



Aalto University
School of Science



Department of
Computer Science

Combinatorics of
Efficient
Computations

Approximation Algorithms

Lecture 4: Linear Programming

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2019

Introduction to Linear Programming

Many approximation algorithms are based on linear programming.

- Linear Programming (LP) and LP-Duality
- Min-Max Relationships
- LP-based Algorithm Designs Techniques

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- Need lower bounds $S \geq \text{OPT} / \alpha$ (approximate “No”-certificates) for approximation algorithms!
- For example:
 - Vertex Cover: lower bound by matchings
 - TSP: lower bound by MST or Cycle Cover

Linear Programming

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

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$$\begin{array}{llllllll} \text{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \text{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

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- Standard form (i.e., using only “ \geq ”)

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- $\mathbf{x} = (2, 1, 3)$ is a feasible solution $\rightsquigarrow S = 30$ is an upper bound on OPT

Linear Programming - Lower Bounds

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- $7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3$

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- $7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \geq 10 \rightsquigarrow \text{OPT} \geq 10$
- $7x_1 + x_2 + 5x_3 \geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \geq 10 + 6$
- $\rightsquigarrow \text{OPT} \geq 16$

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$$\begin{array}{llll} y_1 + 5y_2 & \leq & 7 \\ -y_1 + 2y_2 & \leq & 1 \\ 3y_1 - y_2 & \leq & 5 \\ & y_1, y_2 & \geq & 0 \end{array}$$

- $10y_1 + 6y_2$ is a lower bound on OPT

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- Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

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- Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.
- $\mathbf{x} = (7/4, 0, 11/4)$ and $\mathbf{y} = (2, 1)$ both provide objective values of 26

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- Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.
- $\mathbf{x} = (7/4, 0, 11/4)$ and $\mathbf{y} = (2, 1)$ both provide objective values of 26 \rightsquigarrow both solutions are optimal!

LP – standard form

minimize $\sum_{j=1}^n c_j x_j$ Primal Program

subject to $\sum_{j=1}^n a_{ij} x_j \geq b_i \quad i = 1, \dots, m$

$x_j \geq 0 \quad j = 1, \dots, n$

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- maximization instances dualize analogously

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**What is the dual
of the dual?**

- maximization instances dualize analogously

LP-Duality

$$\begin{array}{ll} \text{Primal:} & \text{minimize} \\ & \text{subject to} \end{array} \quad \begin{array}{l} \sum_{j=1}^n c_j x_j \\ \sum_{j=1}^n a_{ij} x_j \geq b_i \quad i = 1, \dots, m \\ x_j \geq 0 \quad j = 1, \dots, n \end{array}$$

$$\begin{array}{ll} \text{Dual:} & \text{maximize} \\ & \text{subject to} \end{array} \quad \begin{array}{l} \sum_{i=1}^m b_i y_i \\ \sum_{i=1}^m a_{ij} y_i \leq c_j \quad j = 1, \dots, n \\ y_i \geq 0 \quad i = 1, \dots, m \end{array}$$

Thm. The primal program has a finite optimum \Leftrightarrow the dual program has a finite optimum. Moreover, if $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ and $\mathbf{y}^* = (y_1^*, \dots, y_m^*)$ are optimal solutions for the primal and dual (respectively), then

$$\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m b_i y_i^* .$$

Weak LP-Duality

$$\begin{array}{ll} \min & \sum_{j=1}^n c_j x_j \\ \text{s.t} & \sum_{j=1}^n a_{ij} x_j \geq b_i \\ & x_j \geq 0 \end{array}$$

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Thm. If $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_m)$ are feasible solutions for the primal and dual programs (resp.), then

$$\sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m b_i y_i .$$

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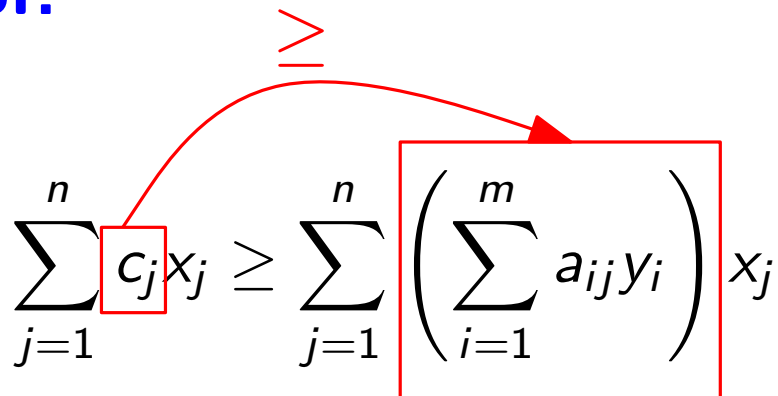
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$$\sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m b_i y_i .$$

Proof.

$$\sum_{j=1}^n c_j x_j \geq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j$$


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$$\sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m b_i y_i .$$

Proof.

$$\sum_{j=1}^n c_j x_j \geq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) y_i$$

Weak LP-Duality

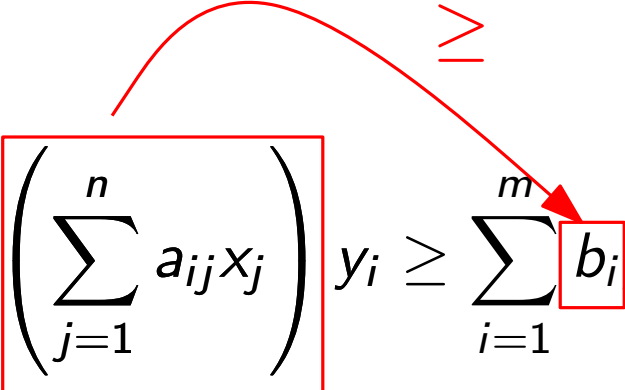
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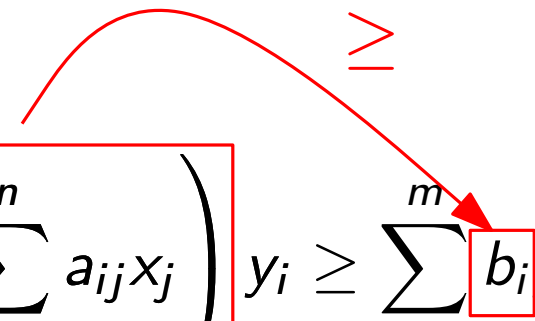
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□

Complementary Slackness

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Thm. Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_m)$ be feasible solutions for the primal and dual Programs (resp.). The solutions \mathbf{x} and \mathbf{y} are optimal if only if the following conditions are met:

Primal CS:

For each $j = 1, \dots, n$: either $x_j = 0$ or $\sum_{i=1}^m a_{ij} y_i = c_j$

Dual CS:

For each $i = 1, \dots, m$: either $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

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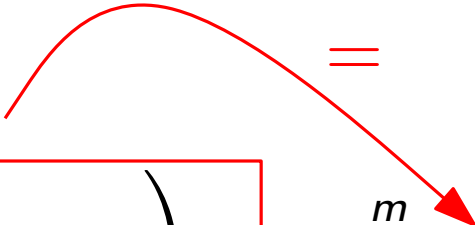
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Dual CS:

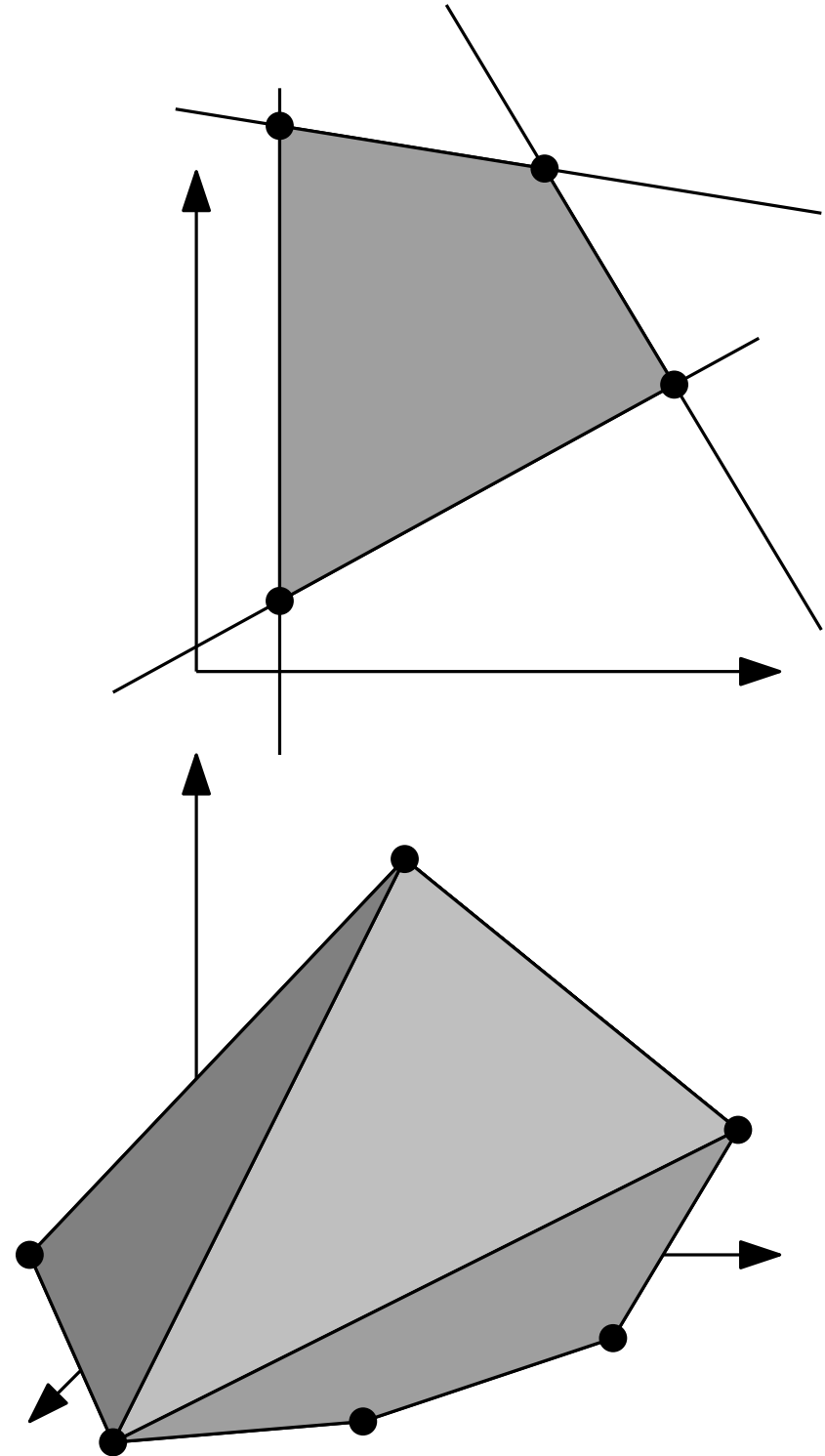
For each $i = 1, \dots, m$: either $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

Proof.

$$\sum_{j=1}^n c_j x_j = \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) y_i = \sum_{i=1}^m b_i y_i. \quad \square$$


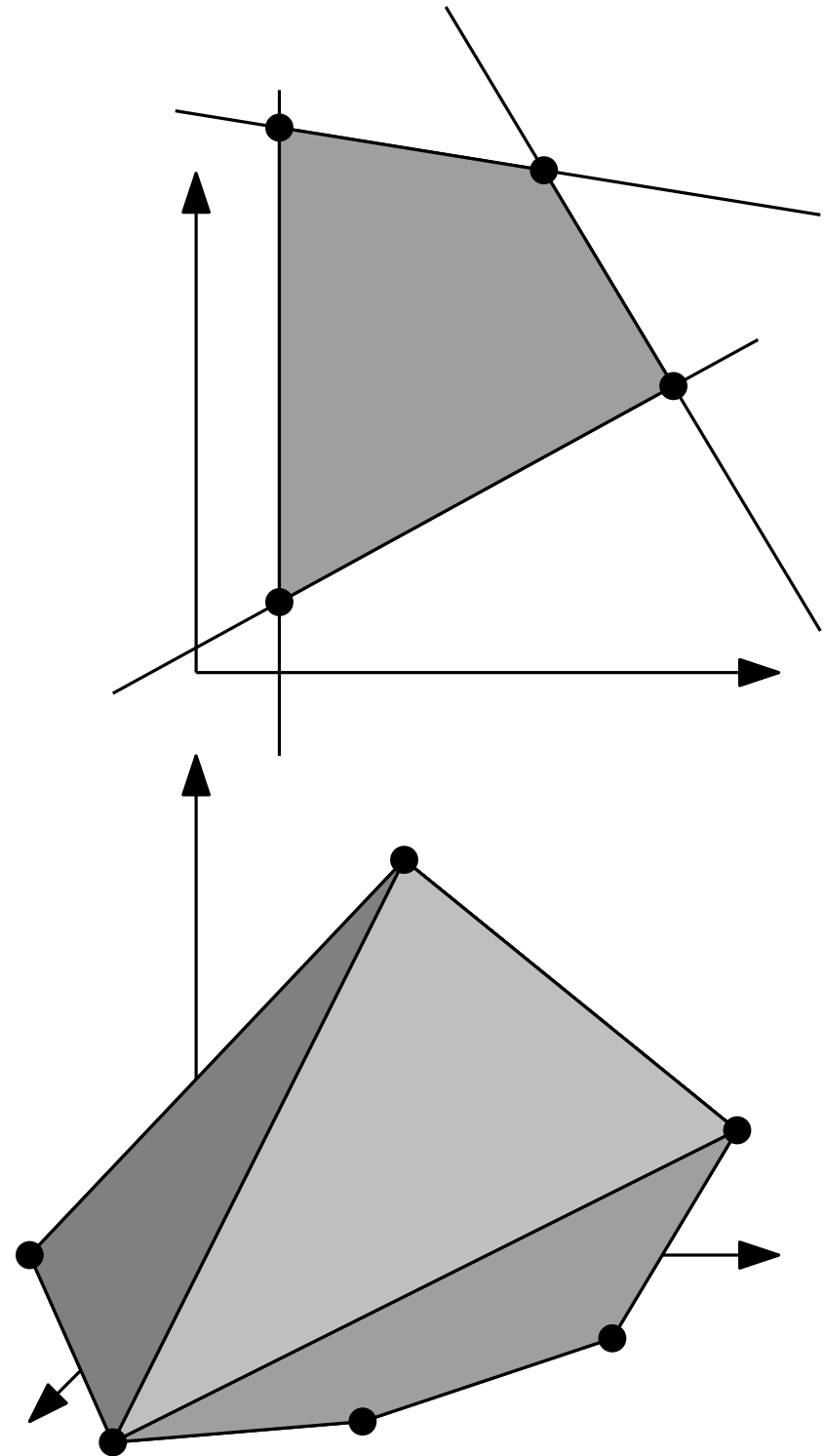
LPs and convex polytopes

- The feasible solutions of an LP with n variables from a **convex polytope** in \mathbb{R}^n (intersection of halfspaces).



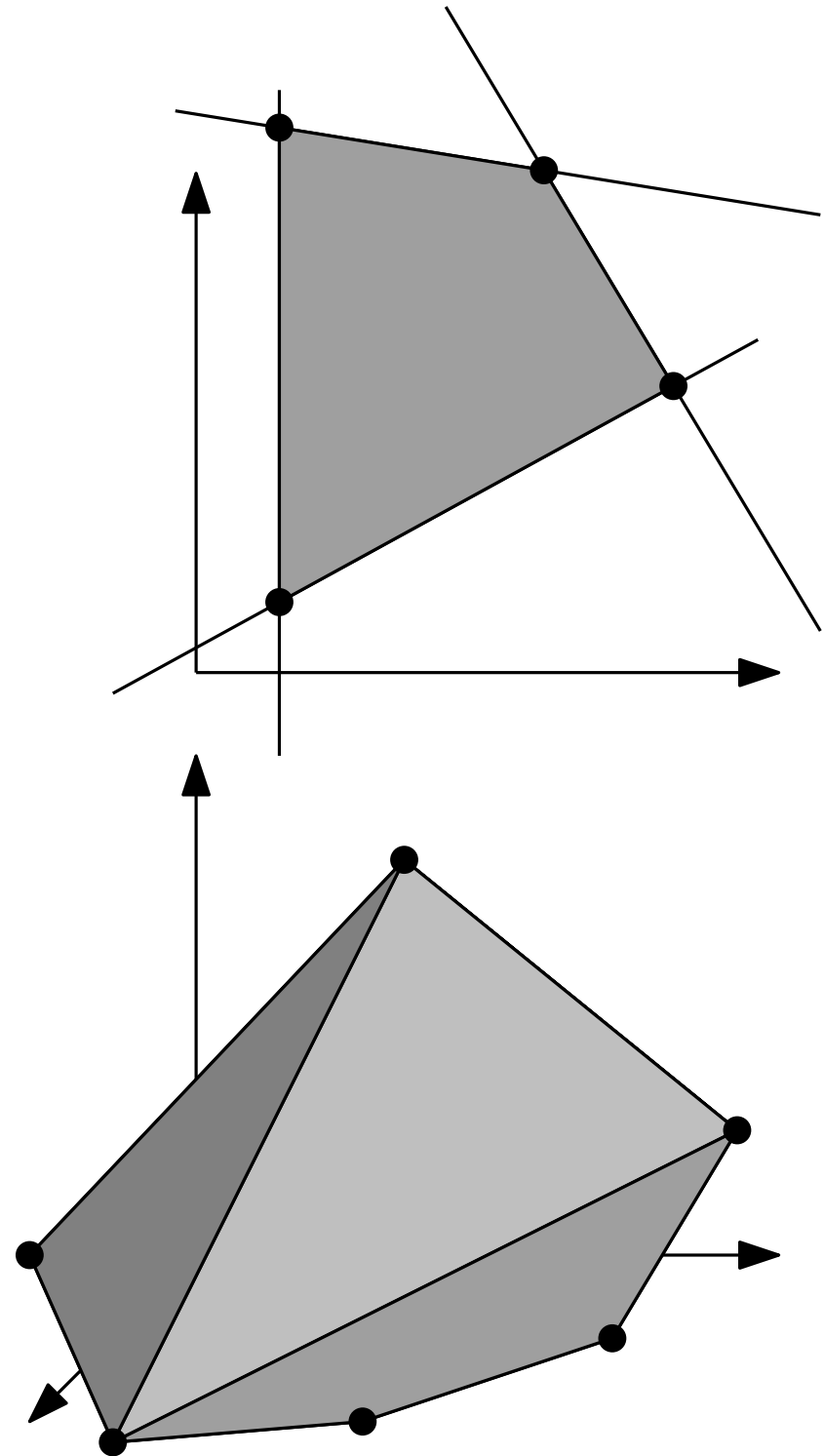
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- When an optimal solution exists, some extreme point will also be optimal.



Integer Linear Programs (ILPs)

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^n c_j x_j \\ \text{subject to} & \sum_{j=1}^n a_{ij} x_j \geq b_i \quad i = 1, \dots, m \\ & x_j \geq 0 \quad x_j \in \mathbb{N} \quad j = 1, \dots, n \end{array}$$

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- e.g., Vertex Cover

Introduction: Linear Programming

Many approximation algorithms are based on linear programming.

- Linear Programming (LP) and LP-Duality
- **Min-Max Relationships**
- LP-based Algorithm Designs Techniques

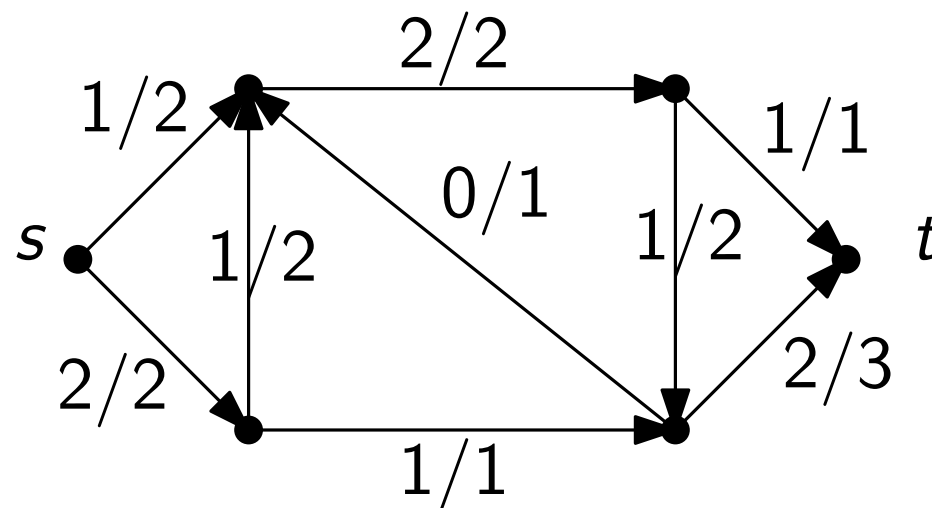
Max-Flow-Problem

Given: A directed graph $G = (V, E)$ with edge capacities $c: E \rightarrow \mathbb{Q}_+$ and two special vertices: the source s and sink t .

Find: A maximum s - t flow (i.e., an assignment of non-negative weights to edges) f , such that

- $f(u, v) \leq c(u, v)$ for each edge $(u, v) \in E$
- $\sum_{(u,v) \in E} f(u, v) = \sum_{(v,z) \in E} f(v, z)$ for each vertex $v \in V - \{s, t\}$

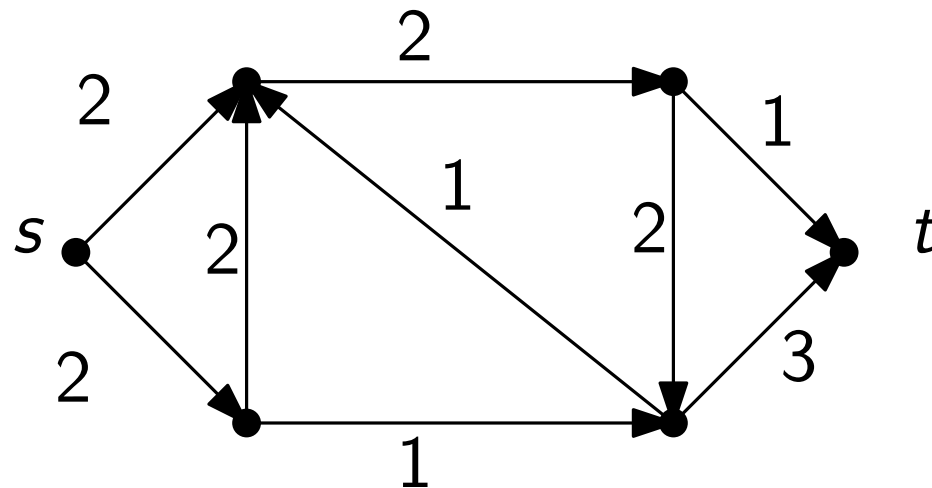
The **flow-value** is the inflow to t minus the outflow from t .



Min-Cut-Problem

Given: A directed graph $G = (V, E)$ with edge capacities $c: E \rightarrow \mathbb{Q}_+$ and two special vertices: the source s and sink t .

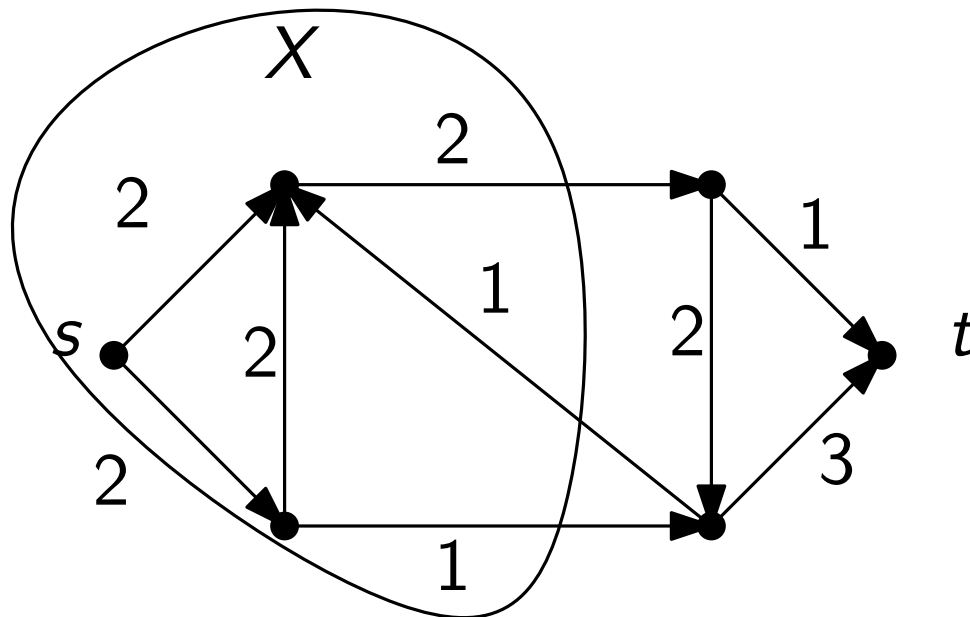
Find: An s - t -cut, i.e., a vertex set X with $s \in X$ and $t \in \bar{X}$, such that the total capacity $c(X, \bar{X})$ of the edges from X to \bar{X} is minimum.



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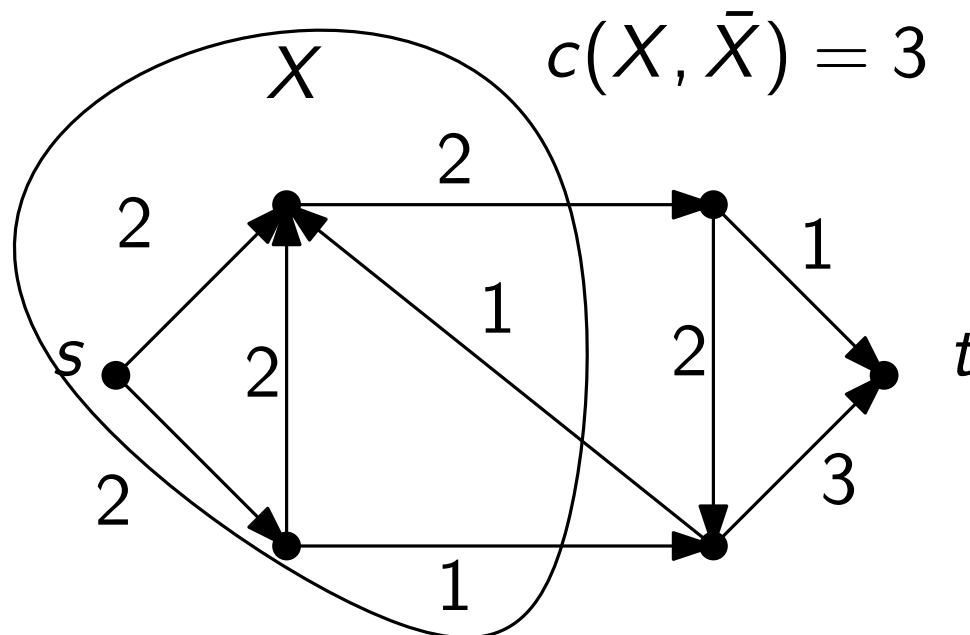
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Max-Flow-Min-Cut-Theorem

Thm. The value of a maximum s – t -flow and the capacity of a minimum s – t -cut are the same.

Max-Flow-Min-Cut-Theorem

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Proof. Special case of LP-duality ...

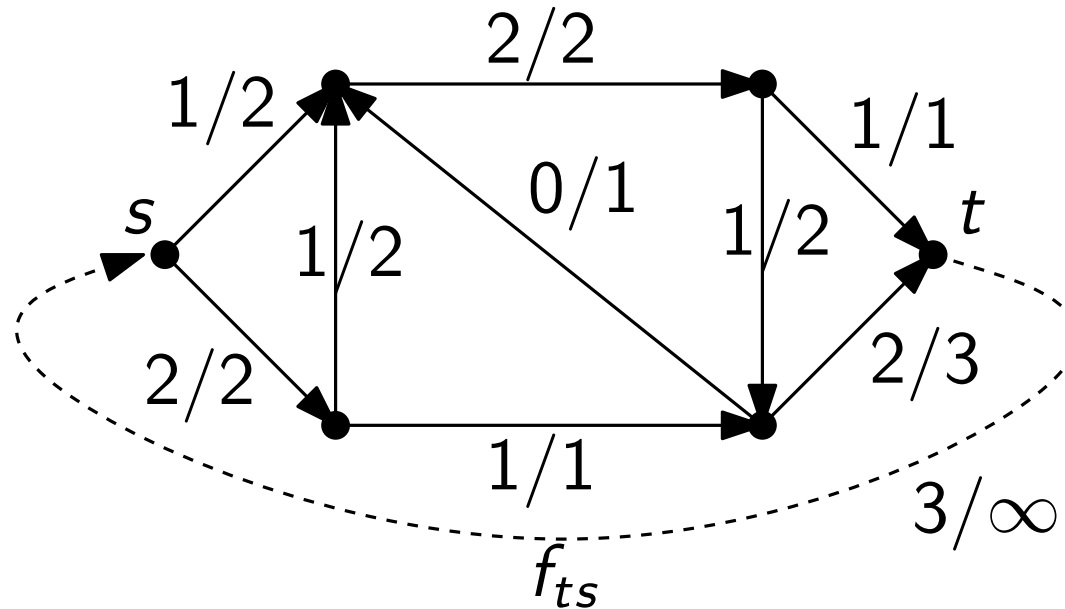
Max-Flow (circulation form) as an LP

maximize f_{ts}

subject to $f_{uv} \leq c_{uv} \quad (u, v) \in E$

$$\sum_{u: (u,v) \in E} f_{uv} - \sum_{z: (v,z) \in E} f_{vz} \leq 0 \quad v \in V$$

$f_{uv} \geq 0 \quad (u, v) \in E$



Max-Flow (circulation form) as an LP

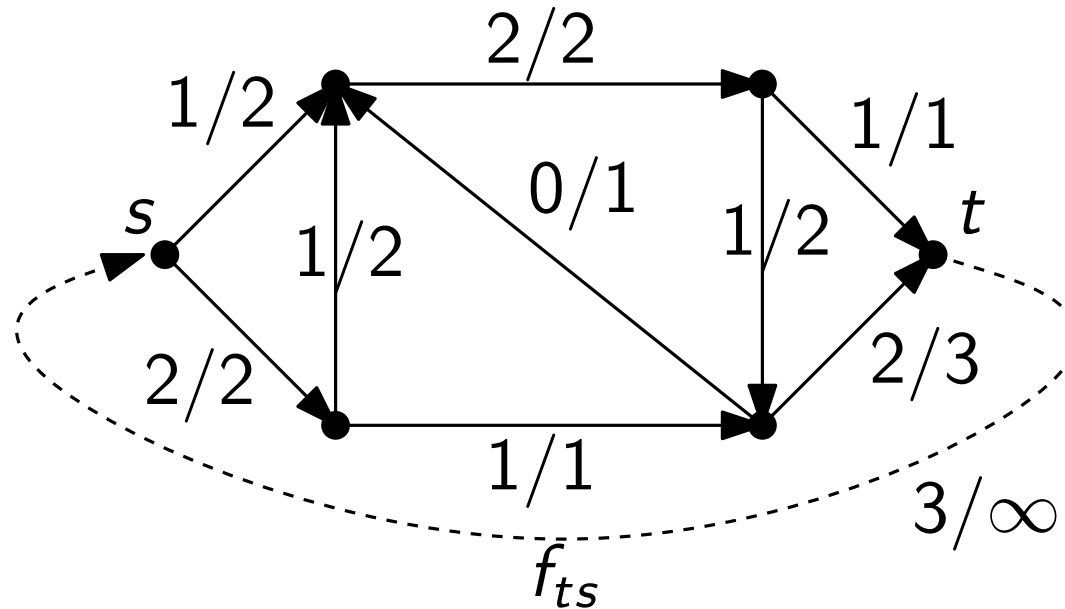
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why does this work?

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Dual LP

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Primal Program

$$(u, v) \in E$$

$$v \in V$$

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Primal Program

$$(u, v) \in E \quad d_{uv}$$

$$v \in V \quad p_v$$

$$(u, v) \in E$$

minimize $\sum_{(u,v) \in E} c_{uv} d_{uv}$

subject to $d_{uv} - p_u + p_v \geq 0 \quad (u, v) \in E$

$$p_s - p_t \geq 1$$

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Dual Program

Dual LP – as an ILP

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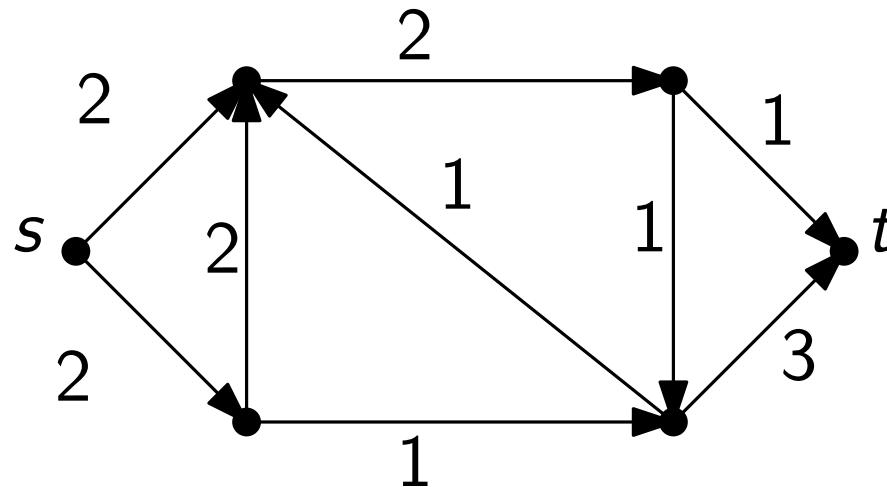
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$$d_{uv} \geq 0 \quad d_{uv} \in \{0, 1\} \quad (u, v) \in E$$

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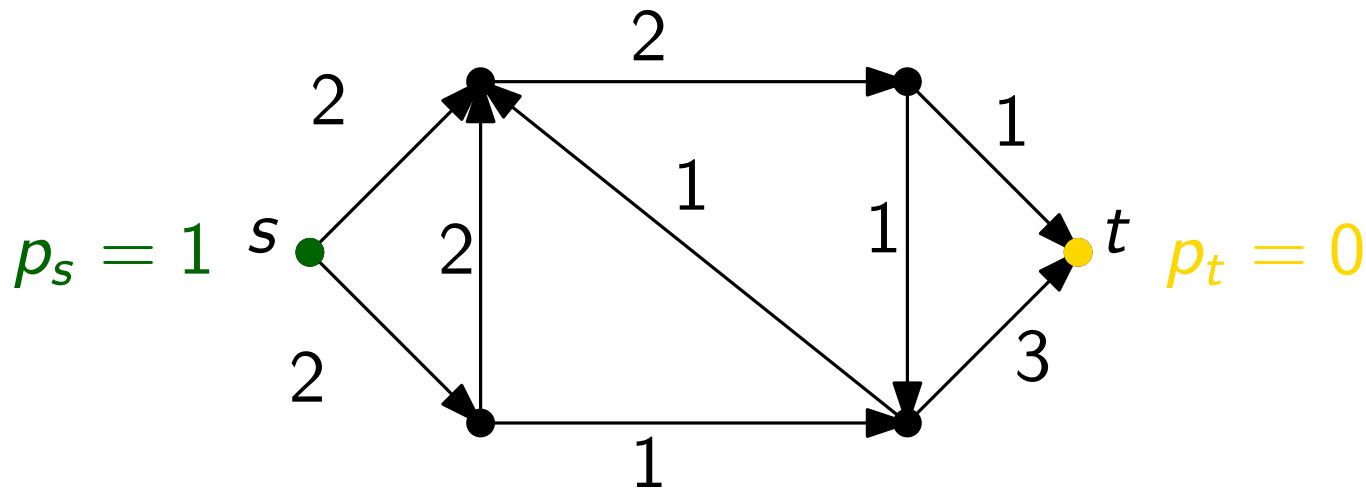
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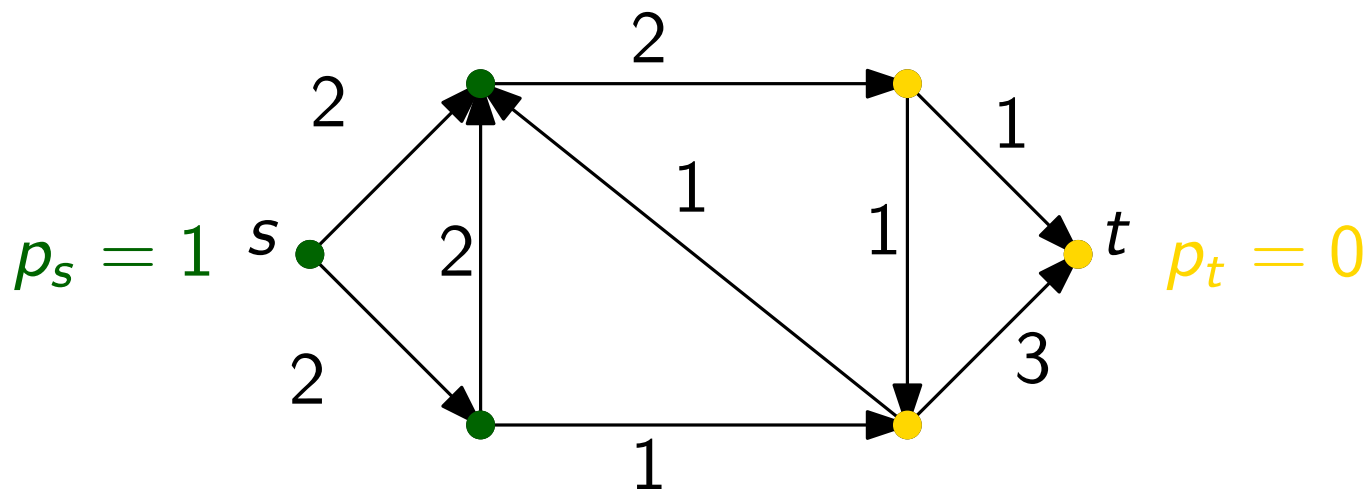
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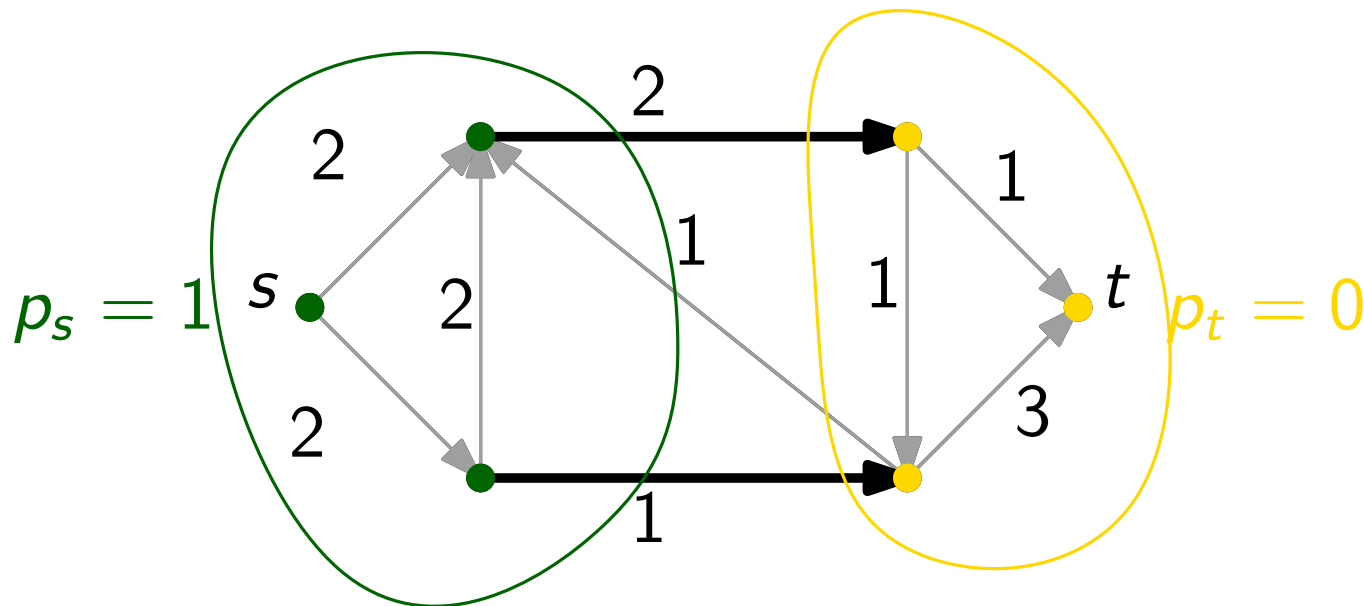
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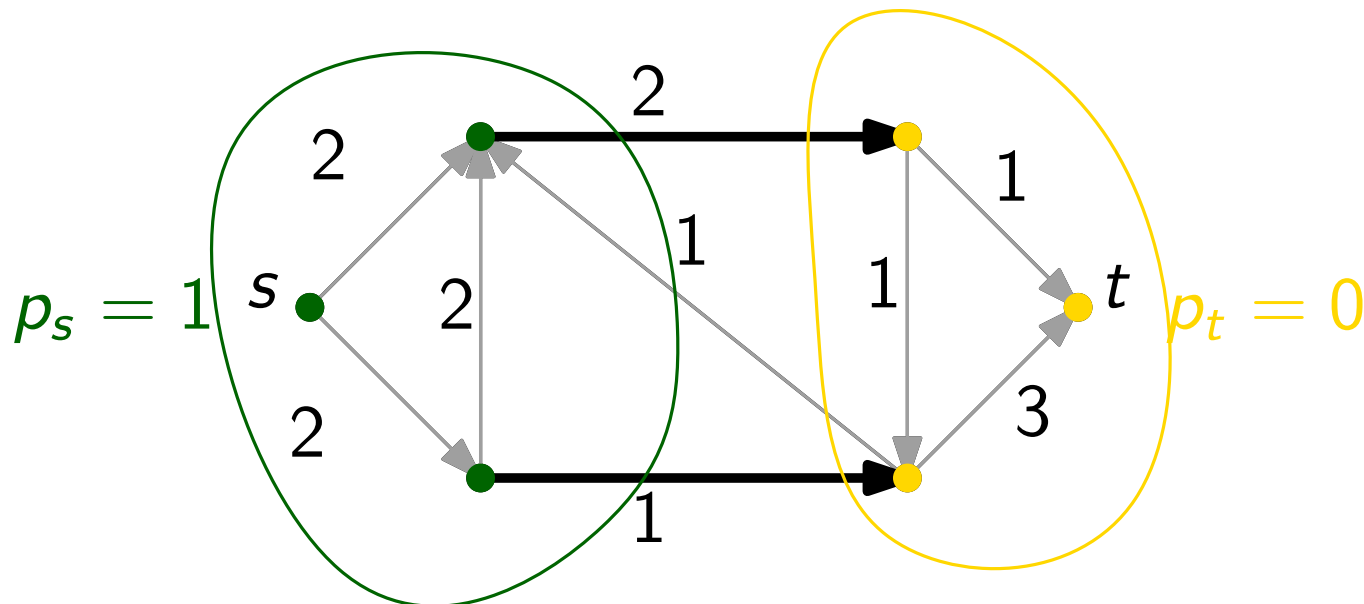
$$\text{subject to } d_{uv} - p_u + p_v \geq 0 \quad (u, v) \in E$$

$$p_s - p_t \geq 1$$

equivalent to Min-Cut!!

~~$$d_{uv} \geq 0 \quad d_{uv} \in \{0, 1\} \quad (u, v) \in E$$~~

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Dual LP – Fractional Cuts

$$\begin{array}{ll} \text{minimize} & \sum_{(u,v) \in E} c_{uv} d_{uv} \\ \text{subject to} & d_{uv} - p_u + p_v \geq 0 \quad (u, v) \in E \\ & p_s - p_t \geq 1 \\ & d_{uv} \geq 0 \quad (u, v) \in E \\ & p_u \geq 0 \quad u \in V \end{array} \quad \equiv \text{LP-relaxation of the ILP}$$

Dual LP – Fractional Cuts

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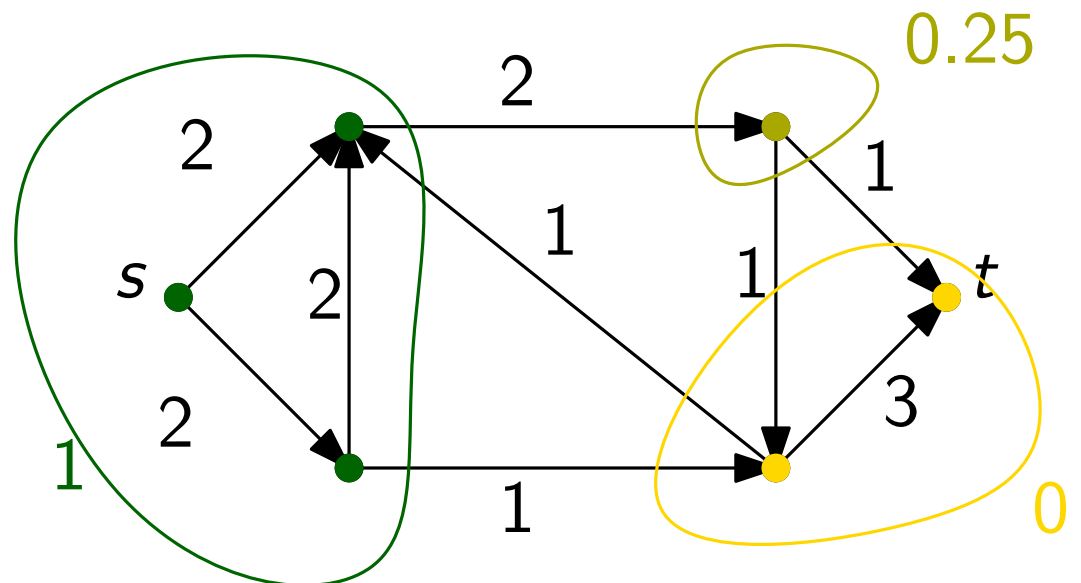
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Dual LP – Fractional Cuts

minimize $\sum_{(u,v) \in E} c_{uv} d_{uv}$

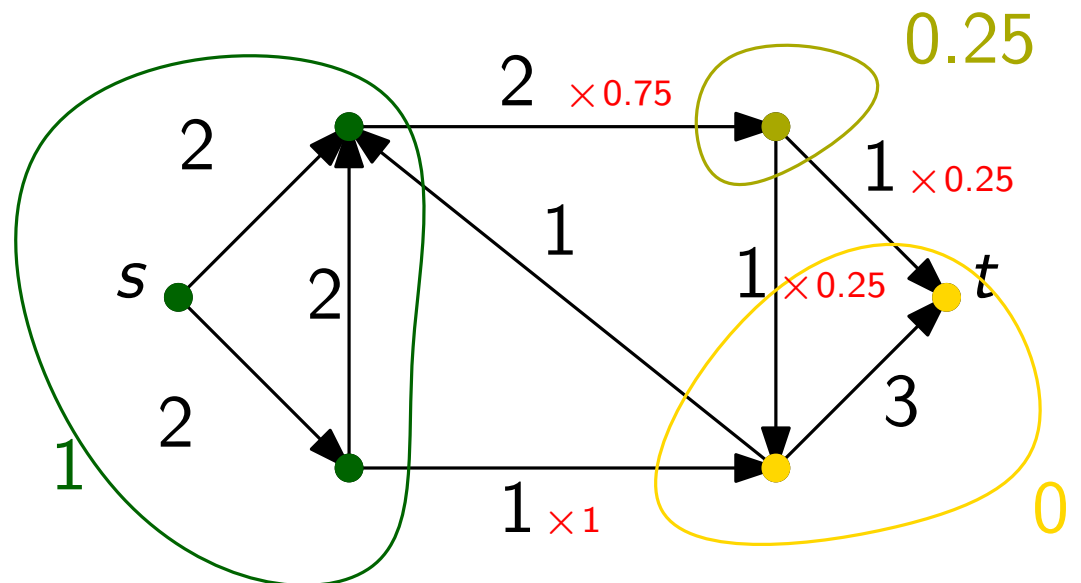
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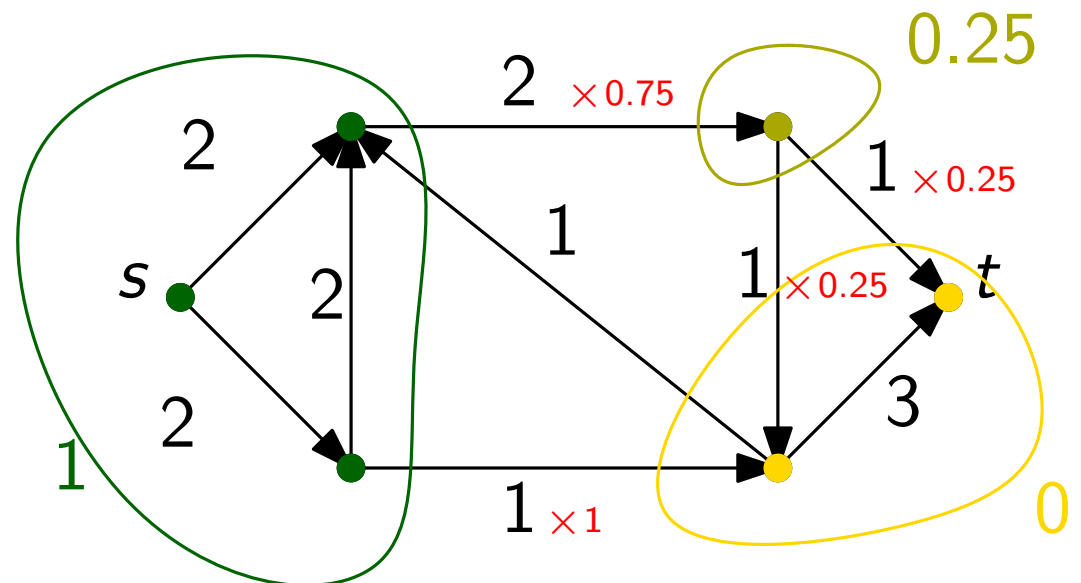
Each s - t -path

$s = v_0, \dots, v_k = t$ has

length ≥ 1 with respect to d

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1} (p_i - p_{i+1})$$

$$= p_s - p_t$$



Dual LP – Fractional Cuts

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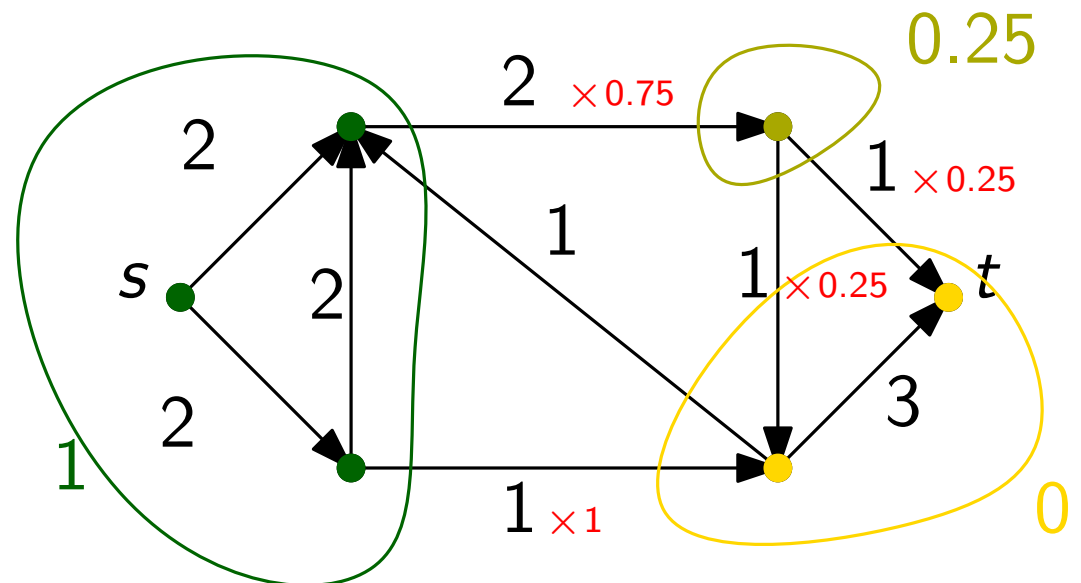
Each extreme-point solution is **integral!**
(exercise)

Each s - t -path

$s = v_0, \dots, v_k = t$ has length ≥ 1 with respect to d

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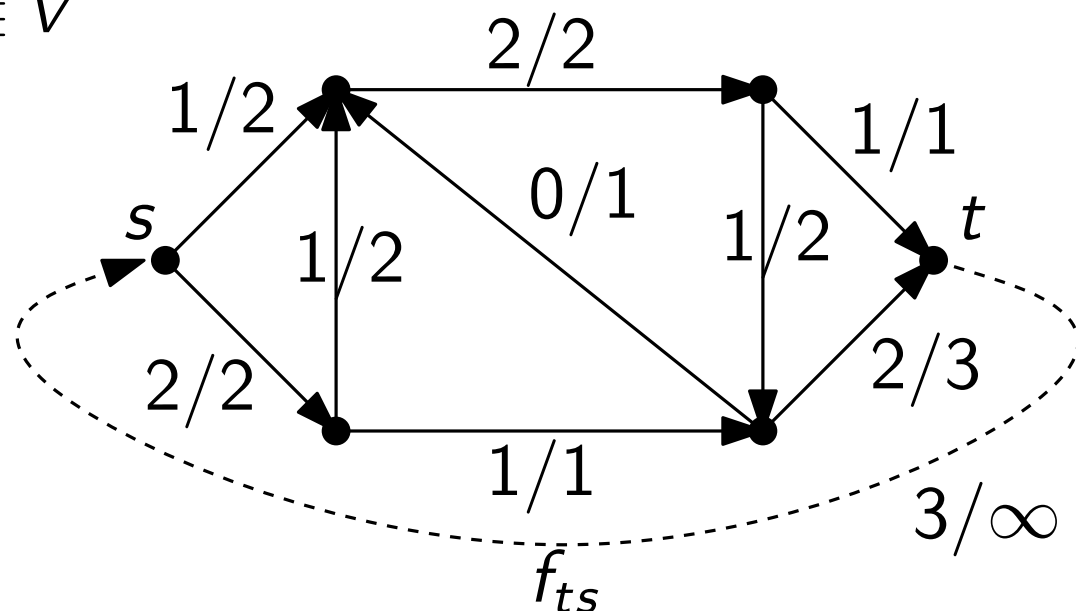
Dual LP – Complementary Slackness

$$\begin{array}{ll}
 \text{maximize} & f_{ts} \\
 \text{subject to} & f_{uv} \leq c_{uv} \quad (u, v) \in E \\
 & \sum_{u: (u,v) \in E} f_{uv} - \sum_{z: (v,z) \in E} f_{vz} \leq 0 \quad v \in V \\
 & f_{uv} \geq 0 \quad (u, v) \in E
 \end{array}$$

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 & p_s - p_t \geq 1 \\
 & d_{uv} \geq 0 \quad (u, v) \in E \\
 & p_u \geq 0 \quad u \in V
 \end{array}$$

For a max. flow and min. cut:

- For each forward edge (u, v) of the cut, $f_{uv} = c_{uv}$
- For each backward edge (u, v) of the cut, $f_{uv} = 0$

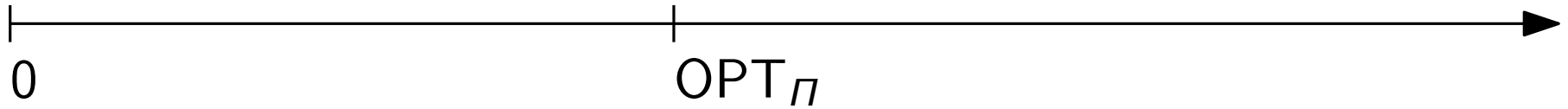


Introduction: Linear Programming

Many approximation algorithms are based on linear programming.

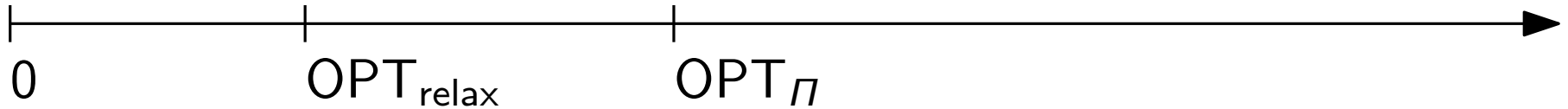
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LP-Rounding



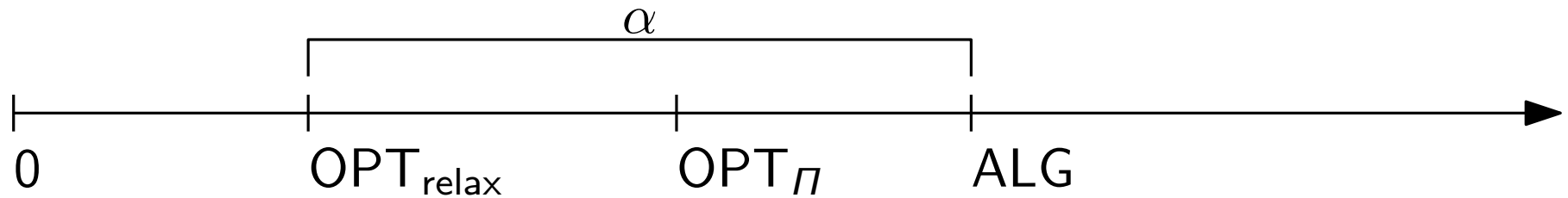
- Consider a minimization problem Π in ILP-form

LP-Rounding



- Consider a minimization problem Π in ILP-form
- Compute a solution to the LP-relaxation

LP-Rounding



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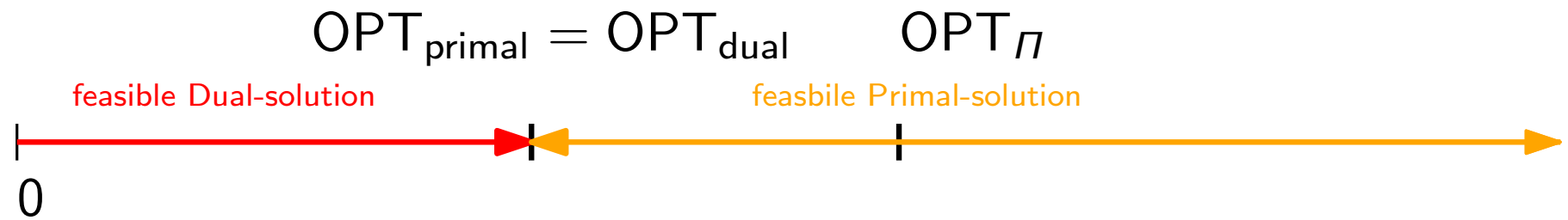
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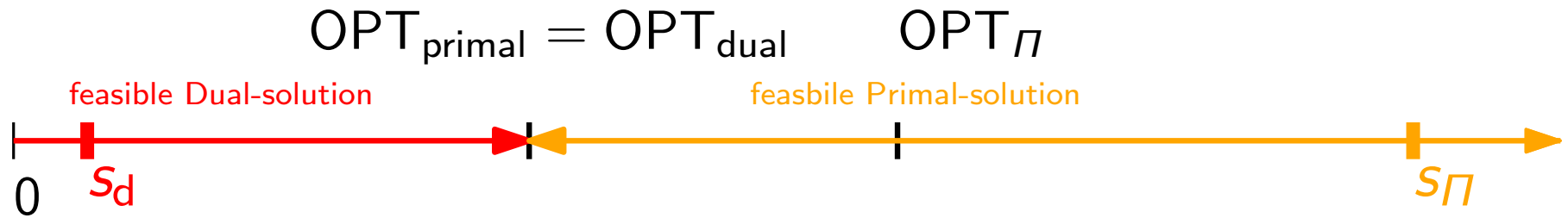
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- Difficulty: ensure **feasible** solution of Π
- Approximation factor $\leq \text{ALG} / \text{OPT}_{\text{relax}}$

Primal-Dual Approach



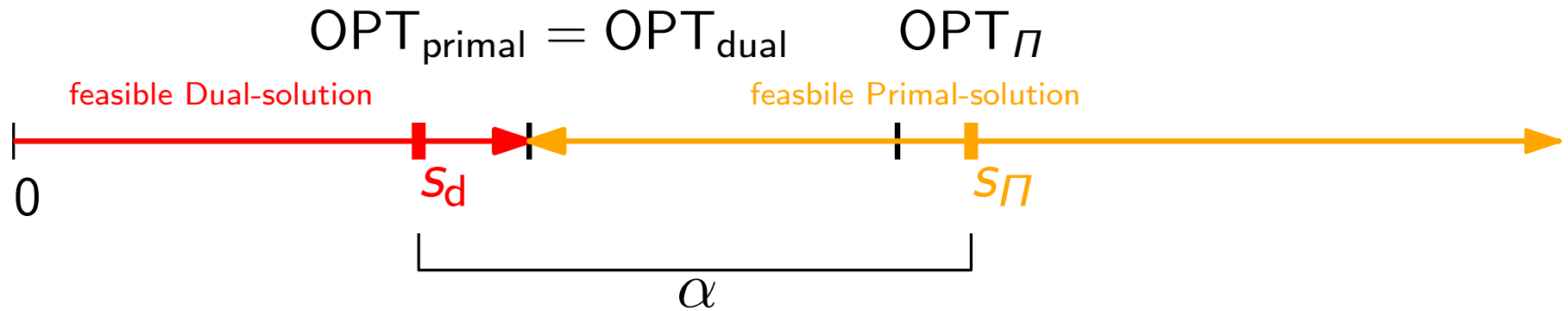
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Primal-Dual Approach



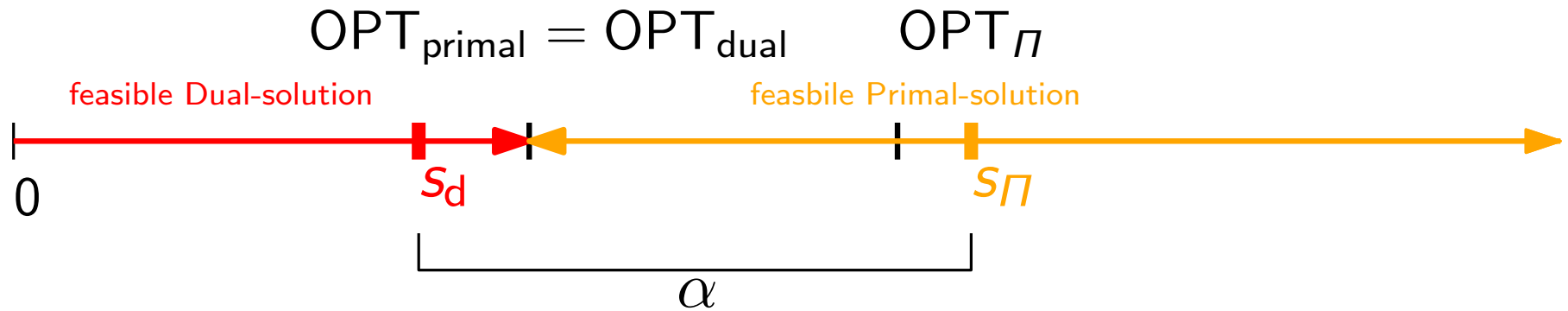
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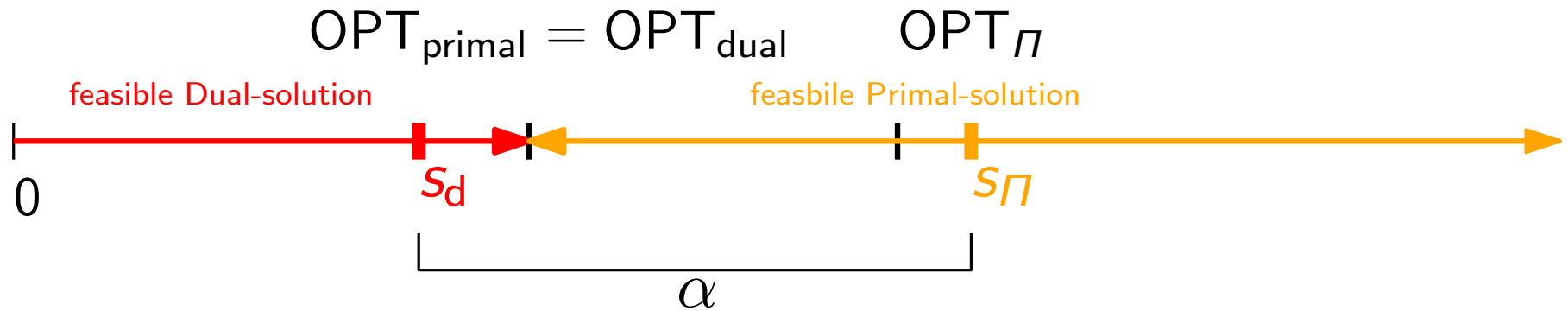
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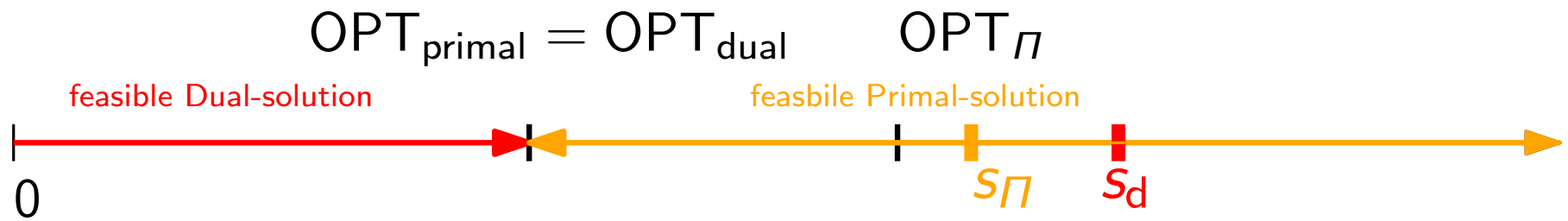
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Primal-Dual Approach



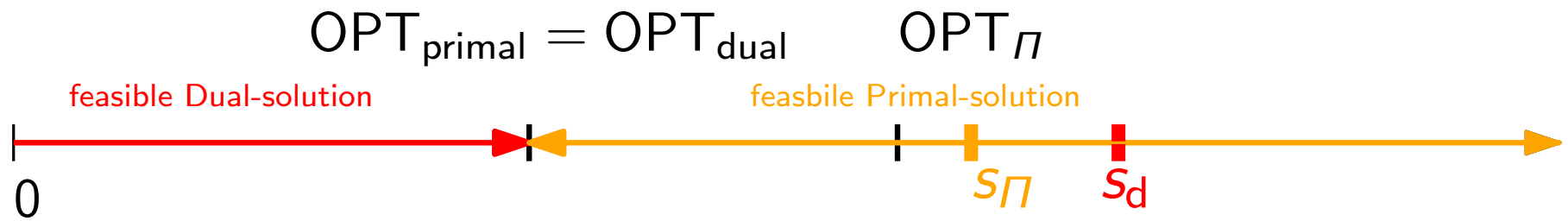
- Consider a minimization problem Π in ILP-form
- Compute **dual** solution s_d and an integral solution s_Π of Π iteratively
- Approximation factor $\leq \text{obj}(s_\Pi)/\text{obj}(s_d)$
- Advantage: don't need LP- "machinery"; possibly faster, more flexible.

Dual Fitting



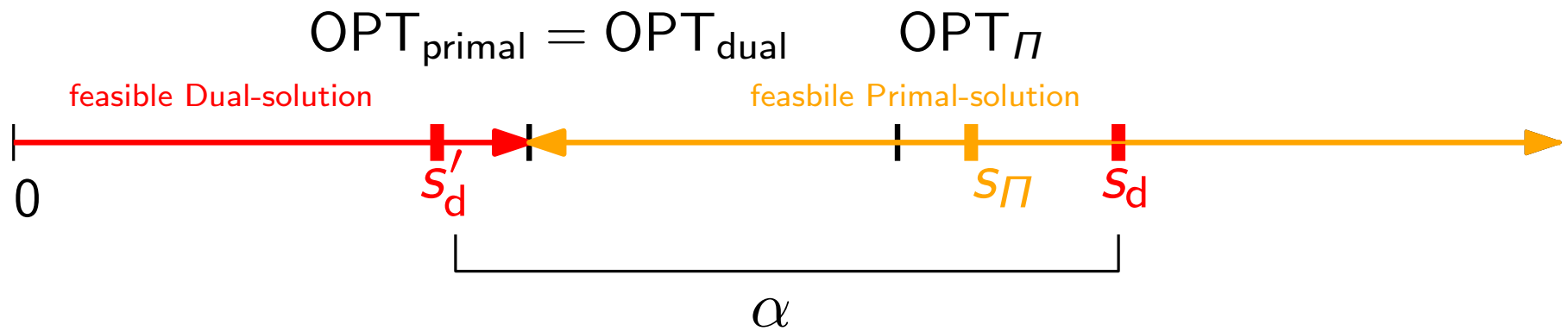
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Dual Fitting



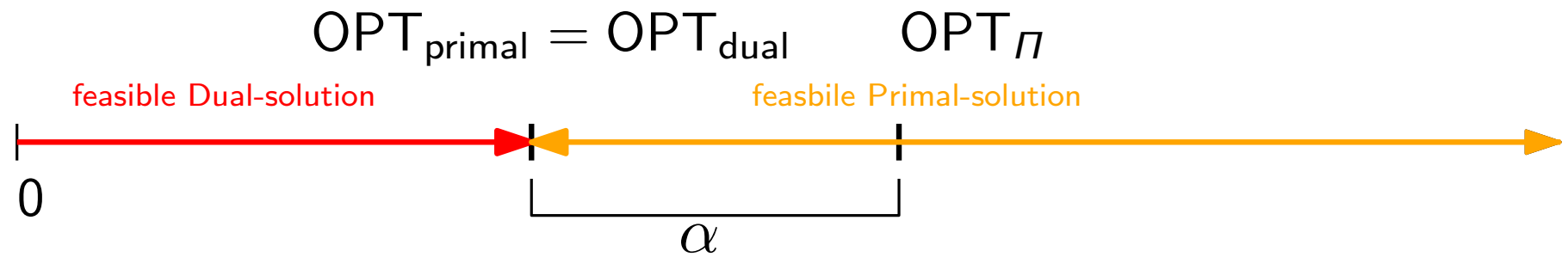
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Dual Fitting



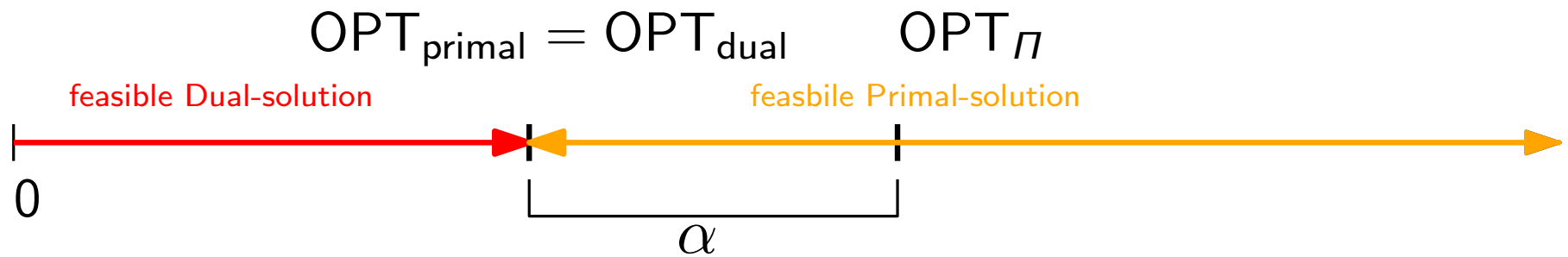
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- Scaling the dual variables \rightsquigarrow feasible dual solution s'_d

Integrality Gap



- Consider a minimization problem Π in ILP-form

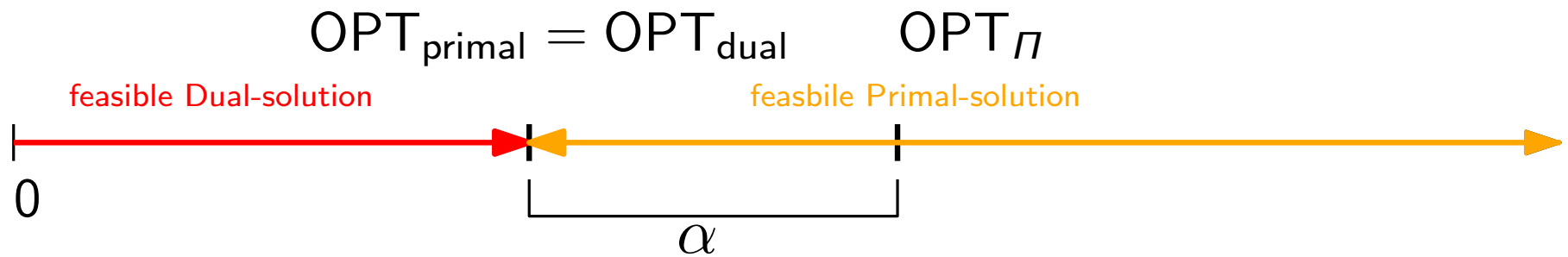
Integrality Gap



- Consider a minimization problem Π in ILP-form
- All the before methods (without additional help) are limited by the **Integrality Gap** of the LP-relaxation

$$\sup_I \frac{OPT(I)}{OPT_{\text{primal}}(I)}$$

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**next class Set Cover
revisited**