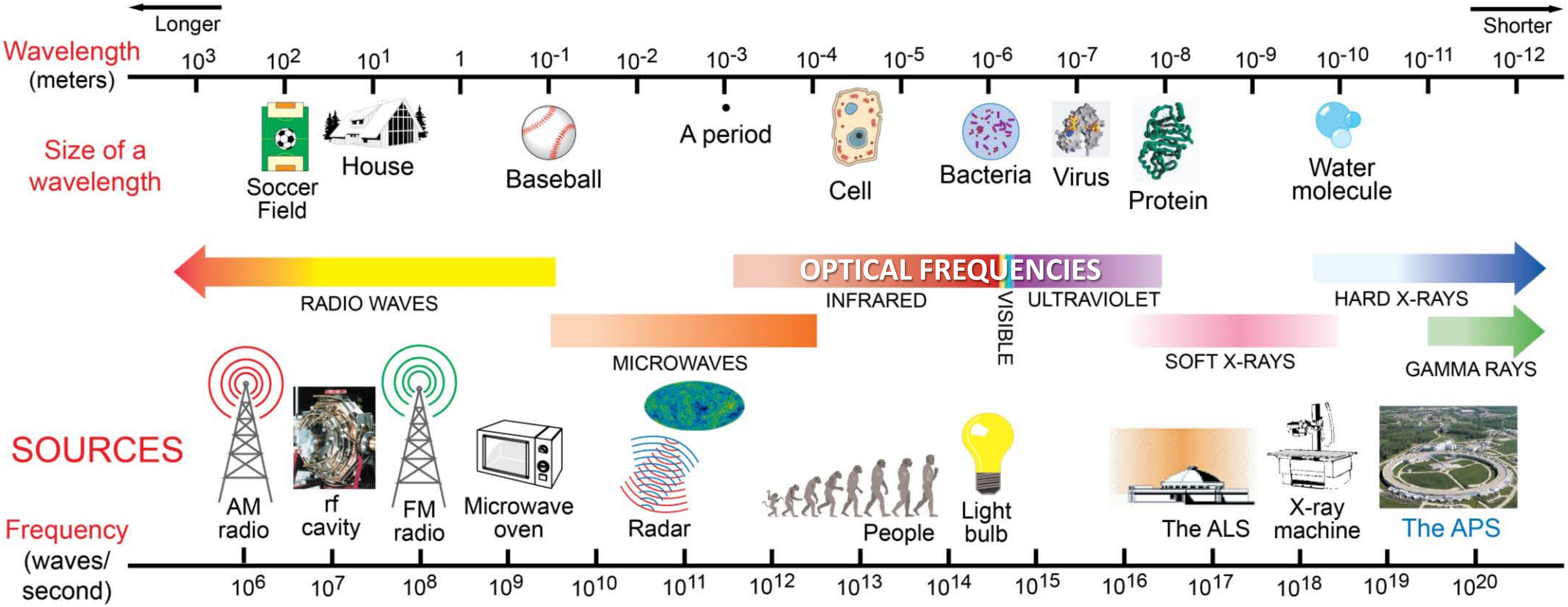
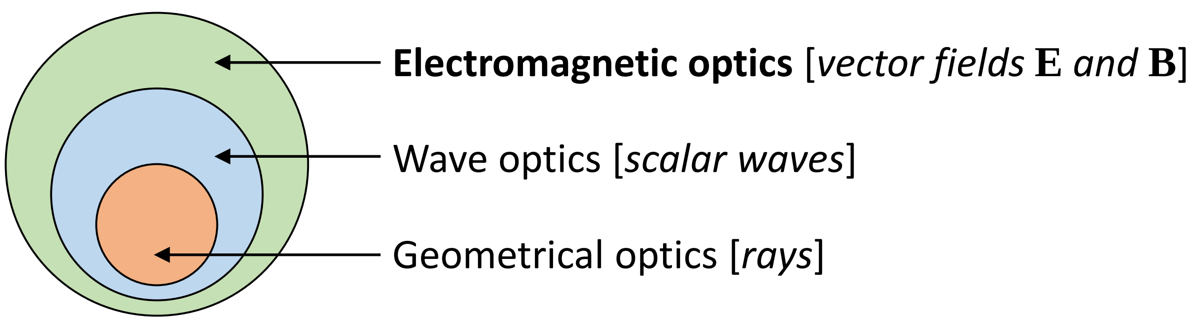
Chapter 5

ELECTROMAGNETIC OPTICS I

Electromagnetic optics





Maxwell's equations in charge-free space

$$\nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t}$$

$$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t}$$

$$\nabla \cdot \mathscr{D} = 0$$

$$\nabla \cdot \mathscr{B} = 0.$$

$\Rightarrow$ Wave equation in a medium ($c = 1/\sqrt{\varepsilon\mu}$)

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

The scalar u is any of the components $(\mathscr{E}_x, \mathscr{E}_y, \mathscr{E}_z)$ and $(\mathscr{H}_x, \mathscr{H}_y, \mathscr{H}_z)$.

The Poynting (power flow) vector is

$$\mathscr{S} = \mathscr{E} \times \mathscr{H}$$

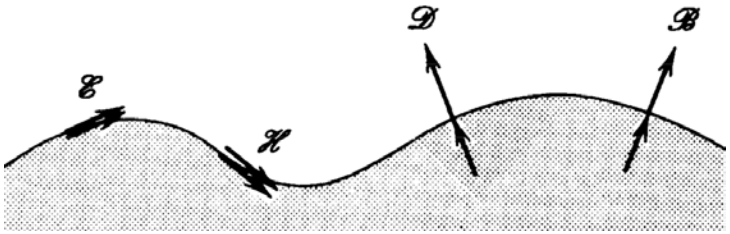
Electric and magnetic flux densities:

$$\mathscr{D} = \epsilon_o \mathscr{E} + \mathscr{P}$$

$$\mathscr{B} = \mu_o \mathscr{H} + \mu_o \mathscr{M}$$

dipole moment per unit volume

Boundary conditions: The tangential components of $\mathscr{E}$ and $\mathscr{H}$ and normal components of $\mathscr{D}$ and $\mathscr{B}$ are continuous



Linear, nondispersive, homogeneous, and isotropic media

$$\mathscr{P} = \epsilon_o \chi \mathscr{E} \qquad \epsilon = \epsilon_o(1+\chi) \qquad n = \sqrt{\epsilon/\epsilon_0} = \sqrt{1+\chi}$$

$$\mathscr{D} = \epsilon \mathscr{E} \qquad \mu \approx \mu_0 \qquad c = c_0/n$$

Inhomogeneous media

$$\chi = \chi(\mathbf{r})$$

$$\epsilon = \epsilon(\mathbf{r})$$

$$\nabla^2 \mathscr{E} - \frac{1}{c^2(\mathbf{r})} \frac{\partial^2 \mathscr{E}}{\partial t^2} + \nabla\left(\frac{1}{\epsilon} \nabla \epsilon \cdot \mathscr{E}\right) = 0$$

Anisotropic media

$$\mathscr{P}_i = \sum_j \epsilon_o \chi_{ij} \mathscr{E}_j \qquad \mathscr{D}_i = \sum_j \epsilon_{ij} \mathscr{E}_j$$

Orthogonally polarized modes with different n_o and n_e

Dispersive media

$$\mathscr{P}(t) = \epsilon_o \int_{-\infty}^{\infty} x(t-t')\mathscr{E}(t')\,dt' \quad \Rightarrow \quad \chi = \chi(\nu) \text{ and } \epsilon = \epsilon(\nu)$$

Impulse-response function

Transfer function (for frequency components of the field)

Nonlinear media

$$\mathscr{D} = \epsilon_o \mathscr{E} + \mathscr{P}(\mathscr{E}) \quad \Rightarrow \quad \nabla^2 \mathscr{E} - \frac{1}{c_o^2} \frac{\partial^2 \mathscr{E}}{\partial t^2} = \mu_o \frac{\partial^2 \mathscr{P}(\mathscr{E})}{\partial t^2}$$

nonlinear

Monochromatic waves

$$\mathscr{E}(\mathbf{r}, t) = \mathrm{Re}\{\mathbf{E}(\mathbf{r}) \exp(j\omega t)\}$$

$$\mathscr{H}(\mathbf{r}, t) = \mathrm{Re}\{\mathbf{H}(\mathbf{r}) \exp(j\omega t)\}$$

complex amplitudes

$$\Rightarrow \quad \frac{\partial}{\partial t} \to j\omega \quad \Rightarrow \quad \begin{cases} \nabla \times \mathbf{H} = j\omega \mathbf{D} \\ \nabla \times \mathbf{E} = -j\omega \mathbf{B} \\ \nabla \cdot \mathbf{D} = 0 \\ \nabla \cdot \mathbf{B} = 0. \end{cases}$$

The Poynting vector is $\mathscr{S} = \mathrm{Re}\{\mathbf{E} e^{j\omega t}\} \times \mathrm{Re}\{\mathbf{H} e^{j\omega t}\}$

$\Rightarrow \langle \mathscr{S} \rangle = \frac{1}{4}(\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}) = \frac{1}{2}(\mathbf{S} + \mathbf{S}^*) = \mathrm{Re}\{\mathbf{S}\}$, where

$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$ — complex

$I = \mathrm{Re}\{S\}$ — intensity

The wave equation is $\nabla^2 U + k^2 U = 0$ (*Helmholtz equation*), where $k = \omega/c$.

Elementary solutions of this equation are *plane waves* with complex amplitudes

$$\left.\begin{aligned} \mathbf{E}(\mathbf{r}) &= \mathbf{E}_0 \exp(-j\mathbf{k} \cdot \mathbf{r}), \\ \mathbf{H}(\mathbf{r}) &= \mathbf{H}_0 \exp(-j\mathbf{k} \cdot \mathbf{r}). \end{aligned}\right\}$$

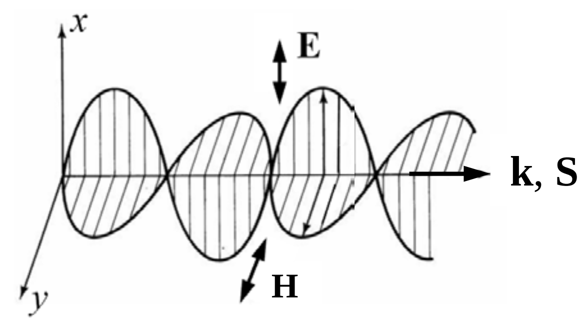
Plane waves

Maxwell's equations:

$$\mathbf{k} \times \mathbf{H}_0 = -\omega\epsilon\mathbf{E}_0$$

$$\mathbf{k} \times \mathbf{E}_0 = \omega\mu \ \mathbf{H}_0.$$

$\Rightarrow$



$$\Rightarrow \frac{E_0}{H_0} = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\epsilon}} = \eta$$ – Impedance ($\eta = \eta_0/n, \ \eta_0 \approx 377\Omega$)

The intensity is $I = \langle \mathrm{Re}\{\mathrm{S}\} \rangle = \frac{|E_0|^2}{2\eta}$

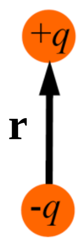
The energy density is $W = I/c = \frac{\epsilon |E_0|^2}{2}$

The linear momentum density is $\mathbf{G} = \epsilon\mu \mathrm{Re}\{\mathbf{S}\} = \hat{\mathbf{s}} I/c^2$

Spherical waves

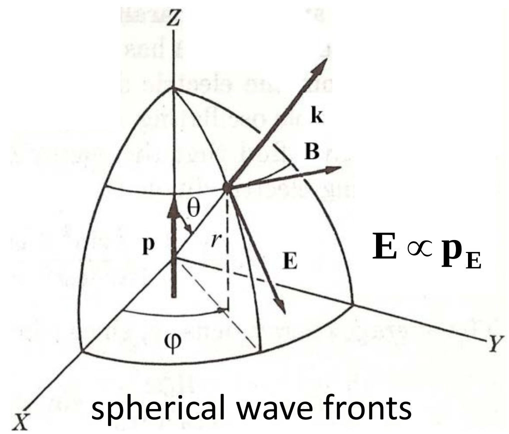
❖ ***Oscillating electric dipoles*** are most common ***elementary sources*** of electromagnetic waves (atoms and molecules also radiate as dipoles)

Dipole moment



$$\mathbf{p} = q\mathbf{r} = q\mathbf{r}_0 \sin(\omega t)$$

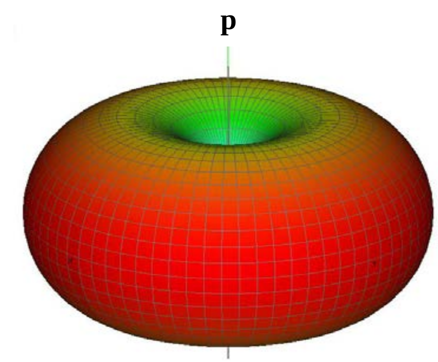
Electric and magnetic fields



spherical wave fronts

$$\mathbf{E} \propto \mathbf{p}_\mathrm{E}$$

Directivity of radiation



donut-shaped diagram

$$\left.\begin{aligned} \mathbf{E} &= \hat{\mathbf{u}}_\theta \frac{k^2}{4\pi\varepsilon} p_0 \sin\theta \frac{\exp(-jkr)}{r} \\ \mathbf{B} &= \hat{\mathbf{u}}_\varphi E/c \end{aligned}\right\} \Rightarrow I(\theta) \propto \frac{\sin^2\theta}{r^2}$$

A scalar spherical wave is given by $U(r) = \dfrac{1}{r}\exp(-jkr)$

The vector Gaussian beam

In the paraxial approximation, the spherical function $U(r) = \frac{A}{r}\exp(-jkr)$ is approximated by a parabolloidal scalar wave

$$U(\mathbf{r}) = A_0 \frac{W_0}{W(z)} \exp\left[-\frac{\rho^2}{W^2(z)}\right] \exp\left[-jkz - jk\frac{\rho^2}{2R(z)} + j\zeta(z)\right]$$

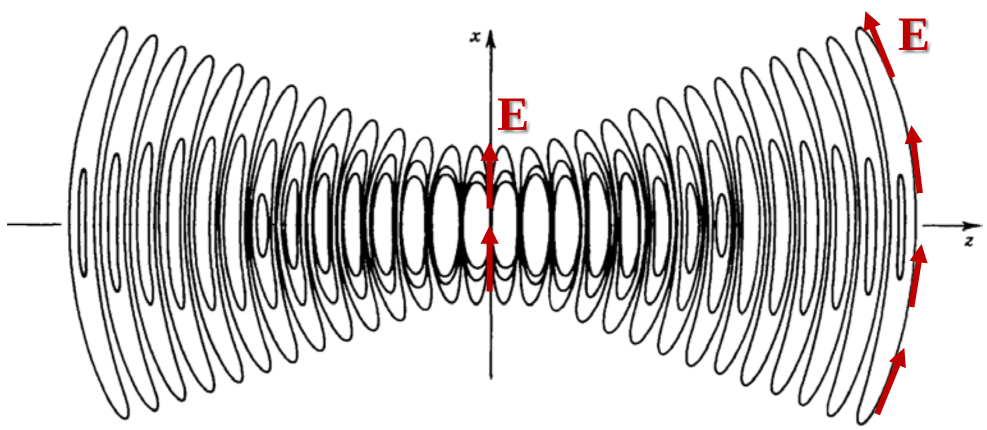
Gaussian beam

that satisfies the paraxial Helmholtz equation. The parameters are

$$W(z) = W_0\sqrt{1+\left(\frac{z}{z_0}\right)^2},\ W_0 = \sqrt{\frac{\lambda z_0}{\pi}},\ R(z) = z\left[1+\left(\frac{z_0}{z}\right)^2\right],\ \zeta(z) = \tan^{-1}\frac{z}{z_0}.$$

The vector Gaussian beam is described by the complex amplitude

$$\mathbf{E}(\mathbf{r}) = E_0\left(-\hat{\mathbf{x}} + \frac{x}{z + jz_0}\hat{\mathbf{z}}\right)U(\mathbf{r}),\ \text{and}\ \mathbf{B} = \hat{\mathbf{k}} \times \mathbf{E}/c.$$



Absorption and dispersion

$$k = k_0\tilde{n} = k_0\sqrt{1+\chi} = k_0\sqrt{1+\chi' + j\chi''} = k_0\left(n - j\frac{\alpha}{2}\right).$$

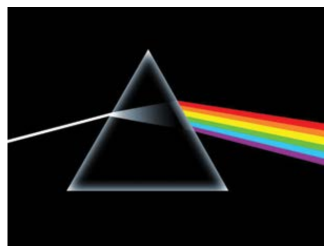
complex

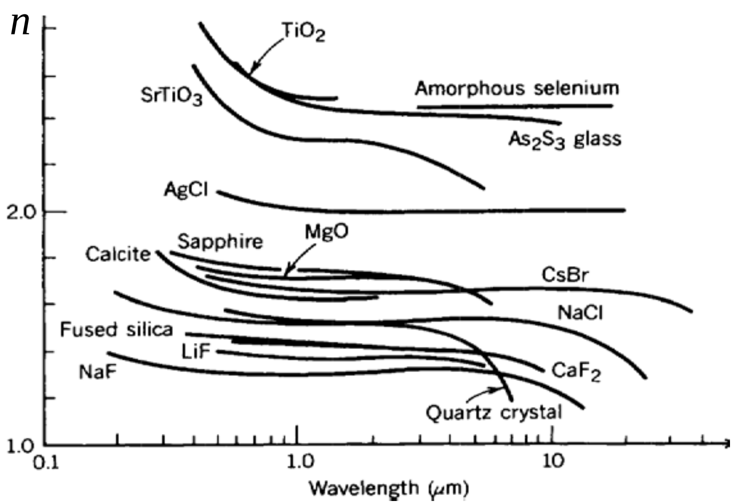
$$\Rightarrow I(z) = I(0)\left|e^{-jkz}\right|^2 = I(0)e^{-\alpha z}$$

For weak absorption, $\chi'' \ll 1+\chi'$, the Taylor series expansion of $\sqrt{1+\chi}$ yields

$$n = \sqrt{1+\chi'} \text{ and } \alpha = -k_0\chi''/n.$$

Dispersion is $\chi = \chi(\nu)$:

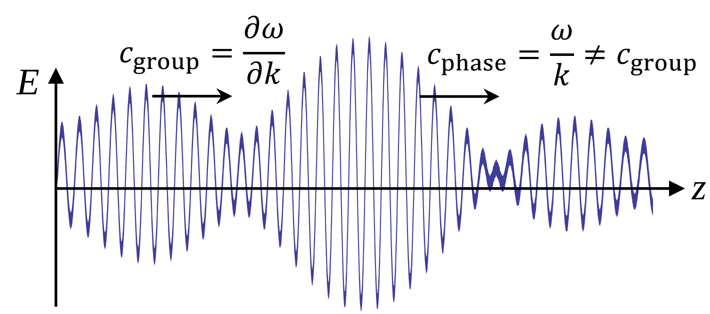




The Kramers-Kronig relations:

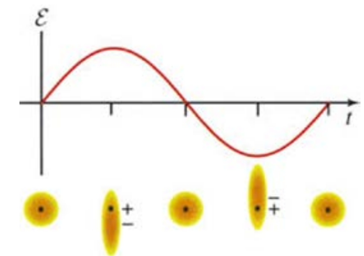
$$\chi'(\nu) = \frac{2}{\pi}\int_0^{\infty} \frac{s\chi''(s)}{s^2-\nu^2}\,ds$$

$$\chi''(\nu) = \frac{2}{\pi}\int_0^{\infty} \frac{\nu\chi'(s)}{\nu^2-s^2}\,ds$$



The resonant medium

The Lorentz harmonic oscillator model:



$$\frac{d^2x}{dt^2} + \sigma\frac{dx}{dt} + \omega_0^2 x = \frac{\mathscr{F}}{m}$$

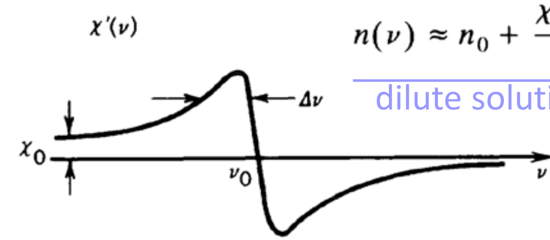
$$\begin{cases} \mathscr{P} = Nex \\ \mathscr{P} = \epsilon_o \chi \mathscr{E} \\ \mathscr{F} = e\mathscr{E} \end{cases} \Rightarrow \quad \frac{d^2\mathscr{P}}{dt^2} + \sigma\frac{d\mathscr{P}}{dt} + \omega_0^2\mathscr{P} = \omega_0^2\epsilon_o\chi_0\mathscr{E}$$

$$\chi_0 = e^2N/m\epsilon_o\omega_0^2$$

For a *harmonic wave*, $\frac{\partial}{\partial t} \to j\omega$, and the complex amplitudes obey

$$\left(-\omega^2 + j\sigma\omega + \omega_0^2\right)P = \omega_0^2\epsilon_o\chi_0 E \leftarrow \frac{P}{\epsilon_0\chi(\nu)}$$

$$\Rightarrow \chi(\nu) = \chi_0\frac{\nu_0^2}{\nu_0^2 - \nu^2 + j\nu\,\Delta\nu}; \quad \Delta\nu = \frac{\sigma}{2\pi}$$

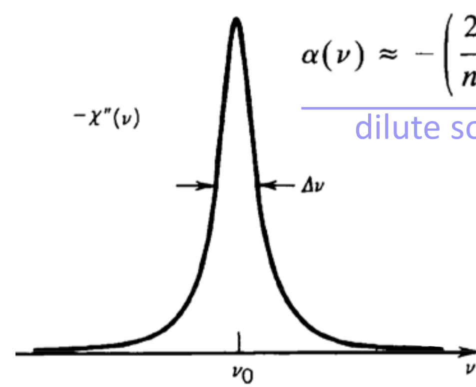


$$n(\nu) \approx n_0 + \frac{\chi'(\nu)}{2n_0}$$

dilute solution

$$\Rightarrow \begin{cases} \chi'(\nu) = \chi_0\dfrac{\nu_0^2(\nu_0^2 - \nu^2)}{(\nu_0^2 - \nu^2)^2 + (\nu\,\Delta\nu)^2} \\ \chi''(\nu) = -\chi_0\dfrac{\nu_0^2\nu\,\Delta\nu}{(\nu_0^2 - \nu^2)^2 + (\nu\,\Delta\nu)^2} \end{cases}$$

Lorenzian lineshape



$$\alpha(\nu) \approx -\left(\frac{2\pi\nu}{n_0 c_o}\right)\chi''(\nu)$$

dilute solution

Media with multiple resonances:

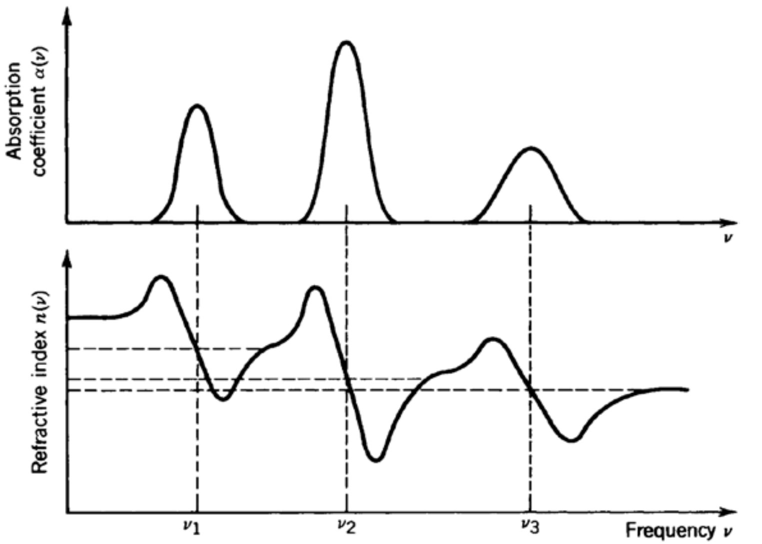
Far from resonance, $\chi(\nu) \approx \chi_0 \frac{\nu_0^2}{\nu_0^2 - \nu^2}$

Sellmeier equation for transparent media:

$$n^2 \approx 1 + \sum_i \chi_{0i} \frac{\nu_i^2}{\nu_i^2 - \nu^2} = 1 + \sum_i \chi_{0i} \frac{\lambda^2}{\lambda^2 - \lambda_i^2}$$

SiO_2 $\quad n^2 = 1 + \frac{0.6962\lambda^2}{\lambda^2 - (0.06840)^2} + \frac{0.4079\lambda^2}{\lambda^2 - (0.1162)^2} + \frac{0.8975\lambda^2}{\lambda^2 - (9.8962)^2}$

Si $\quad n^2 = 1 + \frac{10.6684\lambda^2}{\lambda^2 - (0.3015)^2} + \frac{0.0030\lambda^2}{\lambda^2 - (1.1347)^2} + \frac{1.5413\lambda^2}{\lambda^2 - (1104.0)^2}$



Optics of conductive media

$$\nabla \times \mathbf{H} = j\omega\mathbf{D} + \mathbf{J} = j\omega\epsilon\mathbf{E} + \sigma\mathbf{E} = (j\omega\epsilon + \sigma)\mathbf{E} = j\omega\epsilon_{\text{eff}}\mathbf{E}$$

$$\epsilon_{\text{eff}} = \epsilon + \frac{\sigma}{j\omega} \;\Rightarrow\; n_{\text{eff}} = \sqrt{\frac{\epsilon_{\text{eff}}}{\epsilon_0}} = n - j\frac{\alpha}{2}$$

In the *Drude model*, σ depends on frequency as $\sigma = \frac{\sigma_0}{1+j\omega\tau_{\text{rel}}}$, and at high ω, we have $\epsilon_{\text{eff}} = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$, where the plasma frequency is $\omega_p = \sqrt{\sigma_0/\epsilon_0\tau_{\text{rel}}} = \sqrt{Ne^2/\epsilon_0 m}$.