Aalto University
School of Science

Department of Computer Science

Combinatorics of Efficient Computations

Approximation Algorithms

Lecture 4: Linear Programming

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Introduction to Linear Programming

Many approximation algorithms are based on linear programming.

- Linear Programming (LP) and LP-Duality
- Min-Max Relationships
- LP-based Algorithm Designs Techniques

Motivation: Upper and Lower Bounds

- Consider an NP-hard minimization problem
- Decision Problem: Is S an **upper bound** of OPT?

  Efficiently verifiable "Yes"-certificates.
- Decision Problem: Is S an **lower bound** of OPT?

  Are "No"-certificates efficiently vertifiable?
  $\rightsquigarrow$ probably not! (NP $\neq$ coNP)
- Need lower bounds $S \geq \text{OPT}/\alpha$ (approximate "No"-certificates) for approximation algorithms!
- For example:
  - Vertex Cover: lower bound by matchings
  - TSP: lower bound by MST or Cycle Cover

Linear Programming

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcrcrcr} \text{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \text{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

- Standard form (i.e., using only "$\geq$")

Linear Programming - Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcrcrcr} \text{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \text{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

- $\mathbf{x} = (2, 1, 3)$ is a feasible solution $\rightsquigarrow$ $S = 30$ is an upper bound on OPT

Linear Programming - Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcrcrcr} \text{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \text{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

- $7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \geq 10 \rightsquigarrow \text{OPT} \geq 10$
- $7x_1 + x_2 + 5x_3 \geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \geq 10 + 6$
- $\rightsquigarrow \text{OPT} \geq 16$

Linear Programming - Lower Bounds

$$
\begin{array}{lrcrcrcrl}
\text{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & & \\
\text{subject to } {\color{red}y_1(} & x_1 & - & x_2 & + & 3x_3 {\color{red})} & \geq & 10 & {\color{red}y_1} \\
{\color{red}y_2(} & 5x_1 & + & 2x_2 & - & x_3 {\color{red})} & \geq & 6 & {\color{red}y_2} \\
 & & & & & x_1, x_2, x_3 & \geq & 0 &
\end{array}
$$

$$
\begin{array}{lrcrcr}
{\color{red}\text{maximize}} & {\color{red}10y_1} & {\color{red}+} & {\color{red}6y_2} & & \\
\text{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & y_1, y_2 & \geq & 0
\end{array}
$$

- Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.
- $\mathbf{x} = (7/4, 0, 11/4)$ and $\mathbf{y} = (2, 1)$ both provide objective values of 26 $\rightsquigarrow$ both solutions are optimal!

LP – standard form

Primal Program

$$\text{minimize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad i = 1, \ldots, m$$

$$x_j \geq 0 \qquad j = 1, \ldots, n$$

Dual Program

$$\text{maximize} \quad \sum_{i=1}^{m} b_i y_i$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij} y_i \leq c_j \qquad j = 1, \ldots, n$$

$$y_i \geq 0 \qquad i = 1, \ldots, m$$

What is the dual of the dual?

- maximization instances dualize analogously

LP-Duality

Primal: minimize $\sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = 1, \ldots, m$

$x_j \geq 0 \quad j = 1, \ldots, n$

Dual: maximize $\sum_{i=1}^{m} b_i y_i$

subject to $\sum_{i=1}^{m} a_{ij} y_i \leq c_j \quad j = 1, \ldots, n$

$y_i \geq 0 \quad i = 1, \ldots, m$

Thm. The primal program has a finite optimum $\Leftrightarrow$ the dual program has a finite optimum. Moreover, if $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ and $\mathbf{y}^* = (y_1^*, \ldots, y_m^*)$ are optimal solutions for the primal and dual (respectively), then

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* .$$

Weak LP-Duality

$$\begin{array}{lll} \min & \sum_{j=1}^{n} c_j x_j \\ \text{s.t} & \sum_{j=1}^{n} a_{ij} x_j \geq b_i \\ & x_j \geq 0 \end{array} \qquad \begin{array}{ll} \max. & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} & \sum_{i=1}^{m} a_{ij} y_i \leq c_j \\ & y_i \geq 0 \end{array}$$

Thm. If $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ are feasible solutions for the primal and dual programs (resp.), then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \, .$$

Proof.

$\geq$

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i \, .$$

$\square$

Complementary Slackness

$$\begin{array}{lll} \min & \sum_{j=1}^{n} c_j x_j \\ \text{s.t} & \sum_{j=1}^{n} a_{ij} x_j \geq b_i \\ & x_j \geq 0 \end{array} \qquad \begin{array}{lll} \max. & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} & \sum_{i=1}^{m} a_{ij} y_i \leq c_j \\ & y_i \geq 0 \end{array}$$

Thm. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ be feasible solutions for the primal and dual Programs (resp.). The solutions $\mathbf{x}$ and $\mathbf{y}$ are optimal if only if the following conditoins are met:

Primal CS:
For each $j = 1, \ldots, n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

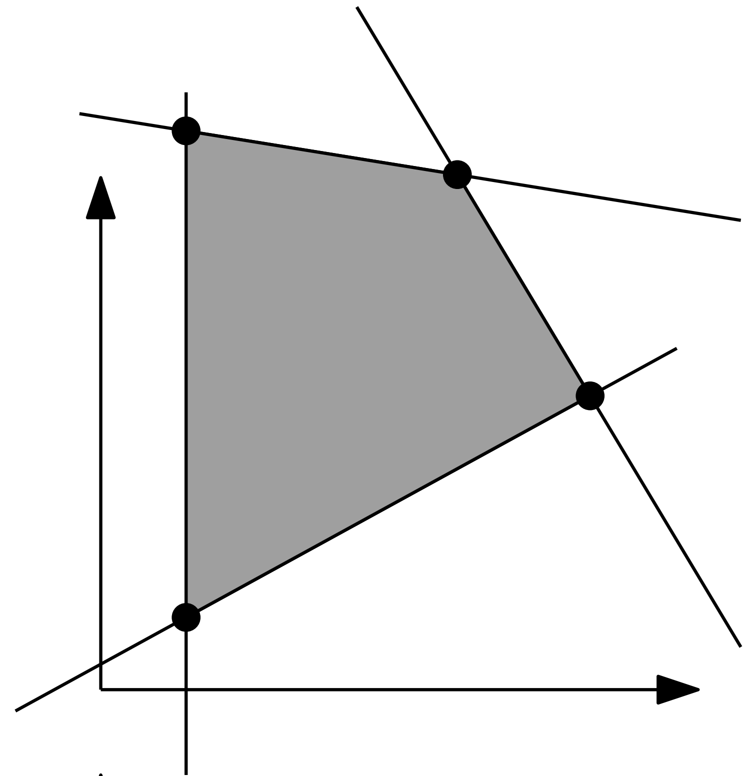
Dual CS:
For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

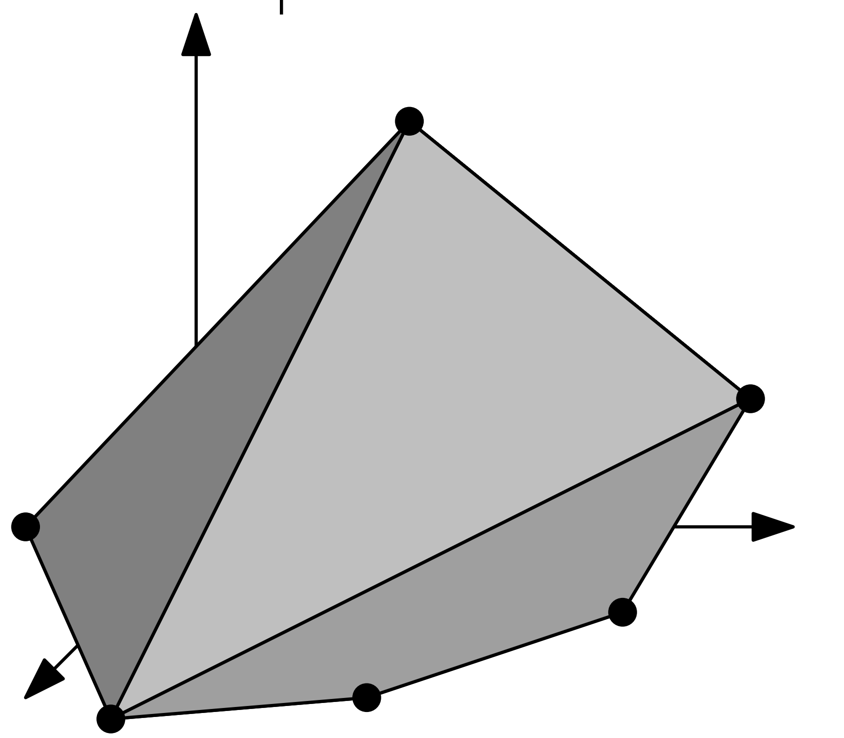
Proof.

$$\sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i = \sum_{i=1}^{m} b_i y_i. \quad \square$$

LPs and convex polytopes

- The feasible solutions of an LP with n variables from a **convex polytope** in $\mathbb{R}^n$ (intersection of halfspaces).
- Corners of the polytope are called **extreme point solutions** $\Leftrightarrow$ n linearly independent inequalities (constraints) are satisfied with equality.
- When an optimal solution exists, some extreme point will also be optimal.





Integer Linear Programs (ILPs)

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \geq b_i & i = 1, \ldots, m \\
& \cancel{x_j \geq 0} \;\; x_j \in \mathbb{N} & j = 1, \ldots, n
\end{array}
$$

- Many NP-optimization problems can be formulated as ILPs.
- NP-hard to solve ILPs.
- LP-relaxation provides a lower bound: $\mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}_{\mathrm{ILP}}$
- e.g., Vertex Cover

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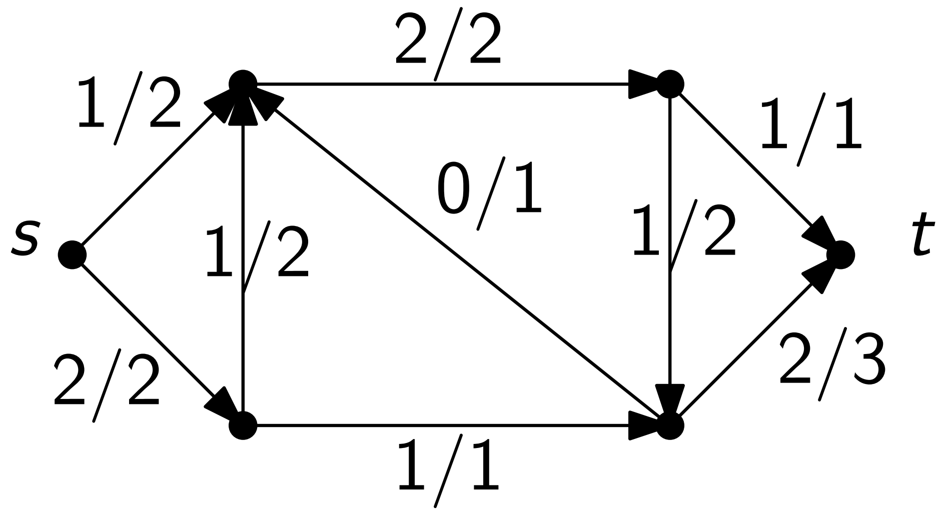
Max-Flow-Problem

Given: A directed graph $G = (V, E)$ with edge capacities $c\colon E \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

Find: A maximum s–t flow (i.e., an assignment of non-negative weights to edges) f, such that

- $f(u, v) \leq c(u, v)$ for each edge $(u, v) \in E$
- $\sum_{(u,v)\in E} f(u, v) = \sum_{(v,z)\in E} f(v, z)$ for each vertex $v \in V - \{s, t\}$

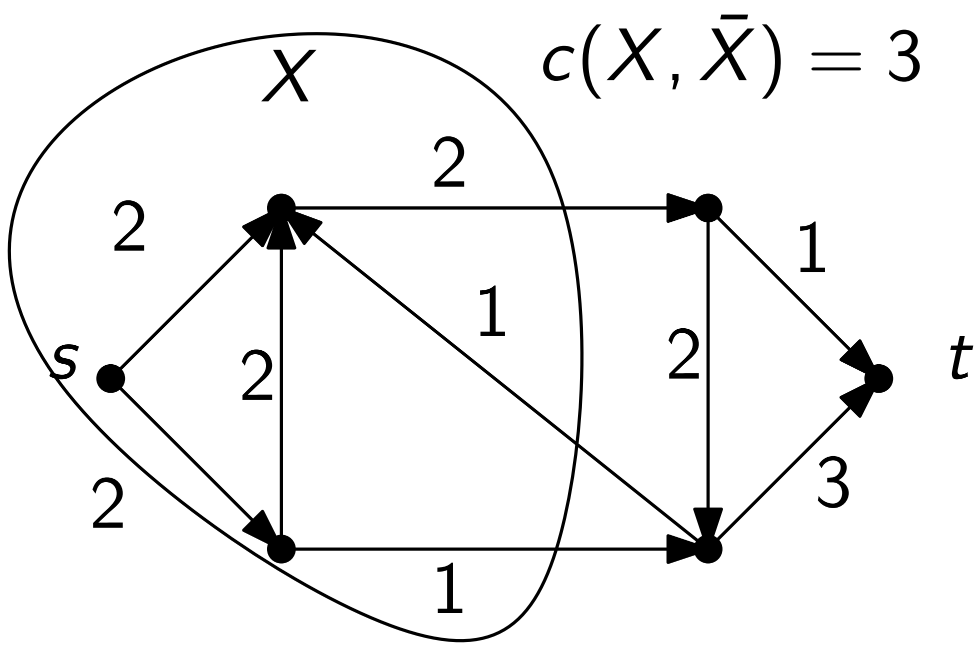
The **flow-value** is the inflow to t minus the outflow from t.



Min-Cut-Problem

Given: A directed graph $G = (V, E)$ with edge capacities $c\colon E \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

Find: An s–t-cut, i.e., a vertex set X with $s \in X$ and $t \in \bar{X}$, such that the total capacity $c(X, \bar{X})$ of the edges from X to $\bar{X}$ is minimum.



Max-Flow-Min-Cut-Theorem

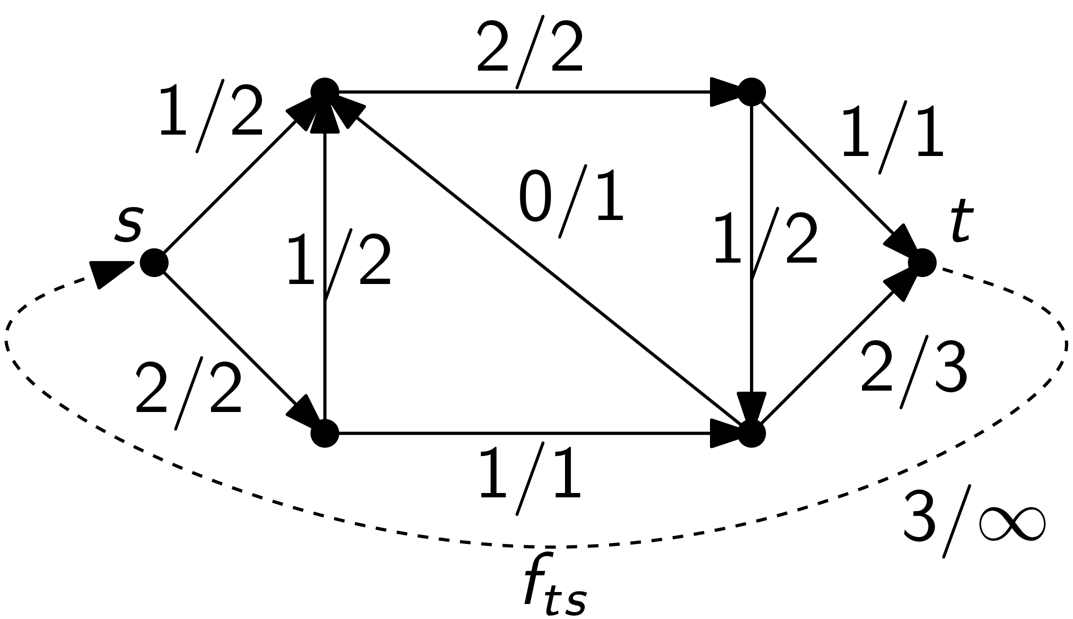
Thm. The value of a maximum s–t-flow and the capacity of a minimum s–t-cut are the same.

Proof. Special case of LP-duality ...

Max-Flow (circulation form) as an LP

$$
\begin{aligned}
\text{maximize} \quad & f_{ts} & \\
\text{subject to} \quad & f_{uv} \le c_{uv} & (u,v) \in E \\
& \sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \le 0 & v \in V \\
& f_{uv} \ge 0 & (u,v) \in E
\end{aligned}
$$

why does this work?



Dual LP

Primal Program

$$\begin{array}{llll}
\text{maximize} & f_{ts} & & \\
\text{subject to} & f_{uv} \leq c_{uv} & (u,v) \in E & \color{red}{d_{uv}} \\
& \displaystyle\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0 & v \in V & \color{red}{p_v} \\
& f_{uv} \geq 0 & (u,v) \in E &
\end{array}$$

Dual Program

$$\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{(u,v)\in E} c_{uv} d_{uv} & \\
\text{subject to} & d_{uv} - p_u + p_v \geq 0 & (u,v) \in E \\
& p_s - p_t \geq 1 & \\
& d_{uv} \geq 0 & (u,v) \in E \\
& p_u \geq 0 & u \in V
\end{array}$$

Dual LP – as an ILP

minimize $\sum_{(u,v)\in E} c_{uv} d_{uv}$

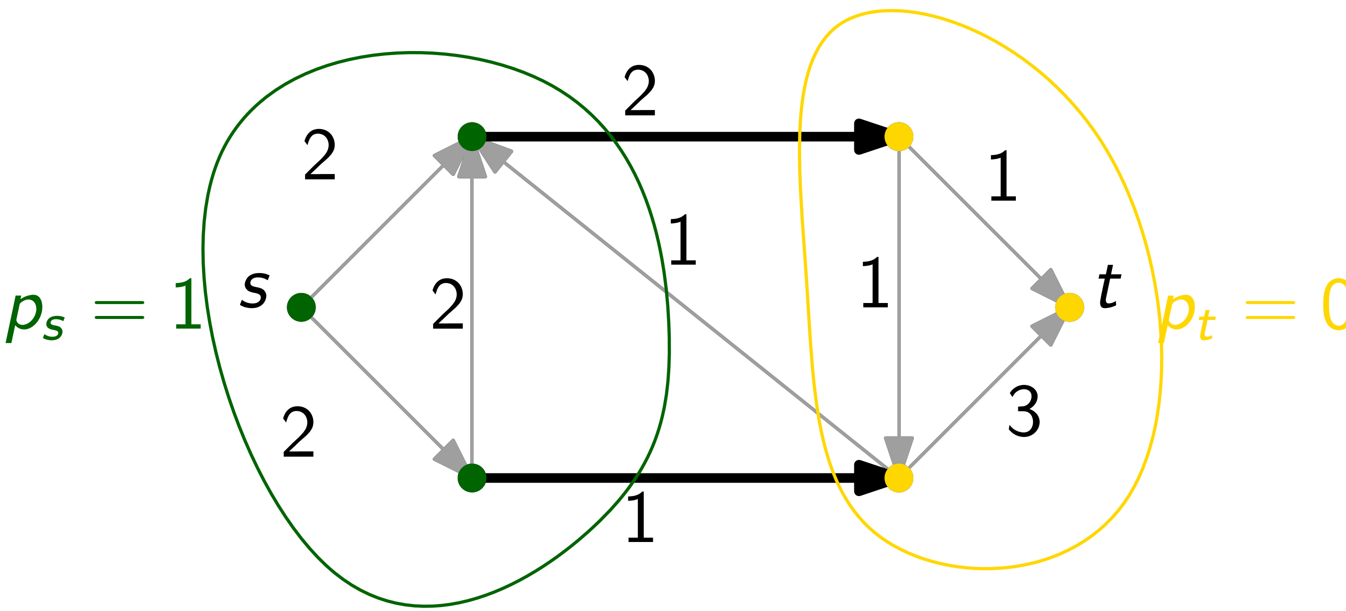
subject to $d_{uv} - p_u + p_v \geq 0 \quad (u,v) \in E$

$p_s - p_t \geq 1$

~~$d_{uv} \geq 0$~~ $d_{uv} \in \{0,1\}$ $(u,v) \in E$

~~$p_u \geq 0$~~ $p_u \in \{0,1\}$ $u \in V$

equivalent to Min-Cut!!



Dual LP – Fractional Cuts

$$\begin{array}{lll} \text{minimize} & \sum_{(u,v)\in E} c_{uv} d_{uv} & \\ \text{subject to} & d_{uv} - p_u + p_v \geq 0 & (u,v) \in E \\ & p_s - p_t \geq 1 & \\ & d_{uv} \geq 0 & (u,v) \in E \\ & p_u \geq 0 & u \in V \end{array}$$

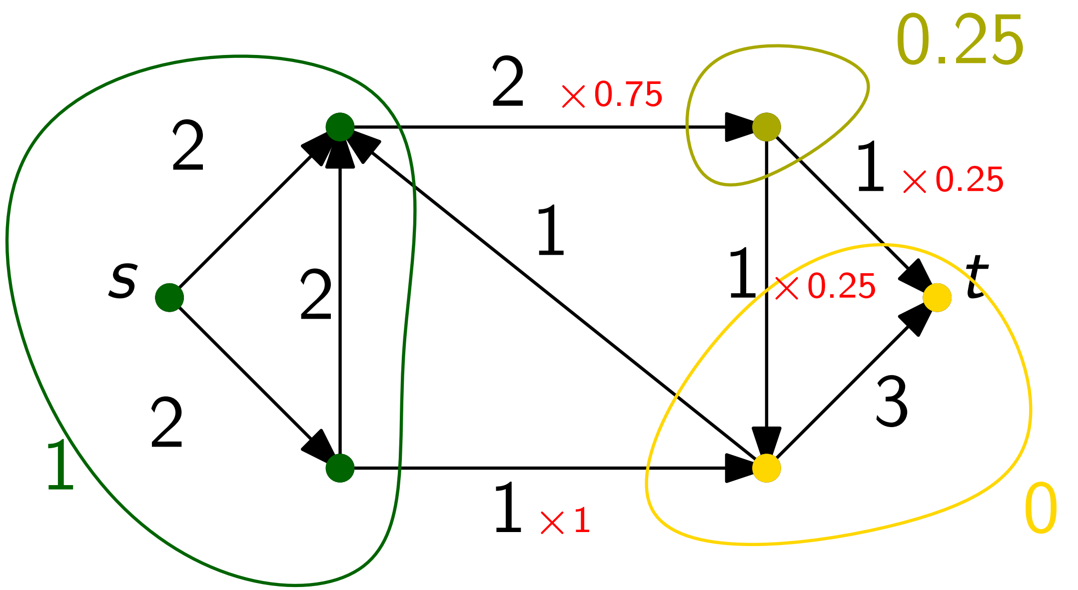
$\equiv$ LP-relaxation of the ILP

Each extreme-point solution is **integral**! (exercise)

Each s–t-path $s = v_0, \ldots, v_k = t$ has length ≥ 1 with respect to d

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1} (p_i - p_{i+1})$$

$$= p_s - p_t$$



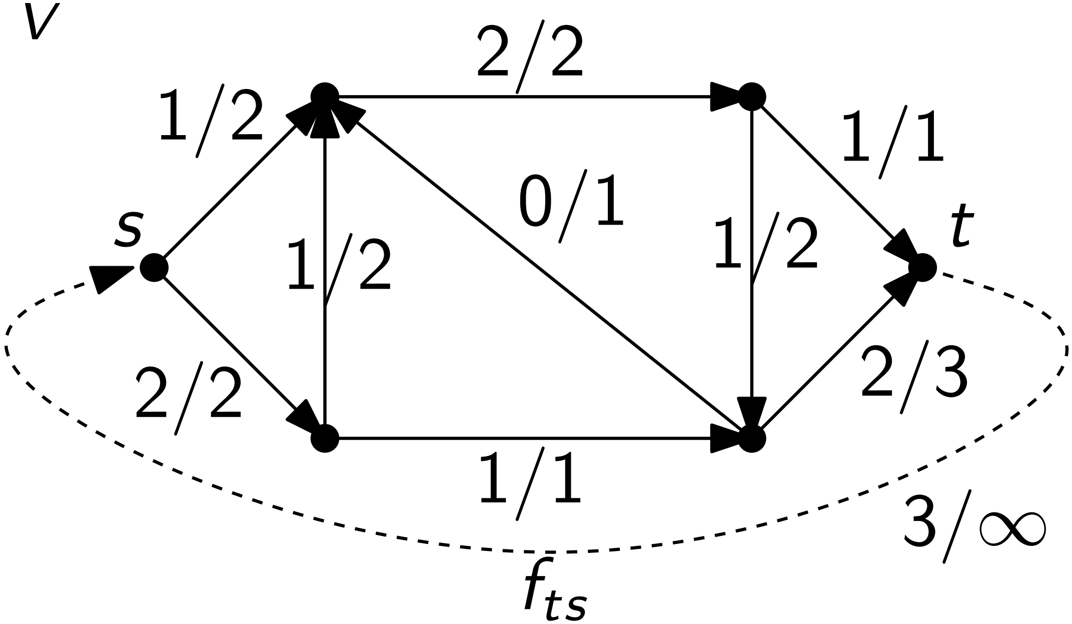
Dual LP – Complementary Slackness

$$
\begin{array}{lll}
\text{maximize} & f_{ts} & \\
\text{subject to} & f_{uv} \le c_{uv} & (u,v) \in E \\
& \displaystyle\sum_{u:\ (u,v)\in E} f_{uv} - \sum_{z:\ (v,z)\in E} f_{vz} \le 0 & v \in V \\
& f_{uv} \ge 0 & (u,v) \in E
\end{array}
$$

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{(u,v)\in E} c_{uv} d_{uv} & \\
\text{subject to} & d_{uv} - p_u + p_v \ge 0 & (u,v) \in E \\
& p_s - p_t \ge 1 & \\
& d_{uv} \ge 0 & (u,v) \in E \\
& p_u \ge 0 & u \in V
\end{array}
$$

For a max. flow and min. cut:

- For each forward edge (u,v) of the cut, $f_{uv} = c_{uv}$
- For each backward edge (u,v) of the cut, $f_{uv} = 0$



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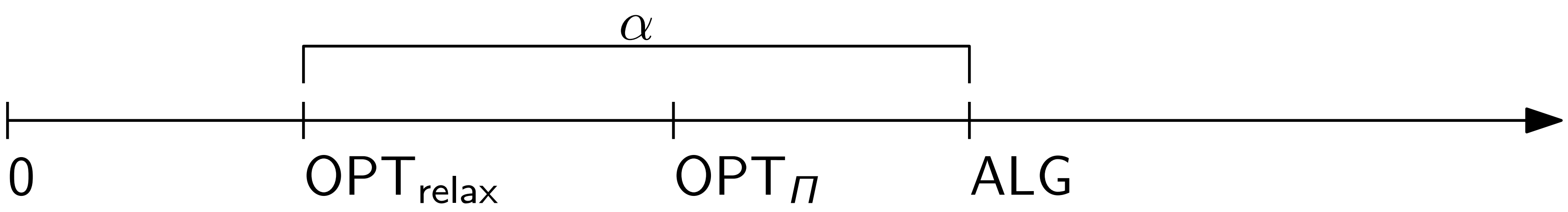
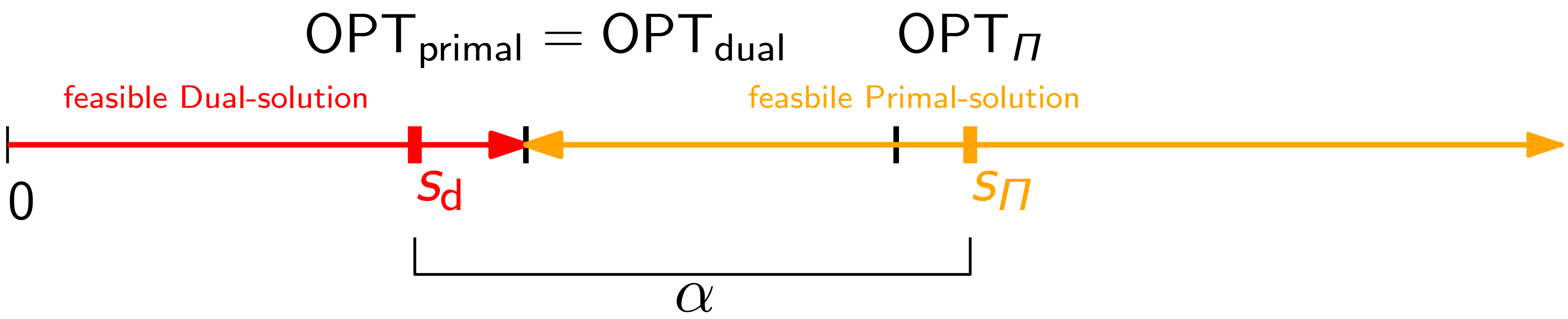
LP-Rounding



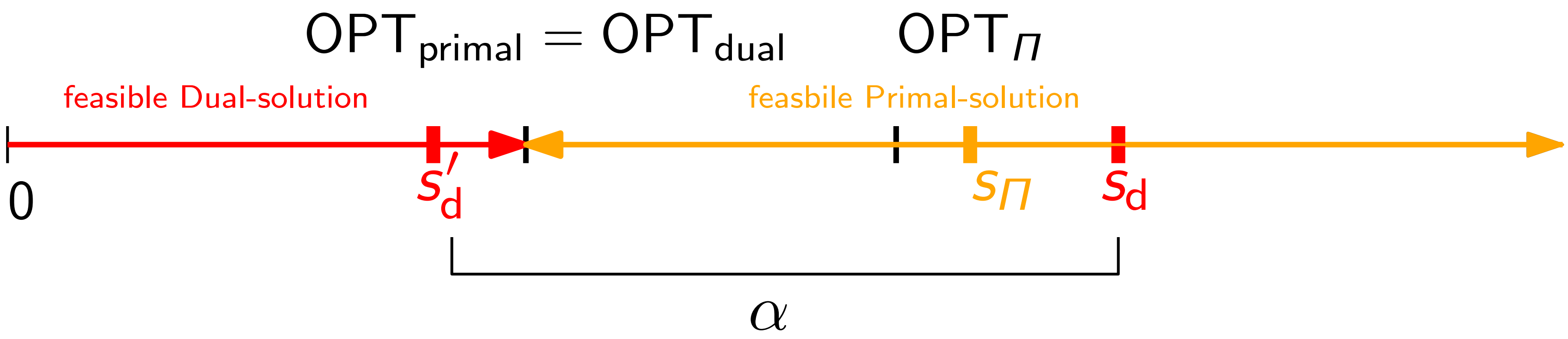
- Consider a minimization problem Π in ILP-form
- Compute a solution to the LP-relaxation
- "Round" to obtain an integer solution for Π
- Difficulty: ensure **feasible** solution of Π
- Approximation factor $\leq \mathrm{ALG}/\,\mathrm{OPT}_{\mathrm{relax}}$

Primal-Dual Approach



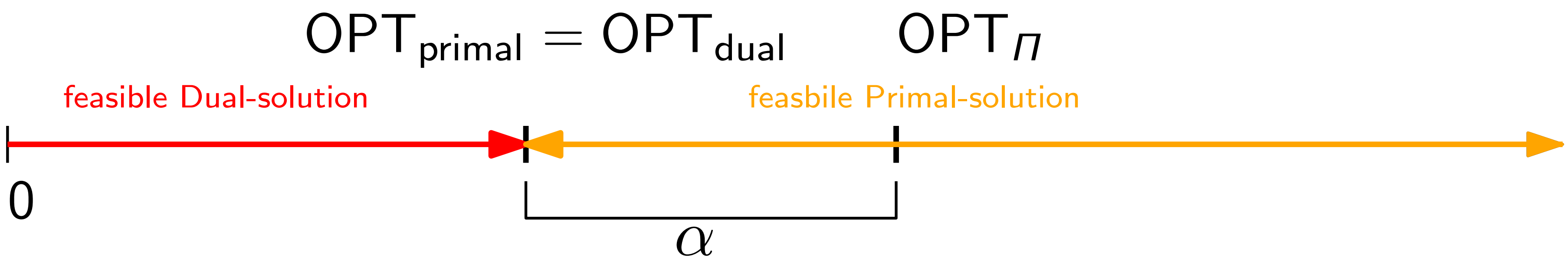
- Consider a minimization problem Π in ILP-form
- Compute **dual** solution s_d and an integral solution s_Π of Π iteratively
- Approximation factor $\leq \text{obj}(s_\Pi)/\text{obj}(s_d)$
- Advantage: don't need LP-"machinery"; possibly faster, more flexible.

Dual Fitting



- Consider a minimization problem Π in ILP-form
- Combinatorial algorithm (e.g., greedy) computes a feasible solution s_{Π} and "infeasible" dual solution s_{d} from s_{Π} which is more expensive than s_{Π}.
- Scaling the dual variables $\rightsquigarrow$ feasible dual solution s'_{d}

Integrality Gap



- Consider a minimization problem Π in ILP-form
- All the before methods (without additional help) are limited by the **Integrality Gap** of the LP-relaxation

$$\sup_{I} \frac{\mathrm{OPT}(I)}{\mathrm{OPT}_{\text{primal}}(I)}$$

next class Set Cover revisited