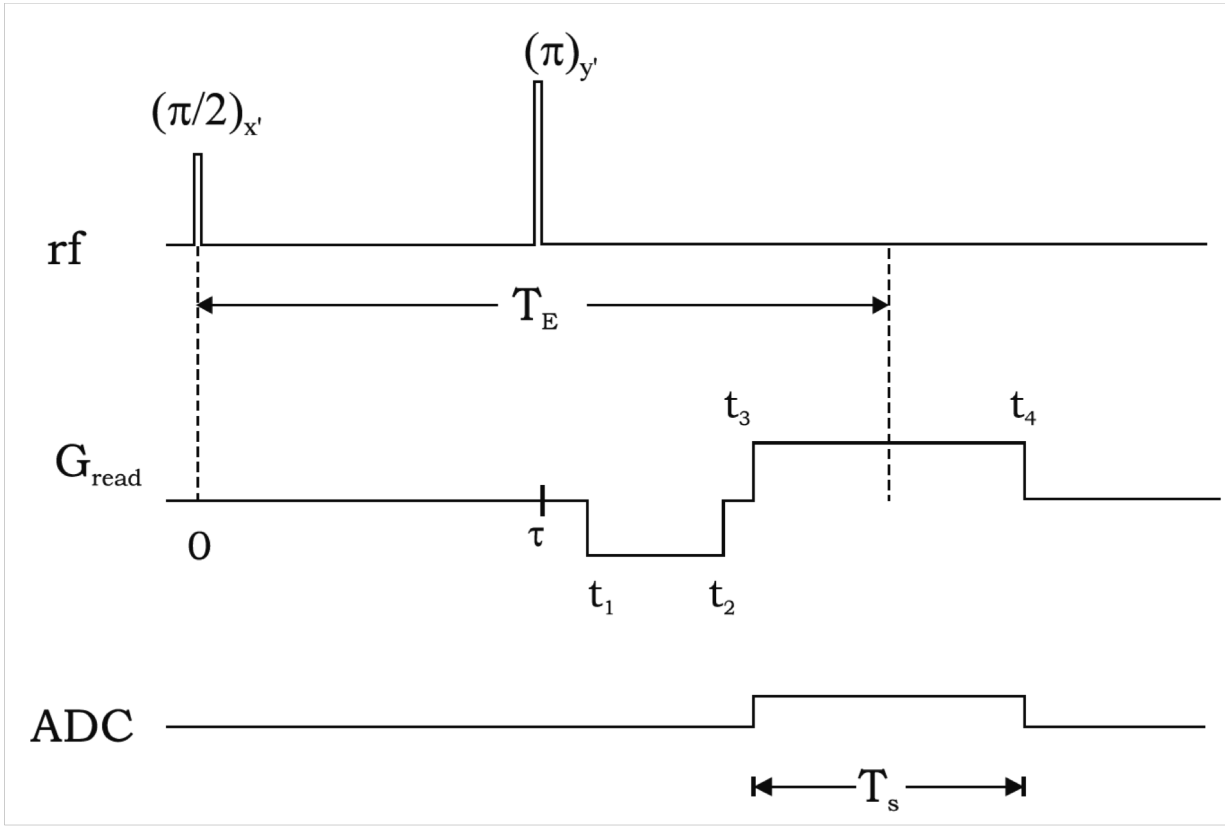
$$\phi(z,t) = \gamma \Delta B(z) \tau + \gamma G z (t_2 - t_1) - \gamma \Delta B(z)(t - \tau) - \gamma G z (t - t_3) \quad t_3 < t < t_4$$

- **Both the echoes then occur at**

$$t = 2\tau,$$

$$T_E \equiv t_3 + (t_2 - t_1)$$

General Spin Echo Imaging v2



$$2\tau = t_3 + (t_2 - t_1)$$

Problem

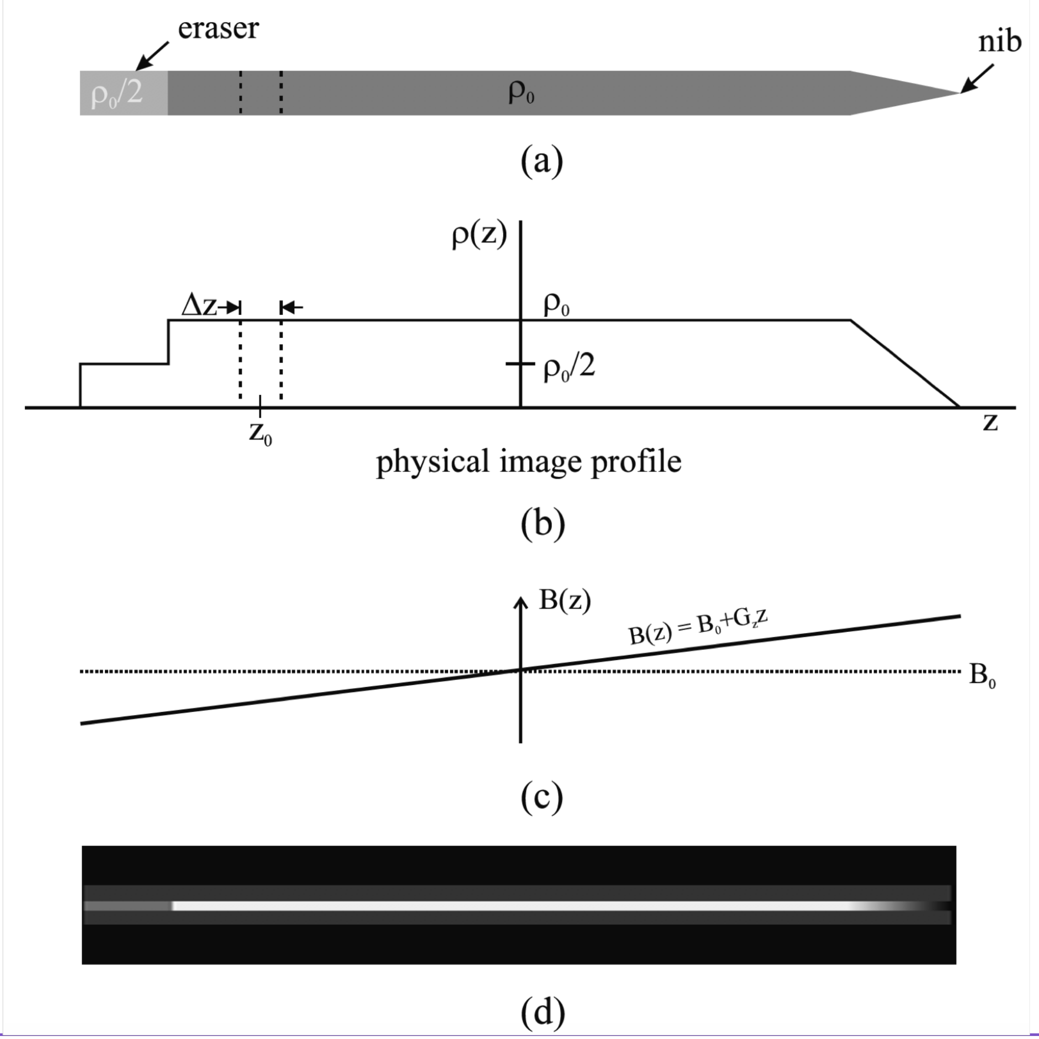
Problem 7.2

Spins with gyromagnetic ratio γ are uniformly distributed with uniform spin density ρ_0 along the z-axis from $-z_0$ to z_0 in a 1D imaging experiment. Suppose that they are excited at $t = 0$ by an rf pulse such that the signal at that instant would be given by

$$s(t = 0) = \int_{-z_0}^{z_0} dz \rho_0 = 2z_0\rho_0 \qquad (9.40)$$

A negative constant gradient field $-G$ is immediately applied at $t = 0^+$ and flipped to the positive gradient field $+G$ at time $t = T$. Find an expression for the signal for $t > T$ and show that it exhibits a gradient echo at time $t = 2T$.

Image profile of a pencil



More on gradients

- **We can also do frequency encoding in other directions:**

$$\begin{aligned}\vec{G}(t) &\equiv \vec{\nabla} B_z^g(\vec{r}) \\ &= \hat{x}\frac{\partial}{\partial x}B_z^g + \hat{y}\frac{\partial}{\partial y}B_z^g + \hat{z}\frac{\partial}{\partial z}B_z^g \\ &\equiv G_x(t)\hat{x} + G_y(t)\hat{y} + G_z(t)\hat{z}\end{aligned}$$

- **Inverse Fourier transform still applies**
- **In practice, gradients are always a bit non-linear**
- **In theory, doesn't matter, but linear gradients are usually attempted**