$$\bar{\bar{A}} \cdot \bar{a} = \bar{e}\,(\bar{f} \cdot \bar{a}) \qquad \bar{a} \cdot \bar{\bar{A}} = \bar{f}\,(\bar{a} \cdot \bar{e})$$

$$\uparrow$$

$$\bar{e}\bar{f} \qquad\qquad = \bar{\bar{A}}^T \cdot \bar{a}$$

$$\bar{\bar{A}} \cdot \bar{\bar{B}} = \left(\bar{\bar{B}}^T \cdot \bar{\bar{A}}^T\right)^T \qquad \left(\bar{\bar{A}} \cdot \bar{\bar{B}}\right) \cdot \bar{\bar{C}} = \bar{\bar{A}} \cdot \left(\bar{\bar{B}} \cdot \bar{\bar{C}}\right)$$

$$\bar{\bar{C}} : \bar{\bar{D}} = \left(\bar{c}_1 \cdot \bar{d}_1\right)\left(\bar{c}_2 \cdot \bar{d}_2\right)$$

$$\bar{c}_1 \bar{c}_2 \qquad \bar{d}_1 \bar{d}_2$$

$$\bar{\bar{C}} \overset{\times}{\times} \bar{\bar{D}} = \left(\bar{c}_1 \times \bar{d}_1\right)\left(\bar{c}_2 \times \bar{d}_2\right)$$

$$\bar{\bar{A}} : \bar{\bar{S}} = \bar{\bar{S}} : \bar{\bar{A}} = \bar{\bar{S}}^T : \bar{\bar{A}}^T = \bar{\bar{A}}^T : \bar{\bar{S}}^T = -\bar{\bar{A}} : \bar{\bar{S}}$$

$\uparrow$ antisymmetric $\qquad \uparrow$ symmetric

$$= 0$$

$$\bar{\bar{I}} : \bar{\bar{I}} = 3$$

$$\bar{\bar{I}} \overset{\times}{\times} \bar{\bar{I}} = \left(\bar{u}_x \bar{u}_x + \bar{u}_y \bar{u}_y + \bar{u}_z \bar{u}_z\right) \overset{\times}{\times} \left(\bar{u}_x \bar{u}_x + \bar{u}_y \bar{u}_y + \bar{u}_z \bar{u}_z\right)$$

$$= \bar{u}_z \bar{u}_z + \bar{u}_y \bar{u}_y + \bar{u}_z \bar{u}_z + \bar{u}_x \bar{u}_x + \bar{u}_y \bar{u}_y + \bar{u}_x \bar{u}_x = 2\bar{\bar{I}}$$

$$\bar{\bar{A}} \cdot \bar{\bar{A}} = \bar{\bar{A}}^{2}$$

$$\frac{1}{2} \bar{\bar{A}} \overset{\times}{\times} \bar{\bar{A}} = \bar{\bar{A}}^{(2)}$$

$$\mathrm{tr}\, \bar{\bar{A}} = \bar{\bar{A}} : \bar{\bar{I}}$$

$$\mathrm{spm}\, \bar{\bar{A}} = \frac{1}{2} \bar{\bar{A}} \overset{\times}{\times} \bar{\bar{A}} : \bar{\bar{I}} = \mathrm{tr}\, \bar{\bar{A}}^{(2)}$$

$$\det \bar{\bar{A}} = \frac{1}{6} \bar{\bar{A}} \overset{\times}{\times} \bar{\bar{A}} : \bar{\bar{A}}$$

$$\mathrm{spm}\, \bar{\bar{I}} = \frac{1}{2} \bar{\bar{I}} \overset{\times}{\times} \bar{\bar{I}} : \bar{\bar{I}} = 3$$

$$\det \bar{\bar{I}} = 1$$

$$\bar{\bar{A}}^{-1} \cdot \bar{\bar{A}} = \bar{\bar{I}}$$

$$\bar{\bar{A}} \cdot \bar{\bar{A}}^{-1} = \bar{\bar{I}}$$

$$\bar{\bar{A}}^{-1} = \frac{\bar{\bar{A}}^{(2)T}}{\det \bar{\bar{A}}}$$

$$\begin{aligned}
\bar{\bar{A}} \overset{\times}{\times} \bar{\bar{B}} : \bar{\bar{I}} &= \bar{a}_1 \bar{a}_2 \overset{\times}{\times} \bar{b}_1 \bar{b}_2 : \bar{\bar{I}} = (\bar{a}_1 \times \bar{b}_1)(\bar{a}_2 \times \bar{b}_2) : \bar{\bar{I}} \\
&= \bar{a}_1 \times \bar{b}_1 \cdot \bar{a}_2 \times \bar{b}_2 = \bar{a}_1 \cdot \bar{b}_1 \times (\bar{a}_2 \times \bar{b}_2) \\
&= \bar{a}_1 \cdot (\bar{a}_2\, \bar{b}_1 \cdot \bar{b}_2 - \bar{b}_2\, \bar{b}_1 \cdot \bar{a}_2) \\
&= (\bar{a}_1 \cdot \bar{a}_2)(\bar{b}_1 \cdot \bar{b}_2) - \underbrace{(\bar{a}_1 \cdot \bar{b}_2)(\bar{a}_2 \cdot \bar{b}_1)}_{\bar{a}_1 \bar{a}_2 : \bar{b}_2 \bar{b}_1} \\
&= \mathrm{tr}\, \bar{\bar{A}}\, \mathrm{tr}\, \bar{\bar{B}} - \underbrace{\bar{\bar{A}} : \bar{\bar{B}}^{T}}_{(\bar{\bar{A}} \cdot \bar{\bar{B}}) : \bar{\bar{I}} = \mathrm{tr}(\bar{\bar{A}} \cdot \bar{\bar{B}})}
\end{aligned}$$

$$\mathrm{tr}\, \bar{\bar{A}} \cdot \bar{\bar{B}} = \mathrm{tr}\, \bar{a}_1 \bar{a}_2 \cdot \bar{b}_1 \bar{b}_2 = \bar{a}_2 \cdot \bar{b}_1\, \bar{a}_1 \cdot \bar{b}_2$$

$$\operatorname{spm} \bar{\bar{A}} = \frac{1}{2} \bar{\bar{A}} \overset{\times}{\times} \bar{\bar{A}} : \bar{\bar{I}} = \frac{1}{2} \left((\operatorname{tr} \bar{\bar{A}})^2 - \operatorname{tr} \bar{\bar{A}}^2 \right)$$

COMPLETE : $\det \bar{\bar{A}} \neq 0$

PLANAR (STRICTLY) : $\det \bar{\bar{A}} = 0$, $\cancel{\operatorname{spm} \bar{\bar{A}} \neq 0}$ $\bar{\bar{A}}^{(2)} \neq 0$

LINEAR : $\det \bar{\bar{A}} = 0$, $\operatorname{spm} \bar{\bar{A}} = 0$

$\bar{a}\bar{b}$ $\qquad \bar{a}\bar{b} \overset{\times}{\times} \bar{a}\bar{b} = 0$

$$(\bar{a}\bar{b} + \bar{c}\bar{d}) \overset{\times}{\times} (\bar{a}\bar{b} + \bar{c}\bar{d}) = 2(\bar{a} \times \bar{c})(\bar{b} \times \bar{d})$$

$$(\qquad) : (\bar{a}\bar{b} + \bar{c}\bar{d})$$

$$\overline{\overline{A}} \overset{\times}{\times} \overline{\overline{B}} =$$

$$[(\overline{\overline{A}}:\overline{\overline{I}})(\overline{\overline{B}}:\overline{\overline{I}}) - \overline{\overline{A}}:\overline{\overline{B}}^T]\overline{\overline{I}} - (\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{B}}^T - (\overline{\overline{B}}:\overline{\overline{I}})\overline{\overline{A}}^T + (\overline{\overline{A}}\cdot\overline{\overline{B}} + \overline{\overline{B}}\cdot\overline{\overline{A}})^T. \quad (2.51)$$

$$\overline{\overline{A}} \overset{\times}{\times} \overline{\overline{A}} = (\mathrm{tr}^2 \overline{\overline{A}} - \mathrm{tr}\, \overline{\overline{A}}^2)\overline{\overline{I}} - 2\,\mathrm{tr}\,\overline{\overline{A}}\; \overline{\overline{A}}^T + 2\,\overline{\overline{A}}^{2\,T}$$

$$\overline{\overline{A}}^{(2)\,T} = \frac{1}{2}(\overline{\overline{A}} \overset{\times}{\times} \overline{\overline{A}})^T = \underbrace{\frac{1}{2}(\mathrm{tr}^2 \overline{\overline{A}} - \mathrm{tr}\,\overline{\overline{A}}^2)}_{\mathrm{spm}\,\overline{\overline{A}}}\overline{\overline{I}} - \mathrm{tr}\,\overline{\overline{A}}\;\overline{\overline{A}} + \overline{\overline{A}}^2$$

$$\ne$$

$$\det \overline{\overline{A}}\; \overline{\overline{A}}^{-1}$$

$$\overline{\overline{A}}^3 - (\mathrm{tr}\,\overline{\overline{A}})\,\overline{\overline{A}}^2 + (\mathrm{spm}\,\overline{\overline{A}})\,\overline{\overline{A}} - (\det \overline{\overline{A}})\,\overline{\overline{I}} = 0$$

CAYLEY - HAMILTON

$$\bar{\bar{\varepsilon}} = \varepsilon_0 \bar{\bar{\varepsilon}}_r$$

$$\mu_0 \bar{\bar{I}}$$

$$\bar{E}(\bar{r}) = \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}}$$

$$\bar{H}(\bar{r}) = \bar{H}_0 e^{-j\bar{k}\cdot\bar{r}}$$

$$\bar{k} = \underbrace{\omega\sqrt{\mu_0\varepsilon_0}}_{k_0}\, n\, \bar{u}$$

$$\nabla e^{-j\bar{k}\cdot\bar{r}} = -j\bar{k}\, e^{-j\bar{k}\cdot\bar{r}}$$

$$\nabla \times \bar{E} = -j\omega\mu_0 \bar{H} = -j\bar{k}\times E$$

$$\bar{k}\times E = \omega\mu_0 \bar{H}$$

$$\nabla \times \bar{H} = j\omega\bar{D} = j\omega\varepsilon_0 \bar{\bar{\varepsilon}}_r\cdot\bar{E} = -j\bar{k}\times\bar{H}$$

$$\bar{k}\times\bar{H} = -\omega\varepsilon_0\bar{\bar{\varepsilon}}_r\cdot E$$

$$\bar{k}\times(\bar{k}\times E) = \omega\mu_0\left(-\omega\varepsilon_0\bar{\bar{\varepsilon}}_r\cdot \bar{E}\right) = -k_0^2\,\bar{\bar{\varepsilon}}_r\cdot\bar{E}$$

$$\left[k_0^2\bar{\bar{\varepsilon}}_r - \underbrace{\bar{k}\bar{k}\overset{\times}{\times}\bar{\bar{I}}}_{\bar{k}\cdot\bar{k}\,\bar{\bar{I}} - \bar{k}\bar{k}}\right]\cdot\bar{E} = 0$$

$$\bar{\bar{D}}\cdot E = 0$$

$$\det \bar{\bar{D}} = 0$$

ISOTROPIC: $\bar{\bar{\varepsilon}}_r = \varepsilon_r\bar{\bar{I}}$

$$\bar{k} = k_0 n\bar{u}$$

$$\bar{\bar{D}} = k_0^2\varepsilon_r\bar{\bar{I}} - \bar{k}\bar{k}\overset{\times}{\times}\bar{\bar{I}}$$

$$= k_0^2\varepsilon_r\bar{\bar{I}} - \bar{k}\cdot\bar{k}\,\bar{\bar{I}} + \bar{k}\bar{k}$$

$$= k_0^2\varepsilon_r\bar{\bar{I}} - k_0^2n^2\underbrace{\bar{u}\cdot\bar{u}}_{1}\,\bar{\bar{I}} + k_0^2n^2\bar{u}\bar{u}$$

$$\det\bar{\bar{D}} = 0 \quad\Rightarrow\quad \det\left((\varepsilon_r - n^2)\bar{\bar{I}} + n^2\bar{u}\bar{u}\right) = 0$$

$$\det \bar{\bar{D}} = 0 \quad \Rightarrow \quad \det\left((\varepsilon_r - n^2)\bar{\bar{I}} + n^2 \bar{u}\bar{u} \right) = 0$$

$$\bar{\bar{I}} = \bar{u}\bar{u} + \bar{\bar{I}}_t \qquad \det\left(\varepsilon_r \bar{\bar{I}} - n^2 \bar{\bar{I}}_t \right) = 0$$

$$\det \begin{pmatrix} \varepsilon_r & 0 & 0 \\ 0 & \varepsilon_r - n^2 & 0 \\ 0 & 0 & \varepsilon_r - n^2 \end{pmatrix} = \varepsilon_r (\varepsilon_r - n^2)^2 = 0$$

$$\nabla \cdot \bar{D} = 0 \quad \Rightarrow \quad -j\bar{k} \cdot \varepsilon \bar{E}$$

$$\bar{k} \times (\bar{k} \times \bar{\bar{I}}) = \bar{k}\bar{k} - \bar{k}\cdot\bar{k}\,\bar{\bar{I}}$$

$$= -\bar{k}\bar{k} \overset{\times}{\times} \bar{\bar{I}}$$

$$\bar{u}_z \times (\bar{u}_z \times \bar{\bar{I}}) = \bar{u}_z \times (\bar{u}_y \bar{u}_x - \bar{u}_x \bar{u}_y) = -\bar{u}_x\bar{u}_x - \bar{u}_y\bar{u}_y$$

$$= -\bar{\bar{I}} + \bar{u}_z\bar{u}_z$$

$$\bar{u}_z\bar{u}_z \overset{\times}{\times} \bar{\bar{I}} = \bar{u}_z\bar{u}_z \overset{\times}{\times} (\bar{u}_x\bar{u}_x + \bar{u}_y\bar{u}_y + \bar{u}_z\bar{u}_z) = \bar{u}_x\bar{u}_x + \bar{u}_y\bar{u}_y$$