Aalto University
School of Science

Decision making and problem solving – Lecture 1

- Decision trees
- Elicitation of probabilities

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Motivation

- You have just revised some key concepts of probability calculus
  - Conditional probability
  - Law of total probability
  - Bayes' rule

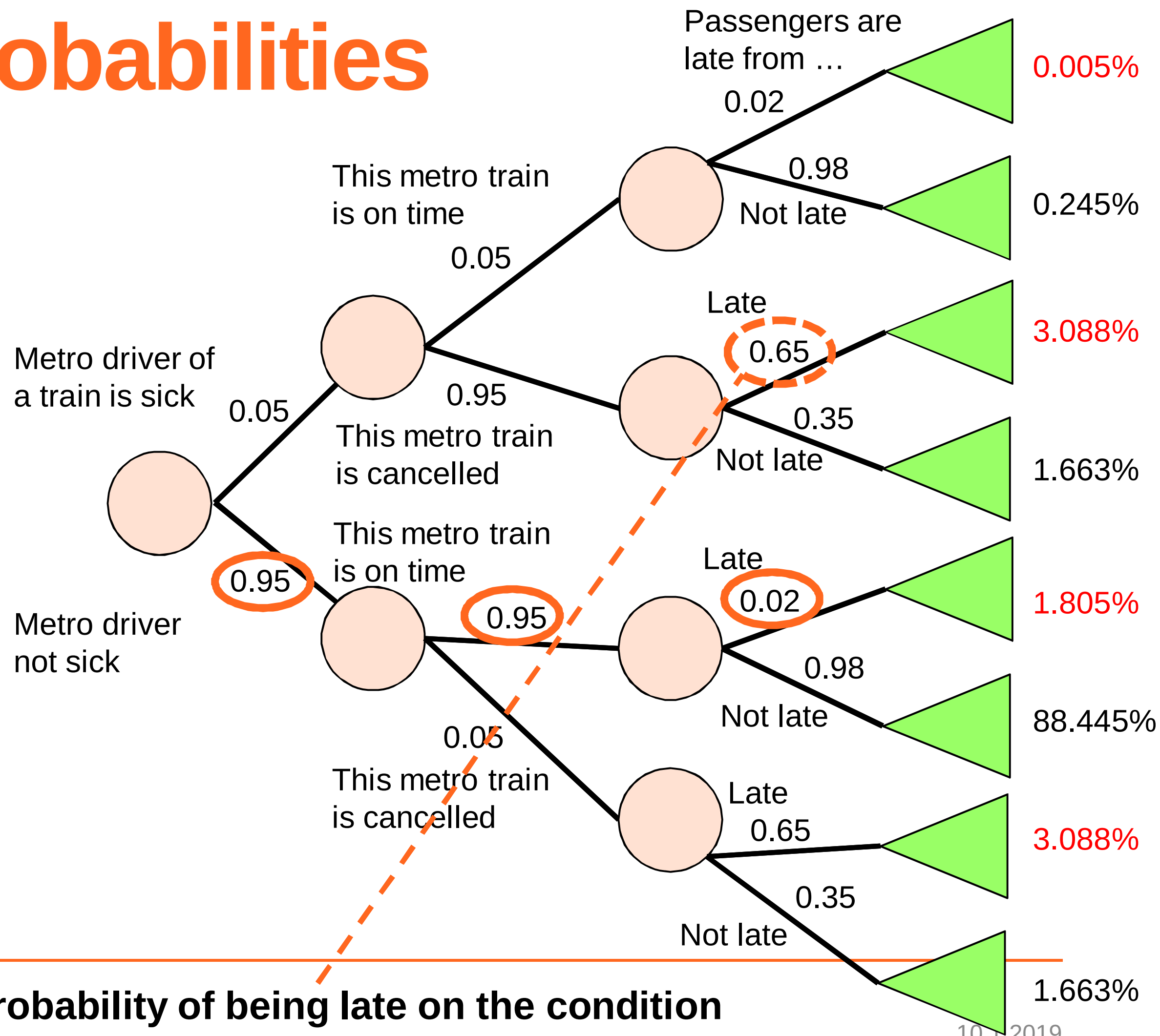
- This time:
  - How to build a probability-based model to support decision-making under uncertainty?
  - How to elicitate the probabilities needed for these models?

Why probabilities for modeling uncertainty?

- Decisions are often made under uncertainty
  - "How many train drivers should be trained, when future traffic is uncertain?"
  - "Should I buy an old or a new car, given that I only need an operational one and want to minimize costs = purchase price, maintenance & repair costs, selling price, etc.?"
  - "Should I buy my first my apartment now or postpone the decision, given that future interest rates, mortgage costs, personal income and apartment prices are uncertain?"
- Probability theory dominates the modeling of uncertainty in decision analysis
  - Well established rules for computations, understandable
  - Other models (e.g., evidence theory, fuzzy sets) exist, too

Conditional probabilities

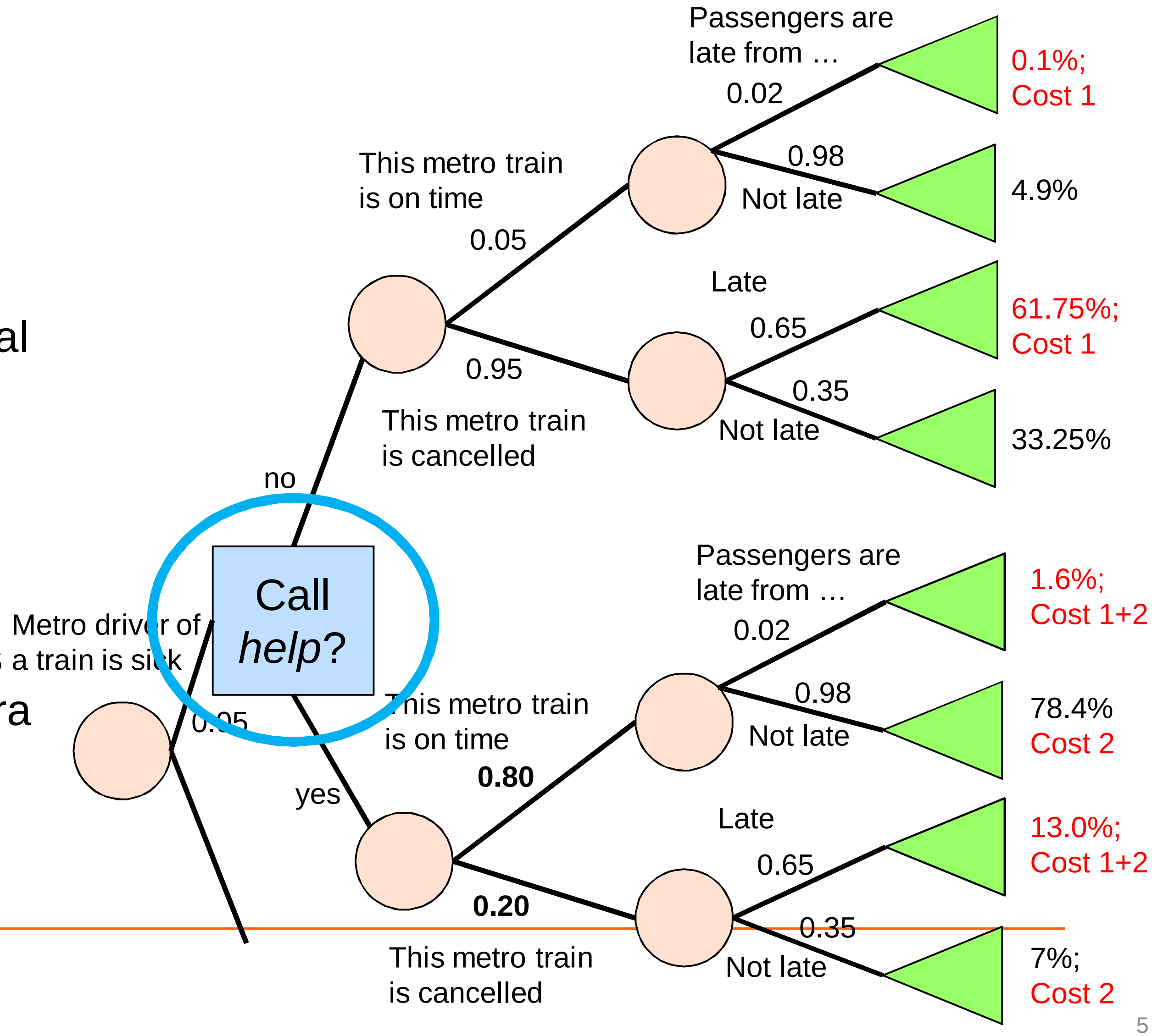
- The probabilities of sequential, mutually exclusive and collectively exhaustive events can be represented if form of a tree
- The probability of a sequence of events is obtained my multiplying the probabilities on the path
  - 0.95 x 0.95 x 0.02 = 1.805 %
- The total probability of being late is 7.985 %



The probability of being late on the condition that the train is cancelled (and the original driver is sick)

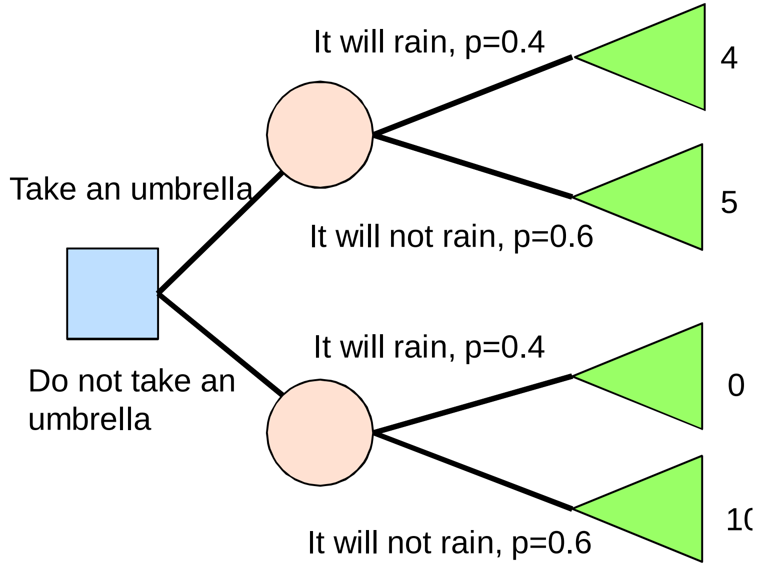
What if…

- We are interested in financial aspects and assume that being late results in unwanted financial consequences (Cost 1)?
  - numerical outcomes for states
- You had a possibility to influence the probability **p(this metro train is on time | metro driver of this train is sick)** by use of extra personnel (*help*) at a cost (Cost 2)?
  - Now the event probabilities depend on your decision



Decision trees

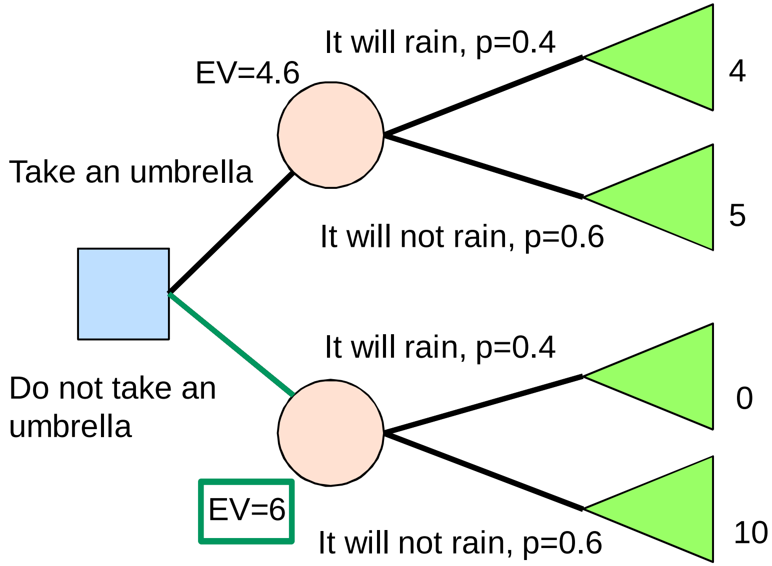
- Decision-making under uncertainty can be modeled by a decision tree
- Decision trees consist of
  - Decision nodes (squares) – DM can choose which arc to follow
  - Chance nodes (circles; cf. states of nature) – chance represented by probabilities dictates which arc will be followed (states of nature). The probabilities following a chance node must sum up to 1
  - Consequence nodes (triangles; resulting consequences) – at the end of the tree; describe the consequence (e.g., profit, cost, revenue, utility) of following the path leading to this node
- Decisions and chance events are displayed **in a logical temporal sequence from left to right**
  - Only chance nodes whose results are known can precede a decision node
- Each chain of decisions and chance events represents a possible outcome



Solving a decision tree

- A decision tree is solved by starting from the leaves (consequence nodes) and going backward toward the root:
  - At each chance node: compute the expected value at the node
  - At each decision node: select the arc with the highest expected value

- The optimal strategy consists of the arcs selected at decision nodes

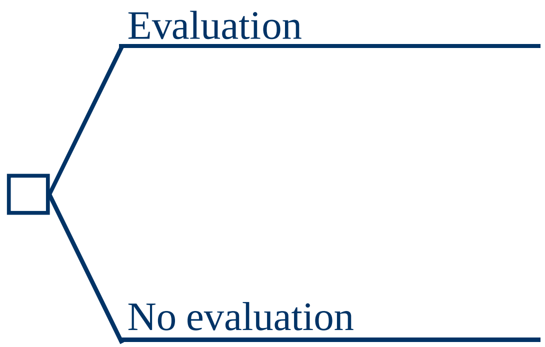


Example: Decision tree (1/12)

- Your uncle is going to buy a tractor. He has two alternatives:
  1. A new tractor (17 000 €)
  2. A used tractor (14 000 €)
- The engine of the old tractor may be defect, which is hard to ascertain. Your uncle estimates a 15 % probability for the defect.
- If the engine is defect, he has to buy a new tractor and gets 2000 € for the old one.
- Before buying the tractor, your uncle can take the old tractor to a garage for an evaluation, which costs 1 500 €.
  - If the engine is OK, the garage can confirm it without exception.
  - If the engine is defect, there is a 20 % chance that the garage does not notice it.
- Your uncle maximizes expected monetary value

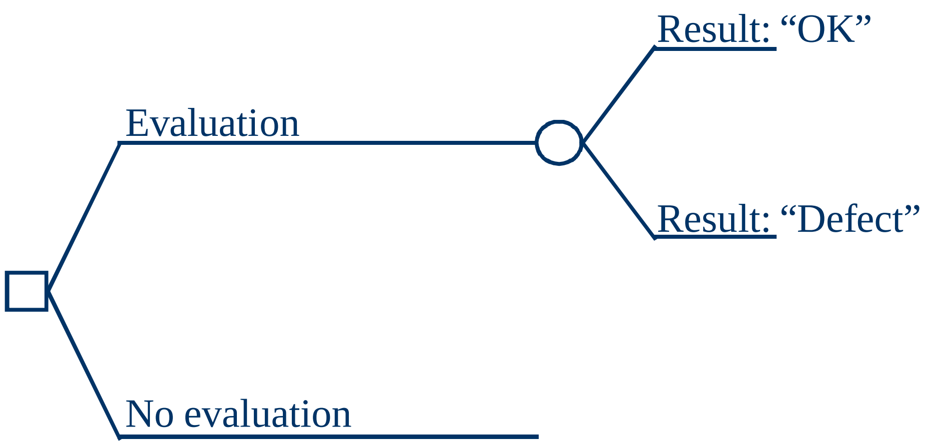
Example: Decision tree (2/12)

- Before making the buying decision and before you get to know the result of any uncertain event, you must **decide** upon taking the old tractor to a garage for an **evaluation**.
- The decision node 'evaluation' is placed leftmost in the tree



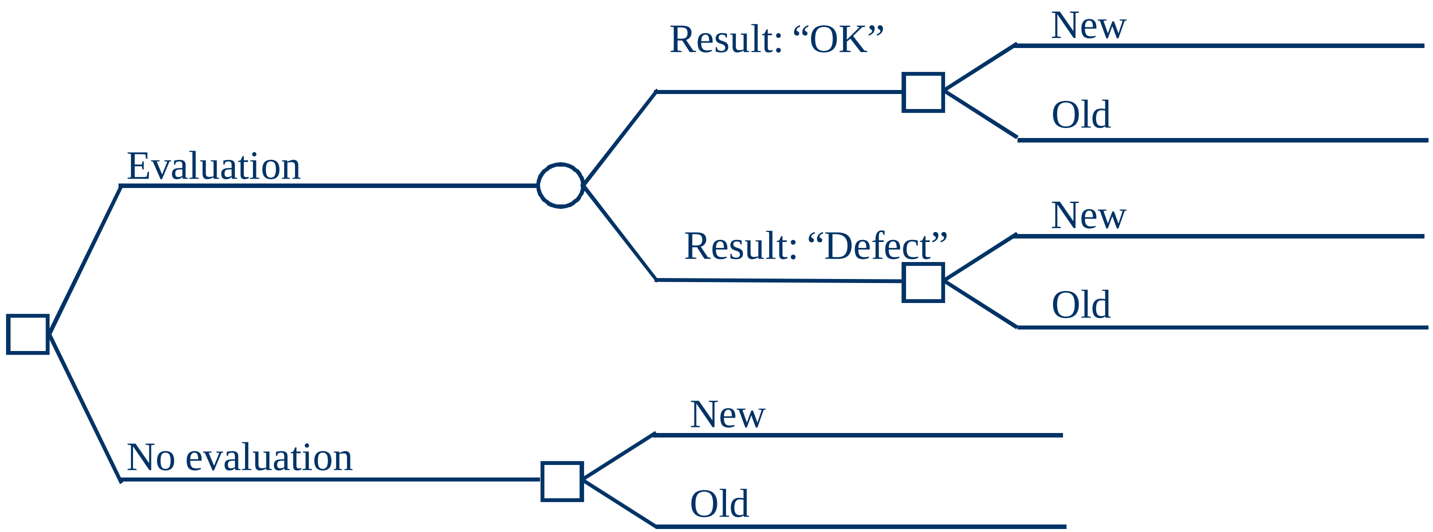
Example: Decision tree (3/12)

- If the old tractor is evaluated, your uncle receives the **results of the evaluation**



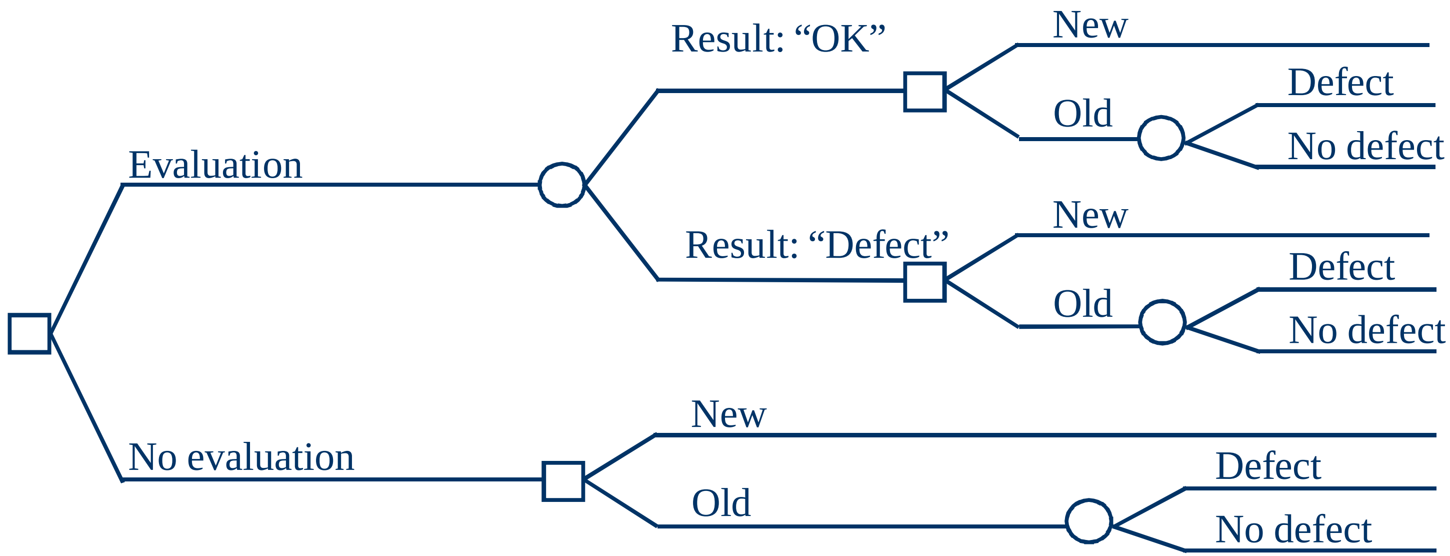
Example: Decision tree (4/12)

- The next step is to **decide which tractor** to buy



Example: Decision tree (5/12)

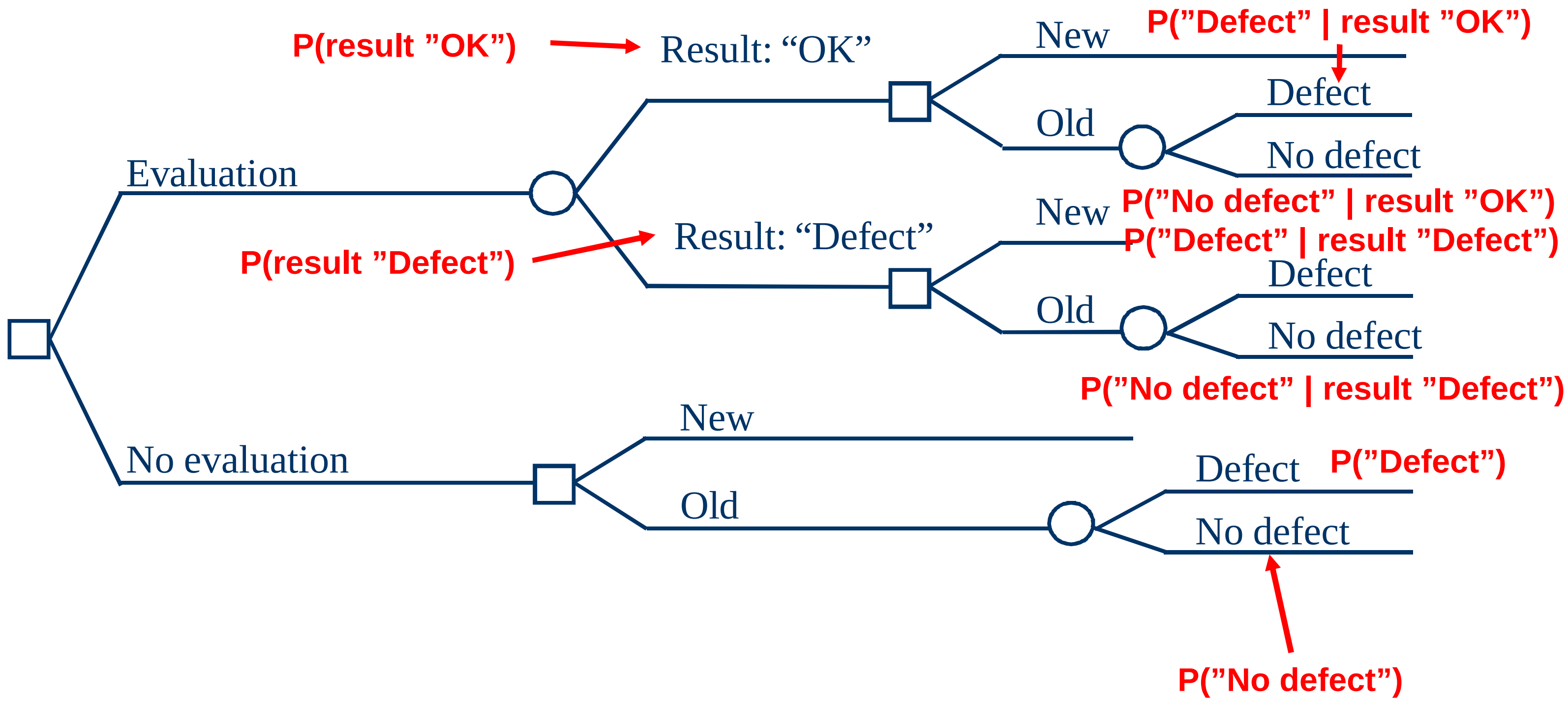
- …But the engine of the old tractor can be defect



- Now all chance nodes and decisions are in chronological order such that in each node, we can follow the path to the left to find out what we know

Example: Decision tree (6/12)

- We next need the probabilities for all outcomes of the chance nodes



Remember: Law of total probability

- If $E_1,\ldots,E_n$ are mutually exclusive and $A = \cup_i E_i$, then

$$P(A)=P(A|E_1)P(E_1)+\ldots+P(A|E_n)P(E_n)$$

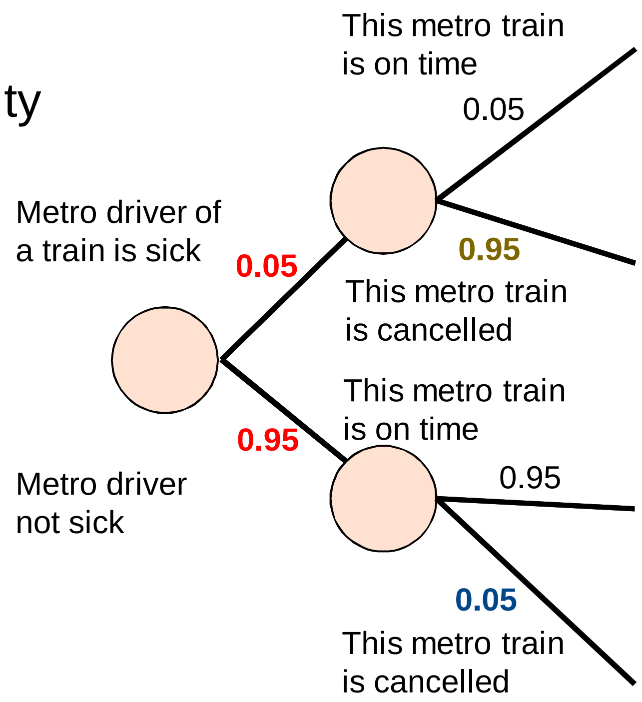
- Most frequent use of this law:
  – Probabilities $P(A|B)$, $P(A|B^c)$, and $P(B)$ are known
  – These can be used to compute $P(A)=P(A|B)P(B)+P(A|B^c)P(B^c)$

Remember: Bayes' rule

- **Bayes' rule:** $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$
- Follows from
  1. The definition of conditional probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$, $P(B|A) = \frac{P(B \cap A)}{P(A)}$,
  2. Commutative laws: $P(B \cap A) = P(A \cap B)$.

Example: Bayes' rule

A metro train is cancelled (event *C*) and we have not had the opportunity to *call help*. What is the probability that the driver originally allocated to drive the train is sick (event *S*)? = What is $P(S|C)$?



Solution:

- $P(S)=0.05$, $P(S^c)=0.95$, $P(C|S)=0.95$, $P(C|S^c)=0.05$

Law of total probability: $\mathbf{P(C)}=P(C|S)P(S)+P(C|S^c)\,P(S^c)=$
$\mathbf{0.95 \times 0.05 + 0.05 \times 0.95 = 0.095}$

Bayes' rule: $P(S|C) = \frac{P(C|S)P(S)}{P(C)} = \frac{0.95 \cdot 0.05}{0.095} = 50\%$

Example: Decision tree (7/12)

- Solve all probabilities. You know that
  - ”Your uncle estimates a 15 % probability for the defect.” => P(Defect)=0.15
  - “If the engine is OK, the garage can confirm it without exception.” => P(result “OK” | No defect)=1
  - “If the engine is defect, there is a 20 % chance that the garage does not notice it.” => P(result “OK” | Defect)=0.20

$$P(\text{result "OK"}) = P(\text{result "OK"} \mid \text{No defect}) \cdot P(\text{No defect}) + P(\text{result "OK"} \mid \text{Defect}) \cdot P(\text{Defect})$$

$$= 1.0 \cdot 0.85 + 0.20 \cdot 0.15 = 0.88$$

$$P(\text{result "defect"}) = 1 - P(\text{result "OK"}) = 0.12$$

$$P(\text{Defect} \mid \text{result "OK"}) = \frac{P(\text{result "OK"} \mid \text{Defect}) \cdot P(\text{Defect})}{P(\text{result "OK"})} = \frac{0.20 \cdot 0.15}{0.88} \approx 0.034$$

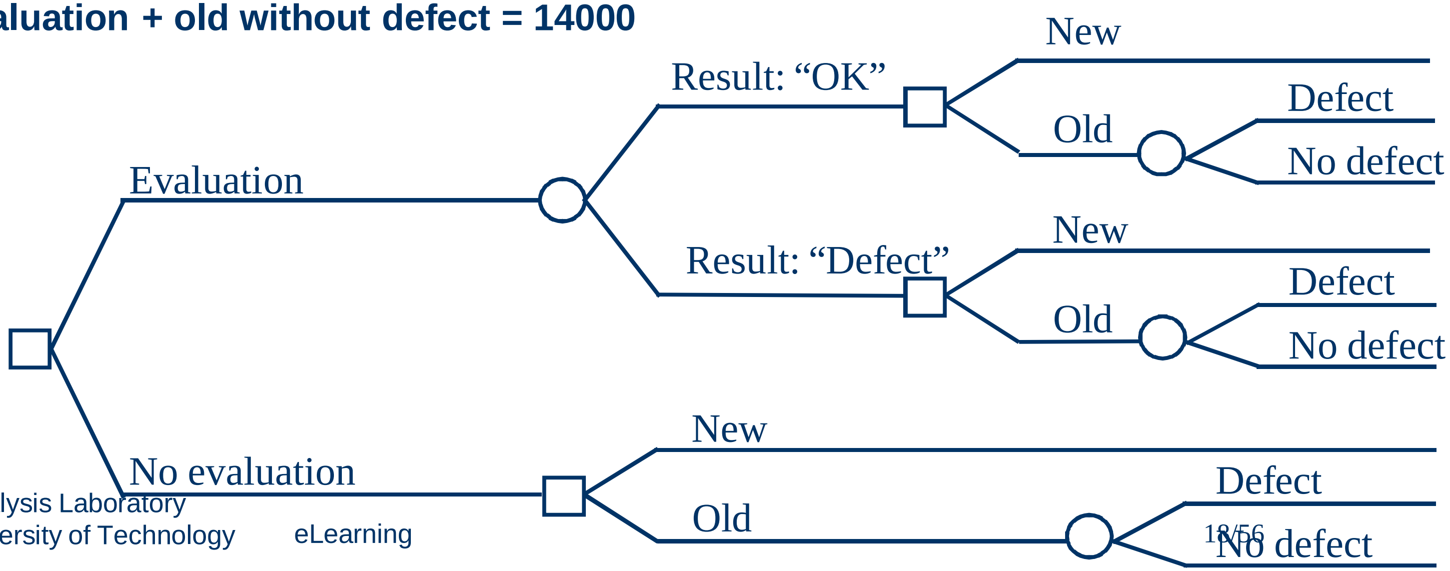
$$P(\text{No defect} \mid \text{result "OK"}) = 1 - 0.034 = 0.966$$

$$P(\text{Defect} \mid \text{result "defect"}) = \frac{P(\text{result "defect"} \mid \text{Defect}) \cdot P(\text{Defect})}{P(\text{result "defect"})} = \frac{0.80 \cdot 0.15}{0.12} = 1.00$$

$$P(\text{No Defect} \mid \text{result "defect"}) = 1 - 1 = 0$$

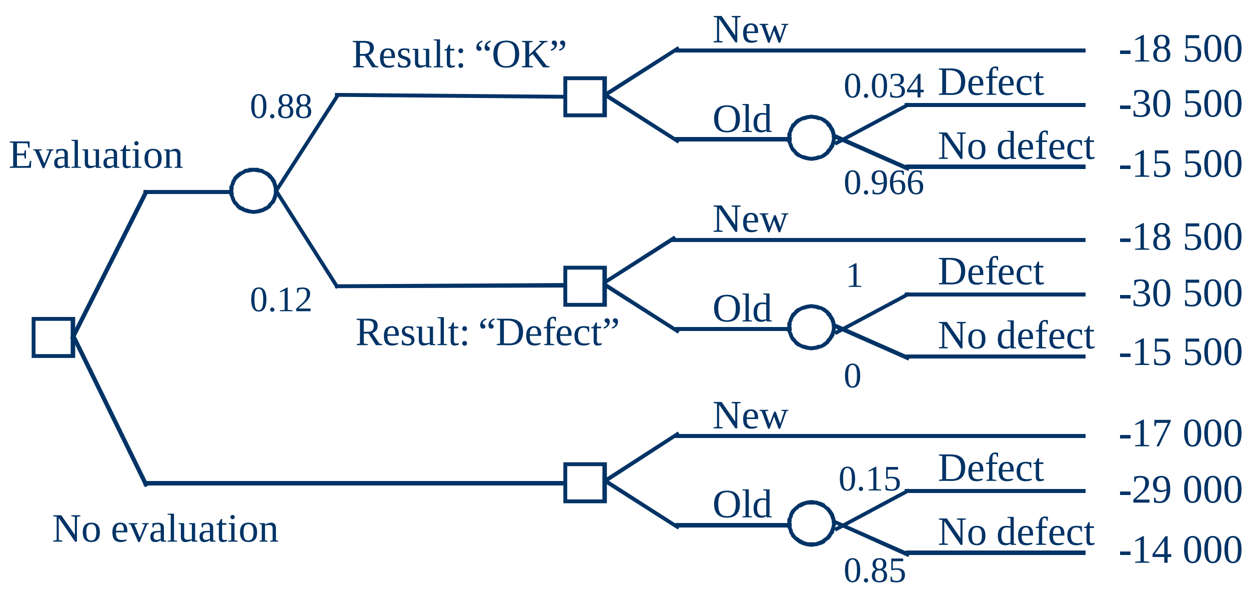
Example: Decision tree (8/12)

- Compute monetary values for each **end node**
  - **Evaluation + new = 1500 + 17000 = 18500**
  - **Evaluation + old with defect = 1500 + 14000 – 2000 + 17000 = 30500**
  - **Evaluation + old without defect = 1500 + 14000 = 15500**
  - **No evaluation + new = 17000**
  - **No evaluation + old with defect = 14000 – 2000 + 17000 = 29000**
  - **No evaluation + old without defect = 14000**



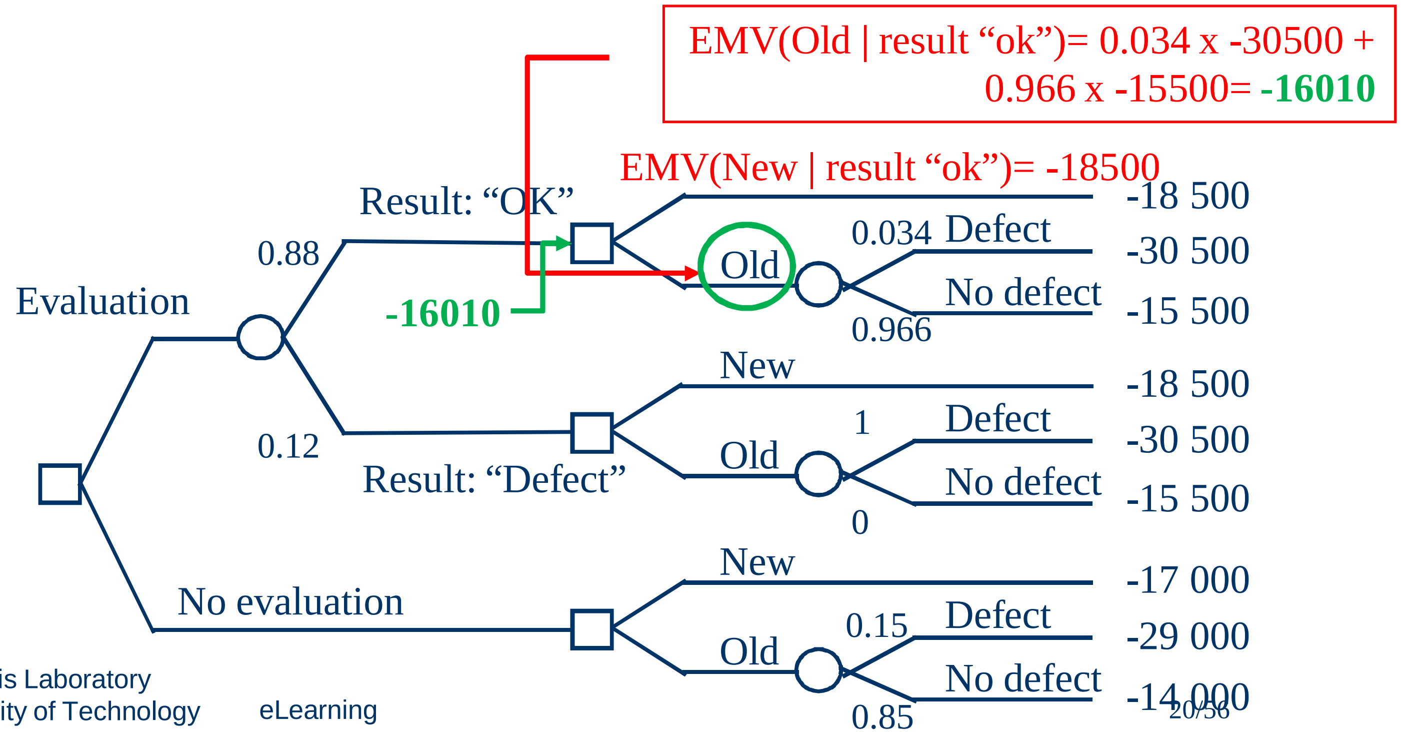
Example: Decision tree (9/12)

- We now have a decision tree presentation of the problem



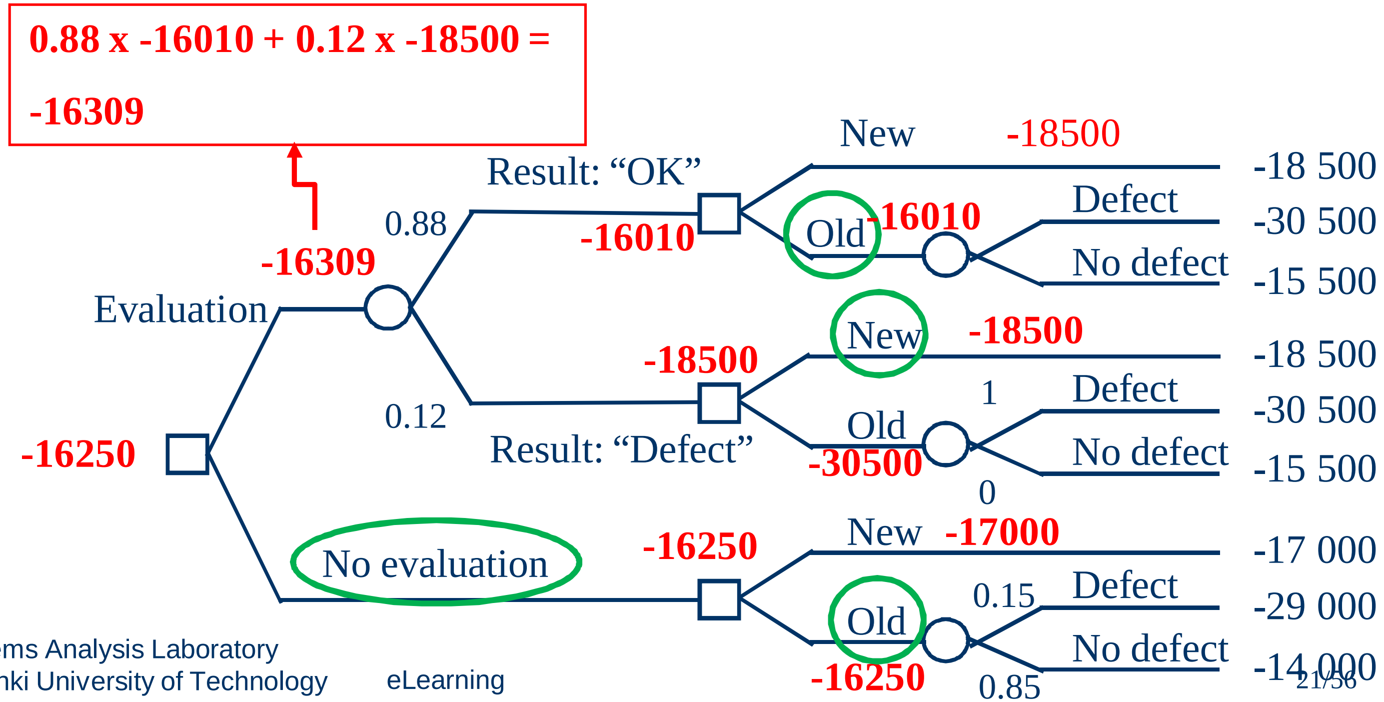
Example: Decision tree (10/12)

- **Starting from the right**, compute expected monetary values for each decision
- Place the value of the better decision to the decision node



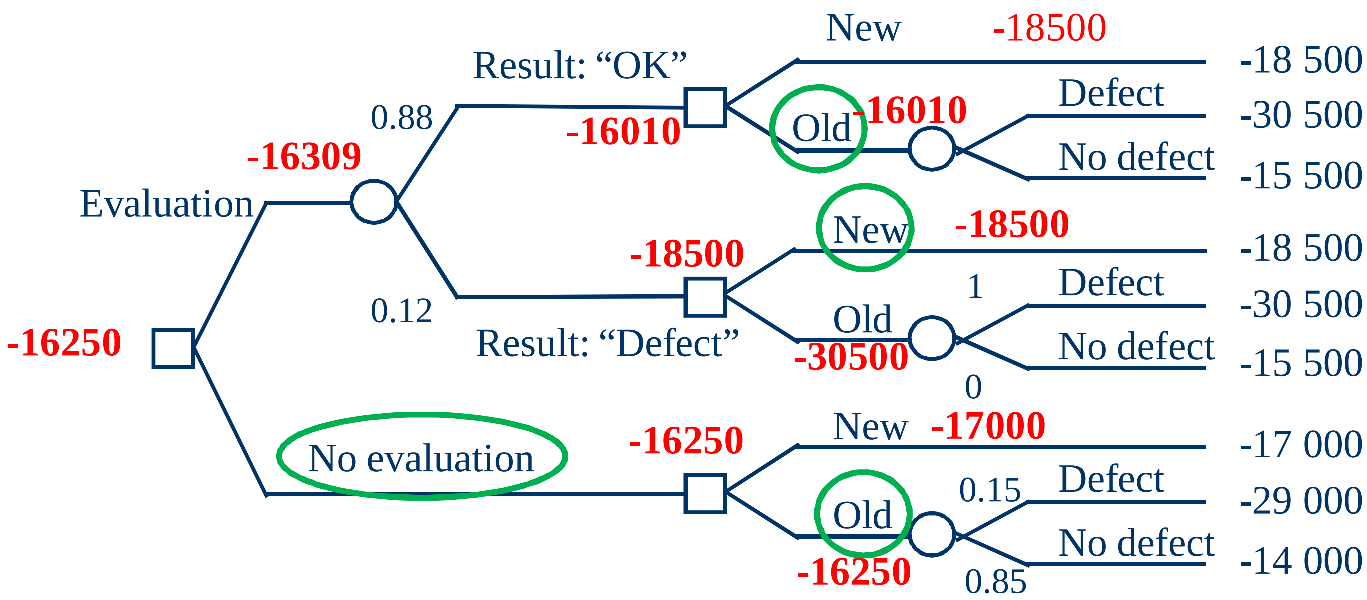
Example: Decision tree (11/12)

- Starting from the right, compute expected monetary values for each decision
- Place the value of the better decision to the decision node



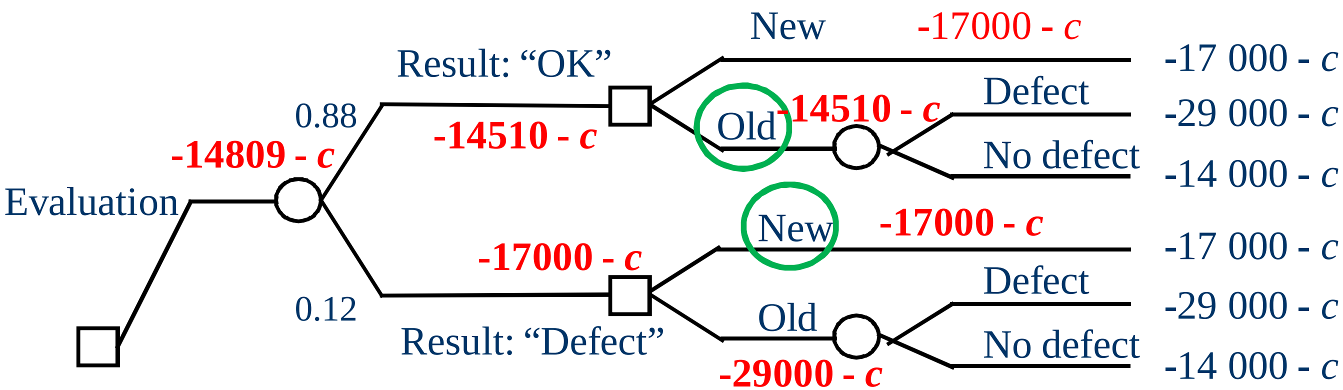
Example: Decision tree (12/12)

- The optimal solution is to buy the old tractor without evaluating it



… How much should we pay for the *sample information* by the garage?

- ❑ The expected monetary value was higher without evaluating the old tractor
- ❑ Determine evaluation cost *c* so that you are **indifferent** between
  1. Not taking the old tractor for an evaluation (EMV = -16250€)
  2. Taking the old tractor for an evaluation



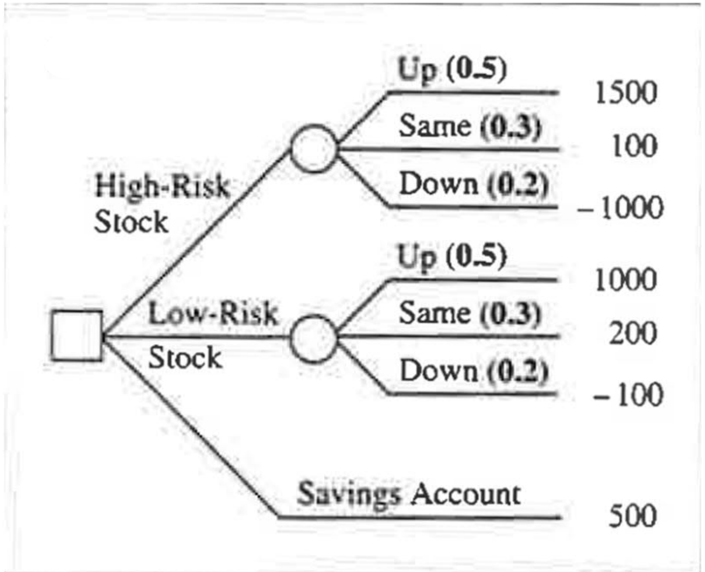
- ❑ Indifference, when EMVs equal: -16250 = -14809 – c => c = 1441€
  - ❑ Expected value of sample information = Expected value with sample information – Expected value without sample information = -14809€ - (-16250€) = 1441€

Example: expected value of *perfect information*

- You are considering between three investment alternatives: high-risk stock, low-risk stock, and savings account
- Savings account: certain payoff of 500€
- Stocks:
  - 200€ brokerage fee
  - Payoffs depend on market conditions

	Up	Same	Down
High-risk	1700	300	-800
Low-risk	1200	400	100
Probability	0.5	0.3	0.1

Decision tree



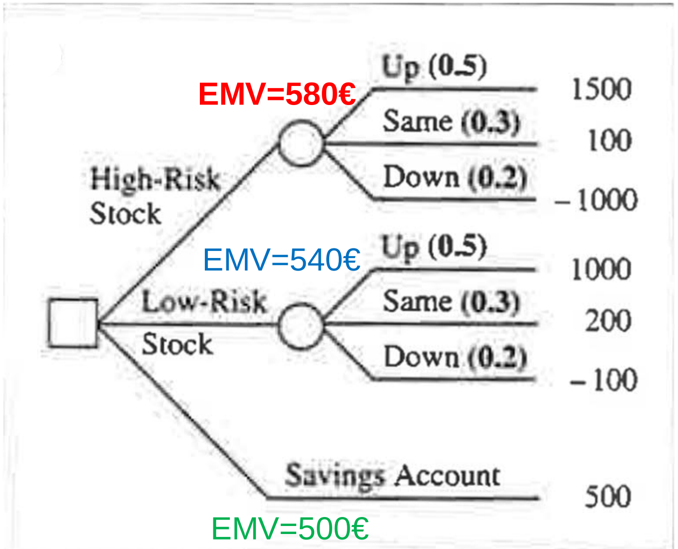
Source: Clemen, R.T. (1996): Making Hard Decisions: An Introduction to Decision Analysis, 2nd edition, Duxbury Press, Belmont.

Example: investing in the stock market

- The expected monetary values (EMVs) for the different alternatives are
  - HRS: 0.5·1500+0.3·100-0.2·1000=580
  - LRS: 0.5·1000+0.3·200-0.2·100=540
  - Savings Account: 500

→ It is optimal* to invest in **high-risk** stock

Decision tree



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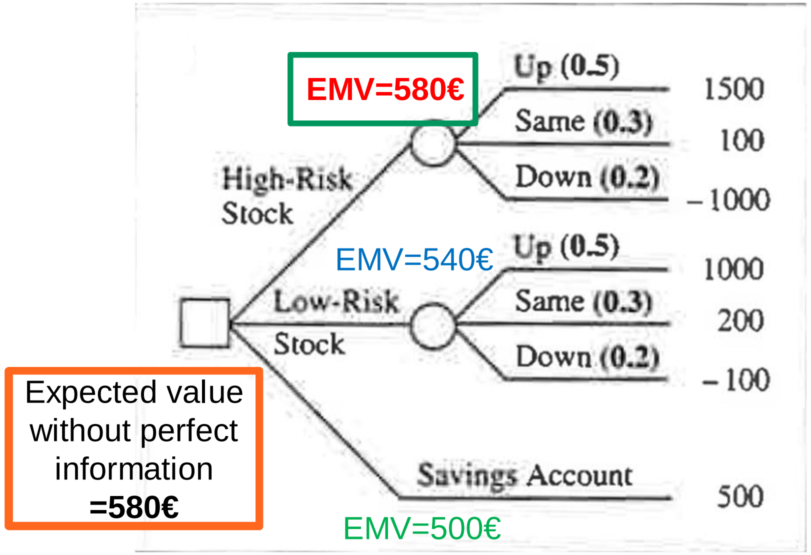
* Assuming you are *risk-neutral* !!! – *risk attitudes* discussed later on this course

Expected value of perfect information

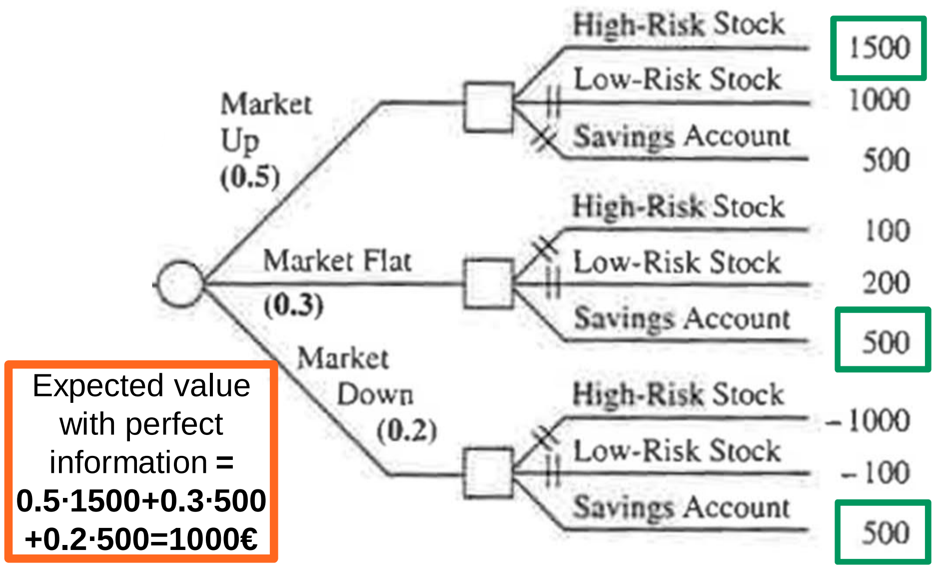
- How much could the expected value be <u>expected</u> to increase, if
  - Additional information about the uncertainties was received before the decision
  - The decision would be made according to this information?
    - Note: this analysis is done <u>before</u> any information is obtained
- Perfect information: certain information about how the uncertainties are resolved – ”if we could choose after we know the state of the world”
  - <u>Expected value of perfect information</u> = Expected value with perfect information – Expected value without perfect information
- **Expected value of perfect information** is computed through a reversed decision tree in which <u>all chance nodes precede all decision nodes</u>

Expected value of perfect information

Decision tree



Reversed decision tree: you know the state of the world when making the decision(s)



Expected value of perfect information

= 1000€ - 580€ = 420€

Probability assessment

- Use a few minutes to answer ten probability assessment questions
  - You have either questionnaire sheet A or B

- Do not communicate with others

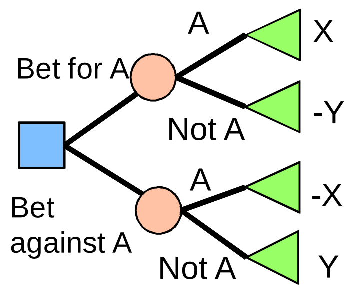
- Do not look up the answers on the internet

Estimation of probabilities

- How to obtain the probabilities needed in decision models?
  1. If possible, use objective data
  2. If objective data is not available, obtain subjective probability estimates from experts through
     - Betting approach
     - Reference lottery
     - Direct judgement

Estimation of probabilities: Betting approach

- **Goal:** to estimate the probability of event A
  - E.g., A="GDP growth is above 3% next year" or A="Sweden will join NATO within the next five years"

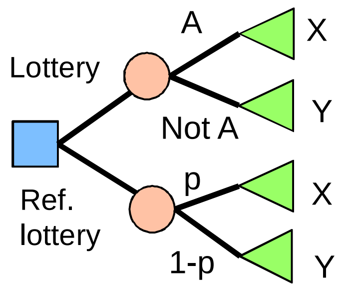


- **Betting approach:**
  - Bet for A: win X € if A happens, lose Y € if not
    - Expected monetary value $X \cdot P(A) - Y \cdot [1 - P(A)]$
  - Bet against A: lose X € if A happens, win Y € if not
    - Expected monetary value $-X \cdot P(A) + Y \cdot [1 - P(A)]$
  - Adjust X and Y until the respondent is indifferent between betting for or against A
  - Assuming risk-neutrality(*, the expected monetary values of betting for or against A must be equal:

$$X \cdot P(A) - Y \cdot [1 - P(A)] = -X \cdot P(A) + Y \cdot [1 - P(A)] \Rightarrow P(A) = \frac{Y}{X + Y}$$

*) A strong assumption

Estimation of probabilities: Reference lottery



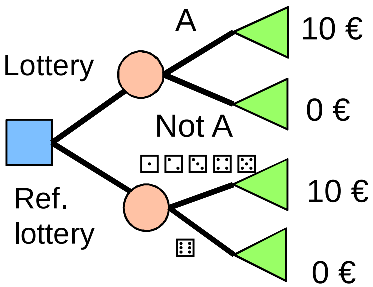
- Lottery:
  - Win X if A happens
  - Win Y if A does not happen
  - X is preferred to Y
- Reference lottery:
  - Win X with (known) probability p
  - Win Y with (known) probability (1-p)
  - Probability p can be visualized with, e.g., a wheel of fortune
- Adjust *p* until the respondent is **indifferent** between the **two lotteries**:

$$X \cdot P(A) + Y \cdot [1 - P(A)] = X \cdot p + Y \cdot [1 - p] \Rightarrow P(A) = p$$

- Here, the respondent's risk attitude does not affect the results (shown later)

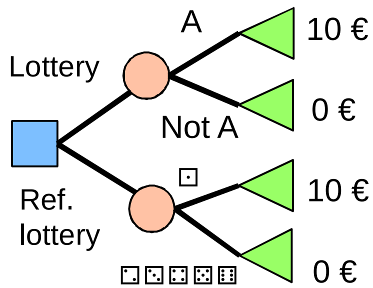
Reference lottery: example

- **Event A:** ”HIFK wins Jokerit”



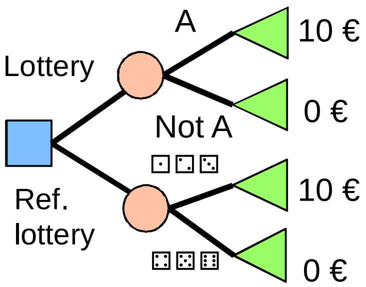
The respondent chooses the reference lottery:

$$10 \cdot P(A) < 10 \cdot \frac{5}{6}$$



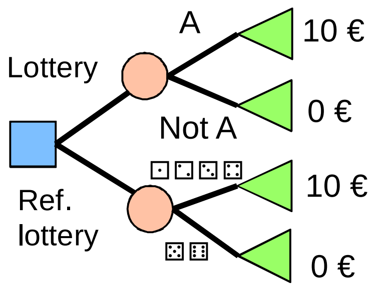
The respondent chooses the lottery:

$$10 \cdot P(A) > 10 \cdot \frac{1}{6}$$



Chooses the lottery:

$$P(A) > \frac{1}{2}$$



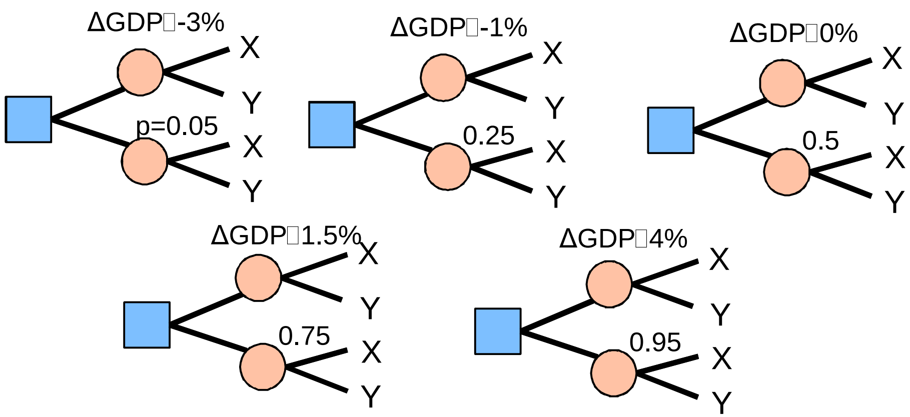
Chooses the reference lottery:

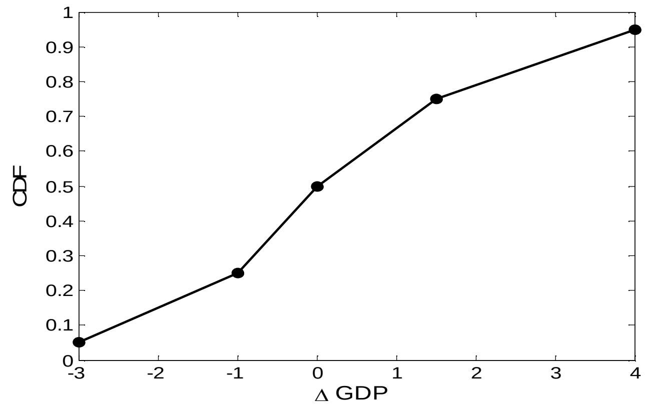
$$P(A) < \frac{2}{3}$$

These four answers revealed to probability estimate of A to be in (0.5, 0.67). Further questions should reveal the respondent’s estimate for P(A)

Estimation of continuous probability distributions

- A continuous distribution can be approximated by estimating several event probabilities (X is preferred to Y)
- Example:
  - Goal: to assess the distribution of the change in GDP (ΔGDP) in Finland next year
  - Means: elicitation of probability p for five different reference lotteries





Estimation of continuous probability distributions

- Often experts assess the descriptive statistics of the distribution directly, e.g.,
  - The feasible range (min, max)
  - Median f_{50} (i.e., $P(X<f_{50})=0.5$)
  - Other quantiles (e.g., 5%, 25%, 75%, 95%)
- In the previous example:
  - "The 5% and 95% quantiles are f_5 =-3% and f_{95} = 4%"
  - "The change in GDP is just as likely to be positive as it is to be negative"
  - "There is a 25% chance that the change in GDP is below -1%"
  - "There is a 25% chance that the change in GDP is above 1.5%"

Summary

- Decision trees are probability-based models to support decision-making under uncertainty
  - Which decision alternative should I choose?
  - How much would I be willing to pay for perfect information or (imperfect) sample information about how the uncertainties are resolved?

- Subjective probability assessments often required
  - Probability elicitation techniques require some effort

Aalto University
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Decision making and problem solving – Lecture 2

- Biases in probability assessment
- Expected Utility Theory (EUT)
- Assessment of utility functions

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Last time

- Decision trees are a visual and easy way to model decision-making problems, which involve uncertainties
  - Paths of decisions and random events
- Probabilities are used to model uncertainty
  - Data to estimate probabilities not necessarily available
- We often need subjective judgements to estimate probabilities

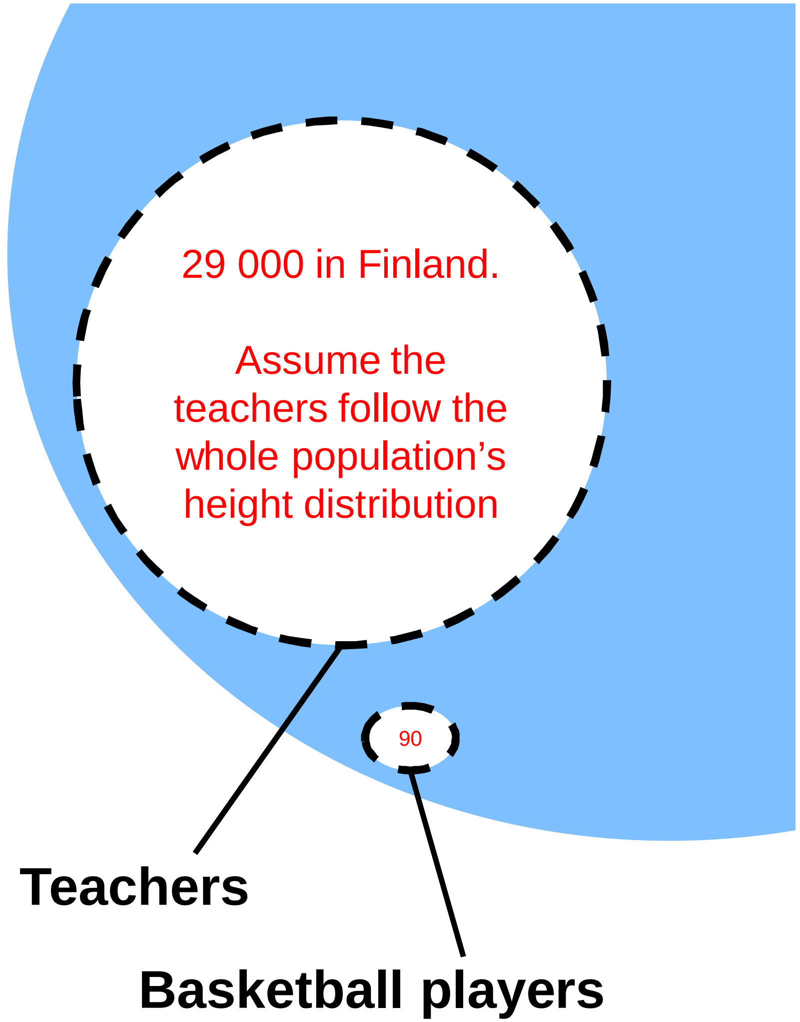
Biases in probability assessment

- Subjective judgements by both ”ordinary people” and ”experts” are prone to numerous biases
  - Cognitive bias: Systematic discrepancy between the ‘correct’ answer and the respondent’s actual answer
    - E.g., assessment of conditional probability differs from the correct value given by Bayes’ rule
  - Motivational biases: judgements are influenced by the desriability or undesirability of events
    - E.g., overoptimism about success probabilities
    - Strategic underestimation of failure probabilities

- Some biases can be easy, some difficult to correct

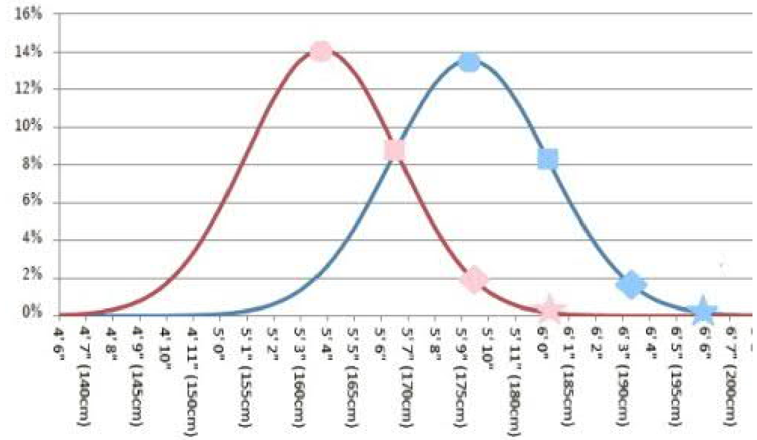
Representativeness bias (cognitive)

- If x fits the description of A well, then P(x∈A) is assumed to be large
- The 'base rate' of A in the population (i.e., the probability of A) is not taken into account
- Example: You see a very tall man in a bar. Is he more likely to be a professional basketball player or a teacher?



Representativeness bias

- ❑ What is ’very tall’?
  - ❑ 195 cm?
  - ❑ Assume all BB players are very tall
- ❑ Based on 30 min of googling[1], the share of Finnish men taller than 195 cm exceeds 0.3 %
- ❑ If BB players go the bar as often as teachers, **it is more probable that the very tall man is a teacher, if the share of very tall men exceeds 0.31%**
  - – 2018 students’ responses: 80% teacher, 20% basketball player
  - – Your responses: 82% teacher, 18 basketball player

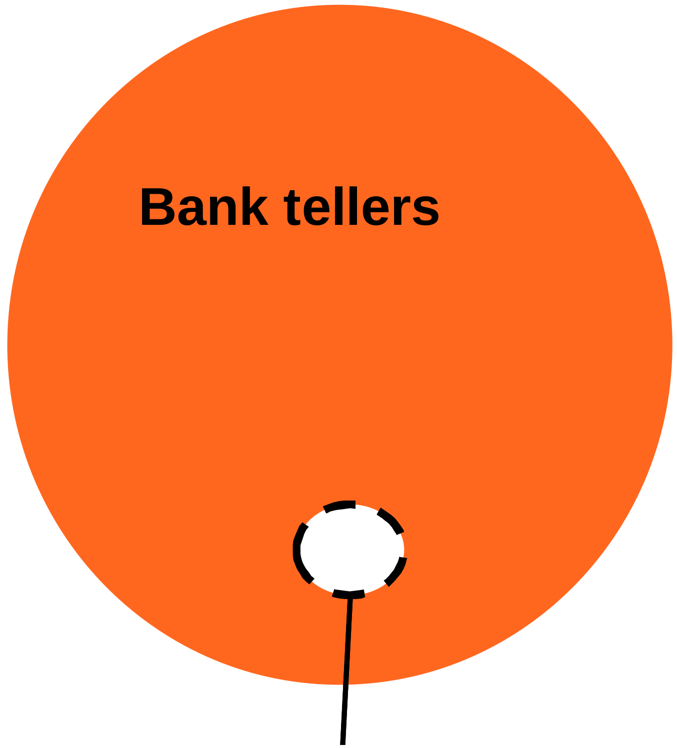


Height	Males					
	20–29 years	30–39 years	40–49 years	50–59 years	60–69 years	70–79 years
Percent under—						
4'10"	–	–	–	(B)	–	–
4'11"	–	–	–	(B)	(B)	–
5'	(B)	–	–	(B)	(B)	–
5'1"	(B)	(B)	(B)	(B)	[1] 0.4	(B)
5'2"	(B)	(B)	(B)	(B)	(B)	(B)
5'3"	(B)	[1] 3.1	[1] 1.9	(B)	[1] 2.3	(B)
5'4"	3.7	[1] 4.4	3.8	[1] 4.3	4.4	5.8
5'5"	7.2	6.7	5.6	7.6	7.8	12.8
5'6"	11.6	13.1	9.8	12.2	14.7	23.0
5'7"	20.6	19.6	19.4	18.6	23.7	35.1
5'8"	33.1	32.2	30.3	30.3	37.7	47.7
5'9"	42.2	45.4	40.4	41.2	50.2	60.3
5'10"	58.6	58.1	54.4	54.3	65.2	75.2
5'11"	70.7	69.4	69.6	70.0	75.0	85.8
6'	79.9	78.5	79.1	81.2	84.3	91.0
6'1"	89.0	89.0	87.4	91.6	93.6	94.9
6'2"	94.1	94.0	92.5	93.7	97.8	98.6
6'3"	98.3	95.8	97.7	96.6	99.9	100.0
6'4"	100.0	97.6	99.0	99.5	100.0	100.0
6'5"	100.0	99.4	99.4	99.6	100.0	100.0
6'6"	100.0	99.5	99.9	100.0	100.0	100.0

[1] note: with the lecturer’s skills

Representativeness bias

- Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in antinuclear demonstrations. Please check the most likely alternative:
  a. Linda is a bank teller.
  b. Linda is a bank teller and active in the feminist movement.
- Many choose b, although $b \subset a$ whereby $P(b)<P(a)$
  – 2018 students’ responses: 67% a, 33% b.
  – Your responses: 74% a, 26% b.



Conservativism bias (cognitive)

- When information about some uncertain event is obtained, people typically do not adjust their initial probability estimate about this event as much as they should based on Bayes' theorem.

- Example: Consider two bags X and Y. Bag X contains 30 white balls and 10 black balls, whereas bag Y contains 30 black balls and 10 white balls. Suppose that you select one of these bags at random, and randomly draw five balls one-by-one by replacing them in the bag after each draw. Suppose you get four white balls and one black. What is the probability that you selected bag X with mainly white balls?

- Typically people answer something between 70-80%. Yet, the correct probability is 27/28 ≈ 96%.
- **2018 students' responses**: mean response 59%. The majority (57%) answered 50%.
- **Your responses**: mean response 68%. Many (32%) answered 50%.

Representativeness and conservativism bias - debiasing

- Demonstrate the logic of joint and conditional probabilities and Bayes’ rule
- Split the task into an assessment of
  – The base rates for the event (i.e., prior probability)
    – E.g., what is the relative share of bank tellers in the population? What are the relative shares of teachers and pro basketball players?
  – The likelihood of the data, given the event (i.e., conditional probabilities)
    – E.g., what is the relative share of people active in the feminist movement? Is this share roughly the same among bank tellers as it is among the general population or higher/lower?
    – What is the likelihood that a male teacher is taller than 195cm? How about a pro basketball player?

Availability bias (cognitive)

- People assess the probability of an event by the ease with which instances or occurences of this event can be brought to mind.
- Example: In a typical sample of English text, is it more likely that a word starts with the letter K or that K is the third letter?
  - Most people think that words beginning with K are more likely, because it is easier to think of words that begin with "K” than words with "K" as the third letter
  - Yet, there are twice as many words with K as the third letter
  - 2018 students’ responses: 13% first letter, 87% third letter.
  - Your responses: 46% first letter, 54% third letter.
- Other examples:
  - Due to media coverage, the number of violent crimes such as child murders seems to have increased
  - Yet, compared to 2000’s, 18 times as many children were killed per capita in 1950’s and twice as many in 1990’s

Availability bias - debiasing

- Conduct probability training
- Provide counterexamples
- Provide statistics

- Based on empirical evidence, availability bias is **difficult to correct**

Anchoring bias (cognitive)

- When assessing probabilities, respondents sometimes consider some reference assessment
- Often, the respondent is *anchored* to the reference assessment
- Example: Is the percentage of African countries in the UN
  A. Greater or less than 65? What is the exact percentage?
     - Average answer: Less, 45%.
     - 2018 students' responses: Less, median 22%, mean 34%.
     - Your responses: Less, median 40%, mean 48%.
  B. Greater or less than 10? What is the exact percentage?
     - Average answer: Greater, 25%.
     - 2018 students' responses: Greater, median 23%, mean 27%.
     - Your responses: Greater, median 20%, mean 27%.

Based on Internet sources, the correct share is (maybe) 54/193 = **28%**

Anchoring bias - debiasing

- Avoid providing anchors
- Provide multiple and counteranchors
  - = if you have to provide an anchor, provide several which differ significantly from each other
- Use different experts who use different anchors

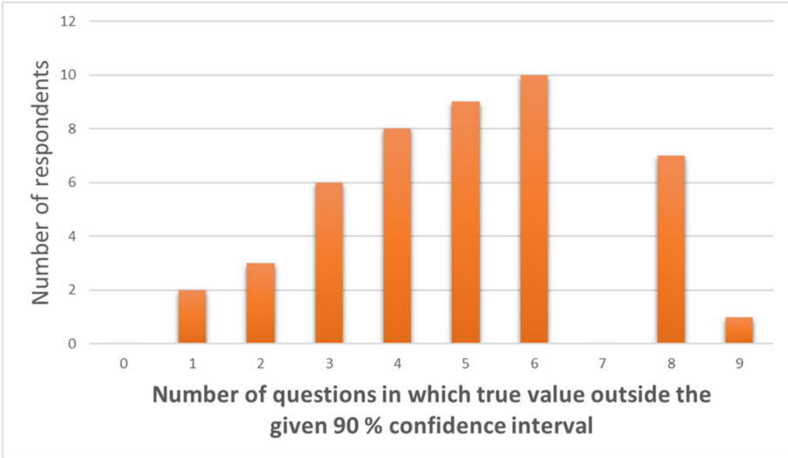
- Based on empirical evidence, anchoring bias is **difficult to correct**

Overconfidence (cognitive)

- People tend to assign overly narrow confidence intervals to their probability estimates

1. Martin Luther King's age at death 39 years
2. Length of the Nile River 6738 km
3. Number of Countries that are members of OPEC 13
4. Number of Books in the Old Testament 39
5. Diameter of the moon 3476 km
6. Weight of an empty Boeing 747 176900 kg
7. Year of Wolfgang Amadeus Mozart's birth 1756
8. Gestation period of an Asian elephant 645 days
9. Air distance from London to Tokyo 9590 km
10. Depth of the deepest known point in the oceans 11033 m

Your responses:



- If 3 or more of your intervals missed the correct value, you have demonstrated overconfidence
  - 89% of you did

Overconfidence - debiasing

- Provide probability training
- Start with extreme estimates (low and high)
- Use fixed values instead of fixed probability elicitations:
  - Do not say: ”Give a value x such that the probability for a change in GDP lower than x is 0.05”
  - Do say: ”What is the probability that the change in GDP is lower than -3%?”
- Based on empirical evidence, overconfidence is **difficult to correct**

Desirability / undesirability of events (motivational)

- People tend to believe that there is a less than 50 % probability that **negative outcomes** will occur compared with peers
  - I am less likely to develop a drinking problem
  - Your responses: 20% (25% in 2018) more likely, 34% (31%) less likely, 46% (44%) equally likely

- People tend to believe that there is a greater than 50 % probability that **positive outcomes** will occur compared with peers
  - I am more likely to become a homeowner / have a starting salary of more than 3,500€
  - Your responses on owning a home: 49% (44%) more likely, 12% (13%) less likely, 39% (44%) equally likely
  - Your responses on salary: 54% (38 %) more likely, 8% (19%) less likely, 38% (44%) equally likely

- People tend to underestimate the probability of negative outcomes and overestimate the probability of positive outcomes

Desirability / undesirability of events - debiasing

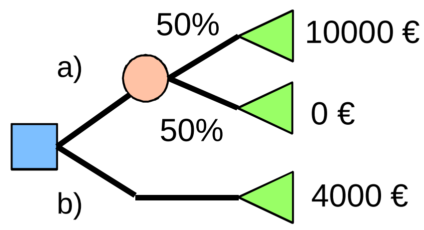
- Use multiple experts with alternative points of view
- Place hypothetical bets against the desired event
  - "Make the respondent's money involved"
- Use decomposition and realistic assessment of partial probabilities
  - "Split the events"
- Yet, empirical evidence suggests that **all motivational biases are difficult to correct**

Further reading: **Montibeller, G., and D. von Winterfeldt**, 2015. Cognitive and Motivational Biases in Decision and Risk Analysis, *Risk Analysis*

Risky or not risky?

https://presemo.aalto.fi/riskattitude1/

- Which one would you choose:
  a) Participate in a lottery, where you have a 50 % chance of getting nothing and 50 % chance of getting 10000 €
  b) Take 4000 €

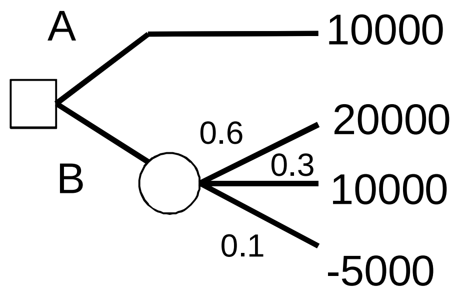


- Many choose the certain outcome of 4000 €, although a)'s expected monetary gain is higher

Option b) involves less risk

How to compare risky alternatives?

- **Last week**
  - We learned how to support decision-making under uncertainty, when the DM's objective is to maximize the expected monetary value
  - Maximizing expected value is rational only if the DM is risk neutral, i.e., indifferent between
    - obtaining x for sure and
    - a gamble with uncertain payoff Y such that x=E[Y]
  - Usually, DMs are risk averse = they prefer obtaining x for sure to a gamble with payoff Y such that x=E[Y]



Expectation = 14500

- **Next:**
  - We learn how to accommodate the DM's risk attitude (=preference over alternatives with uncertain outcomes) in decision models

Expected utility theory (EUT)

- John von Neumann and Oscar Morgenstern (1944) in Theory of Games and Economic Behavior:
  - Axioms of rationality for preferences over alternatives with uncertain outcomes
  - If the DM follows these axioms, she should prefer the alternative with the highest expected utility

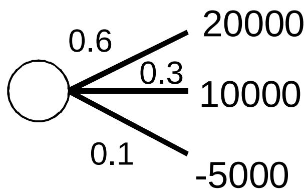
- Elements of EUT
  - Set of outcomes and lotteries
  - Preference relation over the lotteries satisfying four axioms
  - Representation of preference relation with expected utility

EUT: Sets of outcomes and lotteries

- Set of possible outcomes T:
  - E.g., revenue T euros / demand T
- Set of all possible lotteries L:
  - A lottery $f \in L$ associates a probability $f(t) \in [0,1]$ with each possible outcome $t \in T$
    - Finite number of outcomes with a positive probability $f(t) > 0$
    - Probabilities sum up to one $\sum_t f(t) = 1$
    - Lotteries are thus discrete PMFs / decision trees with a single chance node
- Deterministic outcomes are modeled as degenerate lotteries

Lottery

Decision tree

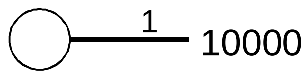


Probability mass function (PMF)

$$f(t) = \begin{cases} 0.6, & t = 20000 \\ 0.3, & t = 10000 \\ 0.1, & t = -5000 \\ 0, & elsewhere \end{cases}$$

Degenerate lottery

Decision tree



PDF

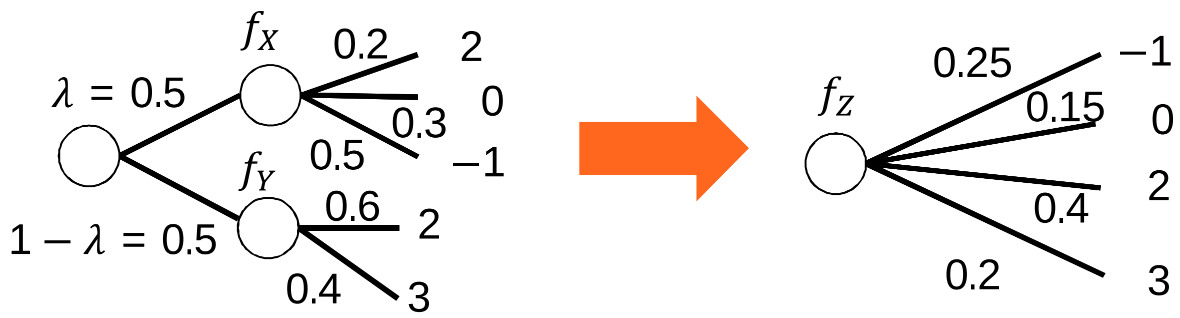
$$f(t) = \begin{cases} 1, & t = 10000 \\ 0, & elsewhere \end{cases}$$

EUT: Compound lotteries

- Compound lottery:
  - Get lottery $f_X \in L$ with probability λ
  - Get lottery $f_Y \in L$ with probability $1-\lambda$
- Compound lottery can be modeled as lottery $f_Z \in L$:

$$f_Z(t) = \lambda f_X(t) + (1-\lambda) f_Y(t) \;\; \forall t \in T \simeq f_Z = \lambda f_X + (1-\lambda) f_Y$$

- Example:
  - You have a 50-50 chance of getting a ticket to lottery $f_X \in L$ or to lottery $f_Y \in L$



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Preference relation

- Let $\succcurlyeq$ be preference relation among lotteries in L
  - Preference $f_X \succcurlyeq f_Y$: f_X at least as preferable as f_Y
  - Strict preference $f_X \succ f_Y$ defined as $\neg(f_Y \succcurlyeq f_X)$
  - Indifference $f_X \sim f_Y$ defined as $f_X \succcurlyeq f_Y \wedge f_Y \succcurlyeq f_X$

EUT axioms A1-A4 for preference relation

- **A1:** $\succcurlyeq$ is complete
  - For any $f_X, f_Y \in L$, either $f_X \succcurlyeq f_Y$ or $f_Y \succcurlyeq f_X$ or both
- **A2:** $\succcurlyeq$ is transitive
  - If $f_X \succcurlyeq f_Y$ and $f_Y \succcurlyeq f_Z$, then $f_X \succcurlyeq f_Z$
- **A3:** Archimedean axiom
  - If $f_X \succ f_Y \succ f_Z$, then $\exists \lambda, \mu \in (0,1)$ such that

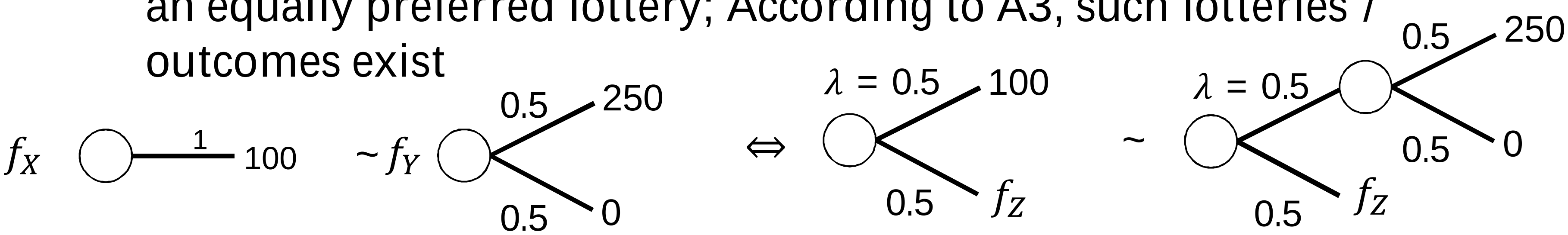
$$\lambda f_X + (1-\lambda) f_Z \succ f_Y \text{ and } f_Y \succ \mu f_X + (1-\mu) f_Z$$

- **A4:** Independence axiom
  - Let $\lambda \in (0,1)$. Then,

$$f_X \succ f_Y \Leftrightarrow \lambda f_X + (1-\lambda) f_Z \succ \lambda f_Y + (1-\lambda) f_Z$$

If the EUT axioms hold for the DM's preferences

- **A3:** Archimedean axiom
  - Let $f_X \succ f_Y \succ f_Z$. Then exists $p \in (0,1)$ so that $f_Y \sim pf_X + (1-p)f_Z$
- **A4:** Independence axiom
  - $f_X \sim f_Y \Leftrightarrow \lambda f_X + (1-\lambda)f_Z \sim \lambda f_Y + (1-\lambda)f_Z$
  - Any lottery (or outcome = a degenerate lottery) can be replaced by an equally preferred lottery; According to A3, such lotteries / outcomes exist



  - NOTE: f_Z can be any lottery and can have several possible outcomes

Main result: Preference representation with Expected Utility

- $\succcurlyeq$ satisfies axioms A1-A4 if and only if there exists a real-valued utility function *u(t)* over the set of outcomes *T* such that

$$f_X \succcurlyeq f_Y \Leftrightarrow \sum_{t \in T} f_X(t)u(t) \geq \sum_{t \in T} f_Y(t)u(t)$$

- Implication: **a rational DM** following axioms A1-A4 **selects the alternative with the highest expected utility**

$$E[u(X)] = \sum_{t \in T} f_X(t)u(t)$$

  – A similar result can be obtained for continuous distributions:
    - $f_X \succcurlyeq f_Y \Leftrightarrow E[u(X)] \geq E[u(Y)]$, where $E[u(X)] = \int f_X(t)u(t)dt$

Computing expected utility

- Example: Joe's utility function for the number of apples is u(1)=2, u(2)=5, u(3)=7. Would he prefer
  - Two apples for certain (X), or
  - A 50-50 gamble between 1 and 3 apples (Y)?

$$E[u(X)] = u(2) = 5$$

$$E[u(Y)] = 0.5u(1) + 0.5u(3) = 0.5 \cdot 2 + 0.5 \cdot 7 = 4.5$$

- Example: Jane's utility function for money is $u(t) = t^2$. Which alternative would she prefer?
  - X: 50-50 gamble between 3 and 5M€
  - Y: A random amount of money from Uni(3,5) distribution
  - What if her utility function was $u(t) = \frac{t^2-9}{25-9}$?

$$E[u(X)] = 0.5u(3) + 0.5u(5) = 0.5 \cdot 9 + 0.5 \cdot 25 = 17$$

$$E[u(Y)] = \int_3^5 f_Y(t)u(t)dt = \int_3^5 \frac{1}{2}t^2 dt = \frac{1}{6}5^3 - \frac{1}{6}3^3 = 16.33333$$

Let's practice!

https://presemo.aalto.fi/drcuckoo

The utility function of Dr. Cuckoo is $u(t) = \sqrt{t}$. Would he

a) Participate in a lottery A with 50-50 chance of getting either 0 or 400 €?

b) Participate in a lottery B in which the probability of getting 900 € is 30% and getting 0 € is 70%?

$u(0) = 0$, $u(400) = 20$, $u(900) = 30$

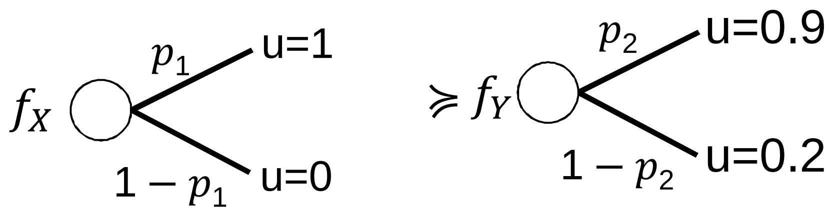
a) $E[u(A)] = 0.5 \cdot 0 + 0.5 \cdot 20 = 10$

b) $E[u(B)] = 0.7 \cdot 0 + 0.3 \cdot 30 = 9$

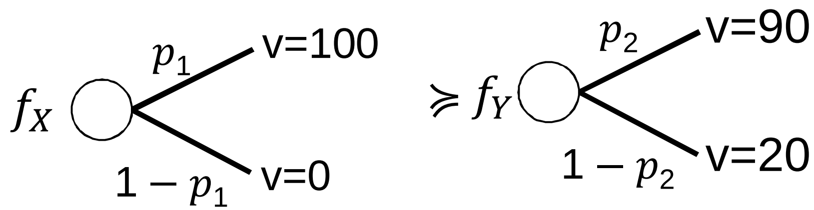
NOTE! the **expectation of lottery A** = 200 € **is smaller** than that of B = 270€

Uniqueness up to positive affine transformations

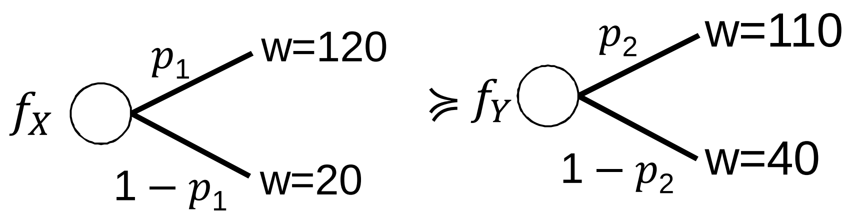
- DM's preferences: $X \succcurlyeq Y$
  - $E[u(X)] = p_1 \geq 0.9p_2 + 0.2(1-p_2)$



- v: Multiply each utility u by 100
  - $E[v(X)] = 100p_1 = 100E[u(X)] \geq 100E[u(Y)] = 90p_2 + 20(1-p_2) = E[v(Y)]$

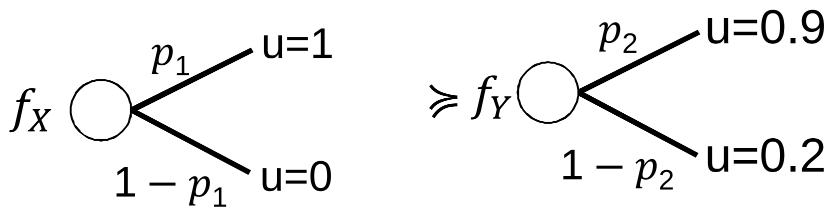


- w: Add 20 to all utilities v
  - $E[w(X)] = 120p_1 + 20(1-p_1) = 100p_1 + 20 = E[v(X)] + 20 \geq E[v(Y)] + 20 = 90p_2 + 20(1-p_2) + 20(1+p_2-p_2) = 110p_2 + 40(1-p_2) = E[w(Y)]$

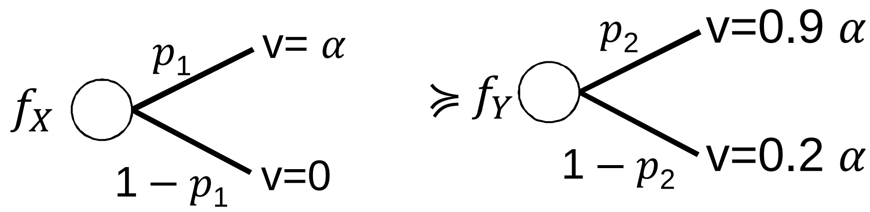


Uniqueness up to positive affine transformations

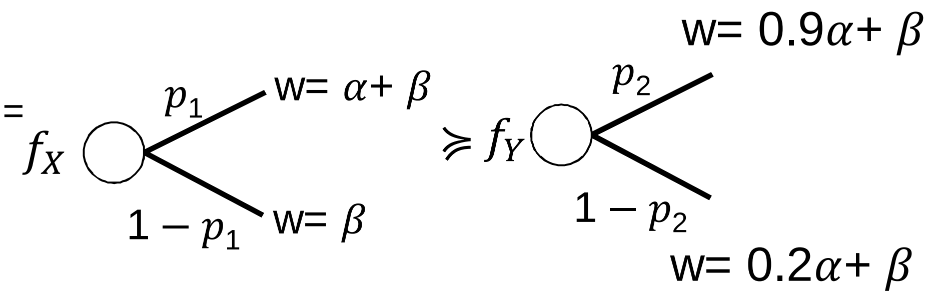
- DM's preferences: $X \succcurlyeq Y$
  - $E[u(X)] = p_1 \geq 0.9p_2 + 0.2(1 - p_2)$



- v: Multiply u by $\alpha > 0$ $E[v(X)] = \alpha p_1 = \alpha E[u(X)] \geq \alpha E[u(Y)] = 0.9\alpha p_2 + 0.2\alpha(1 - p_2) = E[v(Y)]$



- w: Add β to all utilities v
  - $E[w(X)] = (1 + \beta)p_1 + \beta(1 - p_1) = \alpha p_1 + \beta = E[v(X)] + \beta \geq E[v(Y)] + \beta = 0.9\alpha p_2 + 0.2\alpha(1 - p_2) + \beta(1 + p_2 - p_2) = (0.9\alpha + \beta)p_2 + (0.2\alpha + \beta)(1 - p_2) = E[w(Y)]$



Uniqueness up to positive affine transformations

- Let $f_X \succcurlyeq f_Y \iff E[u(X)] \geq E[u(Y)]$. Then $E[\alpha u(X) + \beta] = \alpha E[u(X)] + \beta \geq \alpha E[u(Y)] + \beta = E[\alpha u(Y) + \beta]$ for any $\alpha > 0$
- **Two utility functions $\boldsymbol{u_1(t)}$ and $\boldsymbol{u_2(t) = \alpha u_1(t) + \beta}$, $\boldsymbol{(\alpha > 0)}$ establish the same preference order among any lotteries**:

$$E[u_2(X)] = E[\alpha u_1(X) + \beta] = \alpha E[u_1(X)] + \beta.$$

- Implications:
  - Any linear utility function $u_L(t) = \alpha t + \beta$, $(\alpha > 0)$ is a positive affine transformation of the identity function $u_1(t) = t \Rightarrow u_L(t)$ establishes the same preference order as expected value
  - Utilities for two outcomes can be freely chosen:
    - E.g., scale utilities represented by u_1 such that and $u_2(t^*) = 1$ and $u_2(t^0) = 0$:

$$u_2(t) = \frac{u_1(t) - u_1(t^0)}{u_1(t^*) - u_1(t^0)} = \underbrace{\frac{1}{u_1(t^*) - u_1(t^0)}}_{=\alpha>0} u_1(t) - \underbrace{\frac{u_1(t^0)}{u_1(t^*) - u_1(t^0)}}_{=\beta}$$

Summary

- Probability elicitation is prone to cognitive and motivational biases
  - Some cognitive biases can be easy to correct, but…
  - Some other cognitive biases and all motivational biases can be difficult to overcome

- The DM's preferences over alternatives with uncertain outcomes can be described by a utility function

- A rational DM (according to the four axioms of rationality) should choose the alternative with the highest expected utility
  - NOT necessarily the alternative with the highest utility of expectation

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Decision making and problem solving – Lecture 3

- Modeling risk preferences
- Stochastic dominance

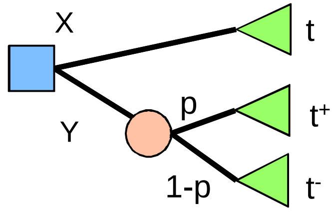
Liesiö, Punkka, Salo, Vilkkumaa

Motivation

- **Last time:**
  - Decisions should be based on expected value of the alternatives' outcomes (if and) only if the DM is risk neutral
  - Under 4 axioms for the DM's preference relation between risky alternatives, there exists a real-valued function ("utility function") so that
    - The DM should choose the alternative with the highest expected utility
    - It is unique up to positive affine transformations -> we can normalize the utility function the way we want
- **This time:**
  - What is this utility function and how to model the DM's preferences with it?
  - We learn how these preferences correspond to the DM's attitude towards risk

Assessment of utility functions

- Utility functions are assessed by asking the DM to choose between a simple lottery and a certain outcome (i.e., a degenerate lottery)
  - X: Certain payoff t
  - Y: Payoff t^+ (t^-) with probability p (1-p)



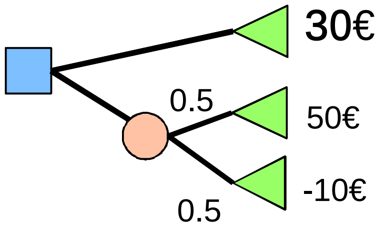
- General idea:
  - Vary the parameters (p,t,t^+, t^-) until the DM is indifferent between X and Y:

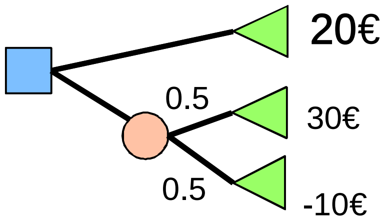
$$E[u(X)] = E[u(Y)] \Leftrightarrow u(t) = pu(t^+) + (1-p)u(t^-)$$

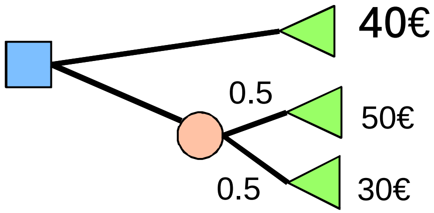
  - Repeat until sufficiently many points for the utility function have been obtained
- Because *u* is unique up to positive affine transformations, *u* can be fixed at two points
  - Usually, u is set at 1 at the most preferred level, and at 0 at the least preferred

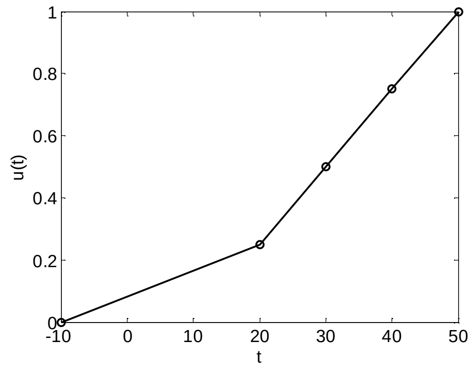
Assessment: The certainty equivalence approach

- The DM assesses t
- Example: Assess utility function for the interval [-10,50] euros
  - Normalization: we can fix u(-10)=0 and u(50)=1









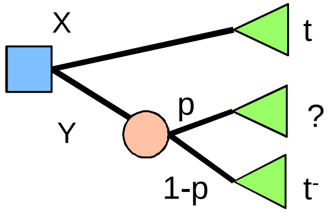
$u(30)$
$= 0.5u(-10) + 0.5u(50)$
$= 0.5 \cdot 0 + 0.5 \cdot 1 = 0.5$

$u(20)$
$= 0.5u(-10) + 0.5u(30)$
$= 0.5 \cdot 0 + 0.5 \cdot 0.5 = 0.25$

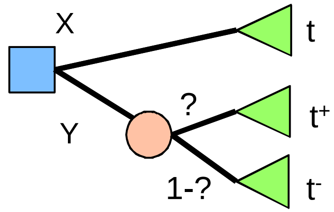
$u(40)$
$= 0.5u(30) + 0.5u(50)$
$= 0.5 \cdot 0.5 + 0.5 \cdot 1$
$= 0.75$

Other approaches to utility assessment

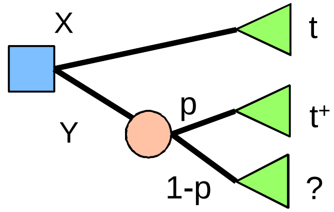
- Probability equivalence:
  - The DM assesses p



- Gain equivalence:
  - The DM assesses t^+



- Loss equivalence:
  - The DM assesses t^-



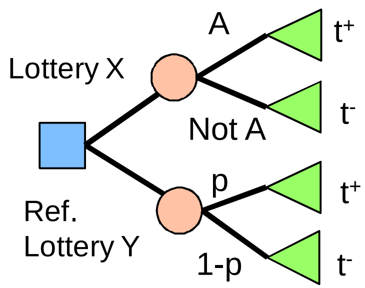
- Often in applications, the analyst chooses a family of utility functions and then asks the DM to compare lotteries to fix the parameter(s)
  - E.g., the exponential utility function (parameter ρ)

$$u(t) = 1 - e^{-\frac{t}{\rho}}, \rho > 0$$

Reference lottery revisited

- Assume that an expected utility maximizer with utility function *u* uses a reference lottery to assess the probability of event A
- She thus adjusts *p* such that she is indifferent between lottery X and reference lottery Y:

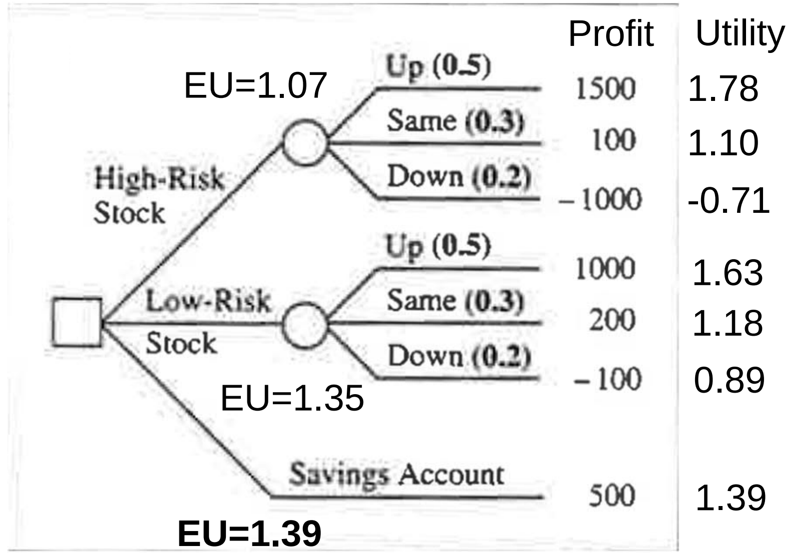
$$E[u(X)] = E[u(Y)]$$
$$\Leftrightarrow P(A)u(t^+) + \big(1 - P(A)\big)u(t^-) = pu(t^+) + (1 - p)u(t^-)$$
$$\Leftrightarrow P(A)\big(u(t^+) - u(t^-)\big) = p\big(u(t^+) - u(t^-)\big)$$
$$\Leftrightarrow P(A) = p$$



- Utility function *u* does not affect the result

Expected utility in decision trees

- Do everything in the usual way, but
  - Chance node: compute the expected utility
  - Decision node: select the alternative corresponding to maximum expected utility
  - Cf. the umbrella example, in which 'some numbers' represented preferences



$$u(t) = 2 - e^{\frac{-t}{1000}}$$

Expected utility in Monte Carlo

- For each sample $x_1, \dots, x_n$ of random variable X, compute utility $u(x_i)$
- Mean of sample utilities $u(x_1), \dots, u(x_n)$ provides an estimate for $E[u(X)]$

=2-EXP(-F12/1000)

C	D	E	F	G	H
		Col.mean	Col.mean	Col.mean	
		0.502964	990.3014	1.580972	
	Sample	u	x	Utility	
	1	0.464077	954.9167	1.615156	
	2	0.704234	1268.308	1.718693	
	3	0.777865	1382.501	1.74905	
	4	0.534927	1043.831	1.647897	
	5	0.4426	927.8094	1.604581	
	6	0.916252	1690.147	1.815508	
	7	0.649453	1191.922	1.696363	
	8	0.65278	1196.418	1.697725	
	9	0.110887	389.0874	1.322325	
	10	0.189275	559.714	1.428628	
	11	0.902882	1649.073	1.807772	

EUT for normative decision support

- EUT is a normative theory: if the DM is rational, she **should** select the alternative with the highest expected utility
  - Not descriptive or predictive: EUT does not describe or predict how people actually do select among alternatives with uncertain outcomes

- The four axioms characterize properties that are required for rational decision support
  - Cf. probability axioms describe a rational model for uncertainty
  - The axioms are not assumptions about the DM's preferences

Question 1

http://presemo.aalto.fi/2134lec2

- Which of the below alternatives would you choose?

1. A sure gain of 1 M€
2. A gamble in which there is a
   - 1% probability of getting nothing,
   - 89% probability of getting 1M€, and
   - 10% probability of getting 5M€

Question 2

http://presemo.aalto.fi/2134lec2

- Imagine that a rare disease is breaking out in a community and is expected to kill 600 people. Two different programs are available to deal with the threat.
  - If Program A is adopted, 200 people will be saved
  - If Program B is adopted, there's a 33% probability that all 600 will be saved and a 67% probability that no one will be saved.

  Which program will you choose?

1. Program A
2. Program B

Question 3

http://presemo.aalto.fi/2134lec2

❑ Which of the below alternatives would you choose?

1. A gamble in which there is a
   - o 89% probability of getting nothing and
   - o 11% probability of getting 1M€

2. A gamble in which there is a
   - o 90% probability of getting nothing, and
   - o 10% probability of getting 5M€

Question 4

http://presemo.aalto.fi/2134lec2

- Imagine that a rare disease is breaking out in some community and is expected to kill 600 people. Two different programs are available to deal with the threat.
  - If Program C is adopted, 400 of the 600 people will die,
  - If Program D is adopted, there is a 33% probability that nobody will die and a 67% probability that 600 people will die.

Which program will you choose?

1. Program C
2. Program D

Allais paradox

- Which of the below alternatives would you choose?
  - A. A sure gain of 1 M€
  - B. A gamble in which there is a
    - o 1% probability of getting nothing,
    - o 89% probability of getting 1M€, and
    - o 10% probability of getting 5M€

Most people choose A; hence E[u(A)]>E[u(B)]:
$u(1) > 0.10u(5)+0.89u(1)+0.01u(0) \Rightarrow$

$$\boxed{0.11u(1) > 0.10u(5)+0.01u(0)}$$

- Which of the below alternatives would you choose?
  - C. A gamble in which there is a
    - o 89% probability of getting nothing and
    - o 11% probability of getting 1M€
  - D. A gamble in which there is a
    - o 90% probability of getting nothing, and
    - o 10% probability of getting 5M€

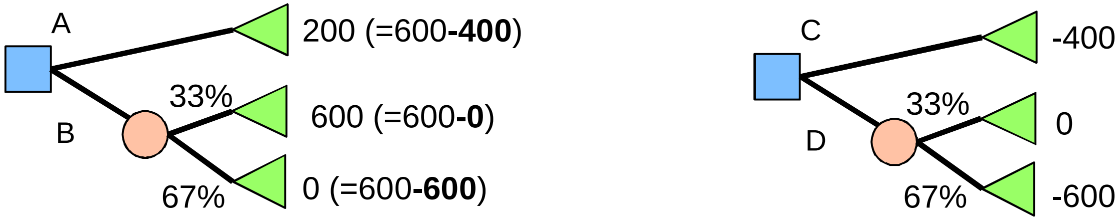
Most people choose D; hence E[u(D)]>E[u(C)]:
$0.10u(5)+0.90u(0) > 0.11u(1)+0.89u(0) \Rightarrow$

$$\boxed{0.11u(1) < 0.10u(5)+0.01u(0)}$$

- **Actual choice behavior is not always consistent with EUT**

Framing effect

- Most people choose A and D
- People tend to be ”risk-averse” about gains and ”risk-seeking” about losses

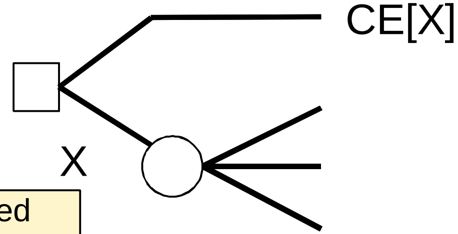


Risk and risk preferences

- Risk: possibility of loss (or some other unpreferred outcome)
  - Characterized by both the probability and magnitude of loss
- Risk preferences:
  - How does the riskiness of a decision alternative affect its desirability?
  - E.g., risk neutrality: choose the alternative with the highest expected (monetary) value, riskiness is not a factor
- Definition of risk preferences requires that outcomes T are quantitative and preferences among them *monotonic*
  - E.g., profits, costs, lives saved etc.
- Here, we assume that more is preferred to less, i.e., $u(t)$ is increasing (and differentiable) for all t

Certainty equivalent in Expected Utility Theory

- **Definition:** Certainty equivalent of a random variable X, denoted by CE[X], is an outcome in T such that



$$u(CE[X]) = E[u(X)] \Leftrightarrow$$
$$CE[X] = u^{-1}(E[u(X)])$$

Allowed because u is monotonic

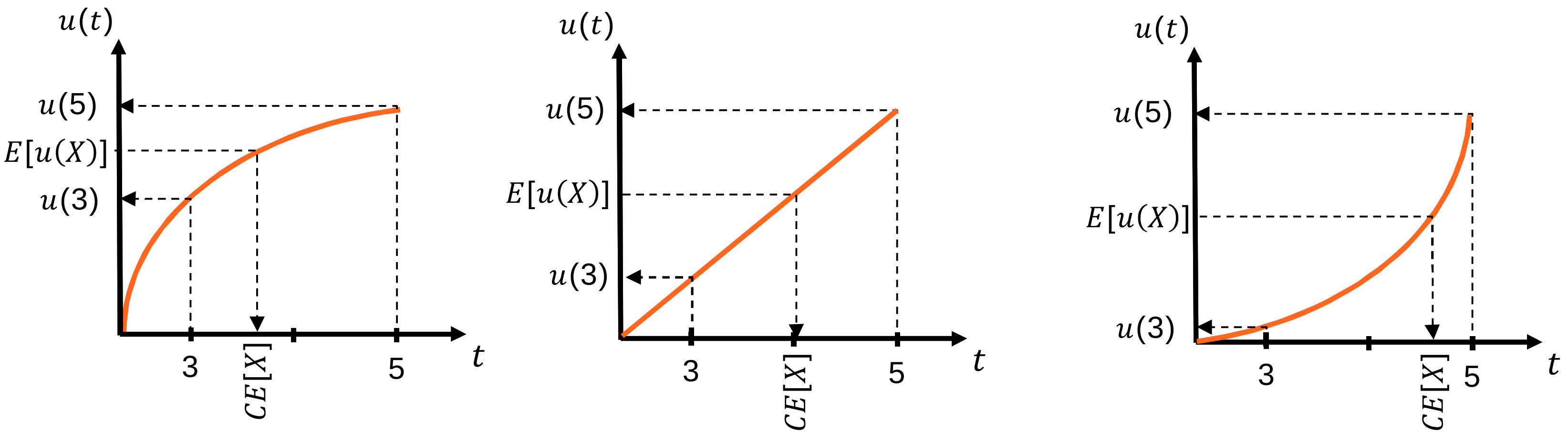
– **IMPORTANT!** CE[X] is the certain outcome such that the DM is indifferent between alternatives X and CE[X]

  – CE[X] depends on both the DM's utility function u (preferences) and the distribution of X (uncertainty)

    - My CE for roulette may be different from yours
    - My CE for roulette may be different from my CE for one-armed bandit

Certainty equivalent - Example

- Consider a decision alternative X with $f_X(3) = 0.5$ and $f_X(5) = 0.5$ and three DMs with the below utility functions
- Compute each DM's certainty equivalent for X



- The shape of the utility function seems to determine whether CE[X] is below, above, or equal to E[X]=4

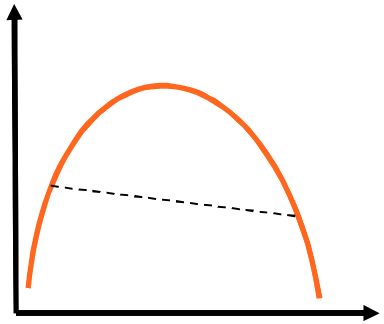
Convex and concave functions

- **Definition:** *u* is concave, if for any $t_{1,} t_2$:

  $$\lambda u(t_1) + (1-\lambda)u(t_2) \leq u(\lambda t_1 + (1-\lambda)t_2) \quad \forall \lambda \in [0,1]$$

  – A line drawn between any two points $u(t_1)$ and $u(t_2)$ is below (or equal to) $u(t)$

  – $u''(t) \leq 0 \; \forall t \in T$, if the second derivative exists

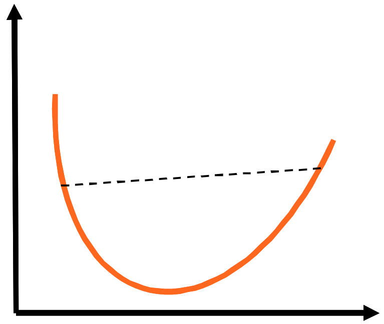


- **Definition:** *u* is convex, if for any $t_{1,} t_2$:

  $$\lambda u(t_1) + (1-\lambda)u(t_2) \geq u(\lambda t_1 + (1-\lambda)t_2) \quad \forall \lambda \in [0,1]$$

  – A line drawn between any two points $u(t_1)$ and $u(t_2)$ is above (or equal to) $u(t)$

  – $u''(t) \geq 0 \; \forall t \in T$, if the second derivative exists



Convex utility functions

- For any utility function *u*, $E[u(X)] = \sum f_X(t_i)\, u(t_i)$ for X with discrete set of outcomes $t_i, i = 1, \ldots, n$
  - Note: $\sum f_X(t_i) = 1$

- Let *u* be convex. Then
  - $\lambda u(t_1) + (1-\lambda)u(t_2) \geq u(\lambda t_1 + (1-\lambda)t_2)\ \ \forall \lambda \in [0,1]$ (by def., previous slide)
  - And, specifically, by applying this definition several times,

$$f_X(t_1)u(t_1) + \ \ldots\ + f_X(t_n)u(t_n) = E[U(X)] \geq u\left(\sum f_X(t_i)t_i\right) = U(E[X])$$

- For convex *u*: Expected utility of X is higher than (expected) utility of E(X)

Jensen's inequality

- For any random variable *X*, if function *u* is
  I. Convex, then $E[u(X)] \geq u(E[X])$
  II. Concave, then $E[u(X)] \leq u(E[X])$

  $\Rightarrow$

$$
\begin{aligned}
&u \text{ concave} \\
&\Rightarrow E[u(X)] \leq u(E[X]) \\
&\Leftrightarrow u^{-1}(E[u(X)]) \leq u^{-1}(u(E[X])) \\
&\Leftrightarrow CE[X] \leq E[X]
\end{aligned}
$$

Allowed because u is increasing

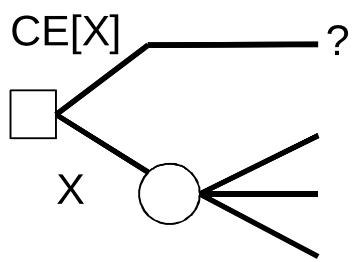
$$
\begin{aligned}
&u \text{ convex} \\
&\Rightarrow E[u(X)] \geq u(E[X]) \\
&\Leftrightarrow u^{-1}(E[u(X)]) \geq u^{-1}(u(E[X])) \\
&\Leftrightarrow CE[X] \geq E[X]
\end{aligned}
$$

Risk attitudes in Expected Utility Theory

I. *u* is concave iff CE[X] ≤ E[X] for all X

II. *u* is convex iff CE[X] ≥ E[X] for all X

III. *u* is linear iff CE[X]=E[X] for all *X*



- A DM with a linear utility function is called *risk neutral*
  – Indifferent between uncertain outcome X and a certain outcome equal to E[X]
- A DM with a concave but not linear utility function is called *risk averse*
  – Prefers a certain outcome smaller than E[X] to uncertain outcome X
- A DM with a convex but not linear utility function is called *risk seeking*
  – Requires a certain outcome larger than E[X] to not choose uncertain outcome X

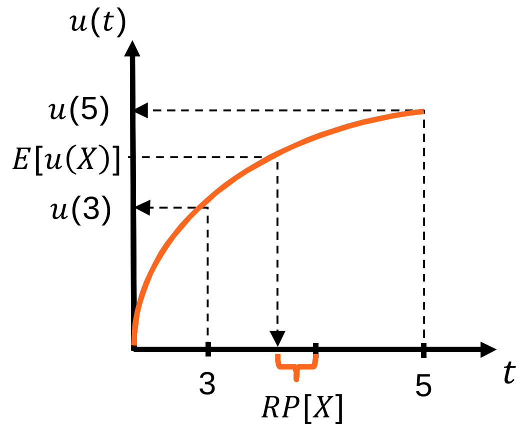
Risk premium in Expected Utility Theory

- **Definition:** Risk premium for random variable X is RP[*X*]=E[*X*]-CE[*X*]
  - RP[X] depends on both the DM's preferences (u) and the uncertainty in the decision alternative (distribution of X)
  - RP[X] is the premium that the DM requires on the expected value to change a certain outcome of CE[X] to an uncertain outcome X

I. DM is risk neutral, iff RP[X]=0 for all X

II. DM is risk averse, iff RP[X] $\geq$ 0 for all X

III. DM is risk seeking, iff RP[X] $\leq$ 0 for all X



Computing CE and RP

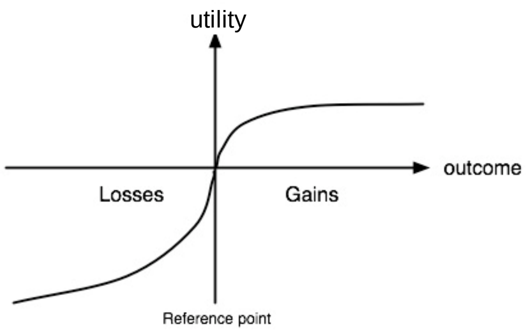
1. Compute E[u(X)] and E(X)
2. Solve $u^{-1}(\cdot)$
3. Compute $CE[X] = u^{-1}(E[u(X)])$
4. Compute RP[*X*]=E[*X*]-CE[*X*]

- Step 2: if $u^{-1}(\cdot)$ cannot be solved analytically, solve it numerically from $u(CE[X]) = E[u(X)]$
  - Trial and error
  - Computer software

Example: Jane's $u(t) = t^2$ and her payoff is *Y~Uni*(3,5)

1. $E[u(X)] = \int_3^5 f_Y(t)u(t)dt = 16.33$
2. $v = u(t) = t^2 \Leftrightarrow t = u^{-1}(v) = \sqrt{v}$
3. $CE[X] = u^{-1}(16.33) = \sqrt{16.33} = 4.04$
4. RP[*X*] = 4 - 4.04 = -0.04

Prospect theory



- Expected Utility Theory assumes that people only care about the outcome in the *absolute* sense
- Yet, empirical evidence suggests that people tend to
  – think of possible outcomes relative to a certain reference point (often the status quo),
  – have different risk attitudes towards gains and losses with regard to the reference point,
  – overweight extreme, but unlikely events, but underweight "average" events.
- Prospect theory seeks to accommodate these empirical findings:

  Tversky, A. and D. Kahneman. "Advances in prospect theory: Cumulative representation of uncertainty." *Journal of Risk and uncertainty* 5.4 (1992): 297-323.
- **NOTE:**
  – EUT is a <u>normative</u> theory: tells what rational people should do
  – Prospect theory is a <u>descriptive</u> theory: tries to describe what people tend to do in real life

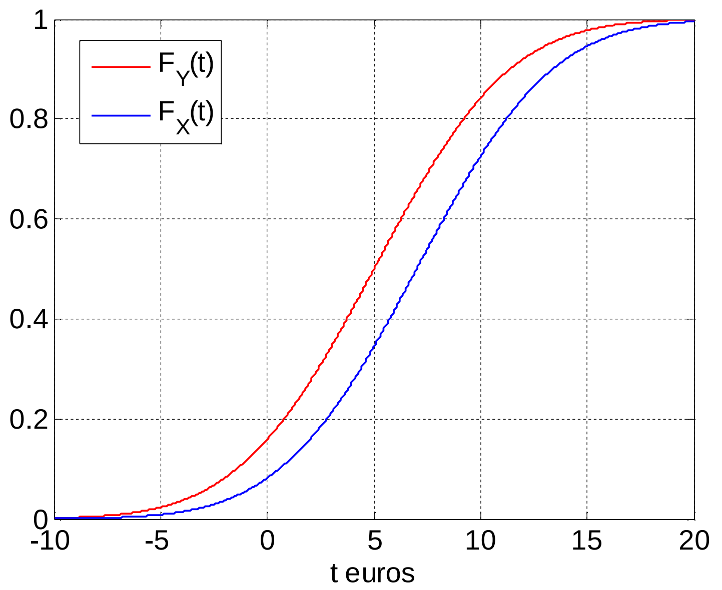
Stochastic dominance

https://presemo.aalto.fi/stocdom/

- **Question:** Which decision alternative would you choose?

1. X
2. Y

$$F_X(t) \leq F_Y(t) \;\; \forall t \in T$$

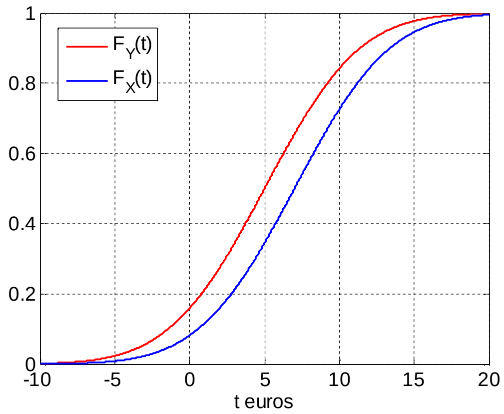


First-degree Stochastic Dominance

Definition: *X* dominates *Y* in the sense of First-degree Stochastic Dominance (denoted $X \succcurlyeq_{FSD} Y$), if

$$F_X(t) \le F_Y(t) \ \ \forall t \in T$$

with strict inequality for some t.

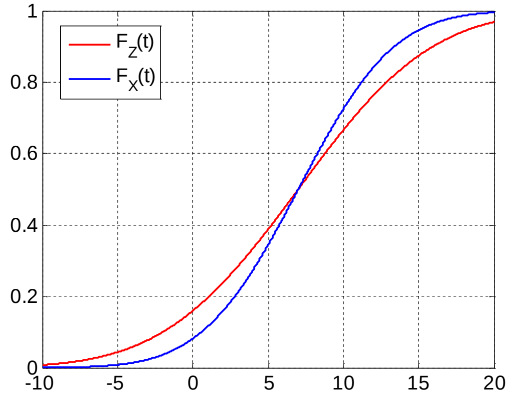


$X \succcurlyeq_{FSD} Y$

Theorem: $X \succcurlyeq_{FSD} Y$ if and only if

$$E[u(X)] \ge E[u(Y)] \ \ \forall u \in U^0,$$

where U^0 is the set of all strictly increasing functions

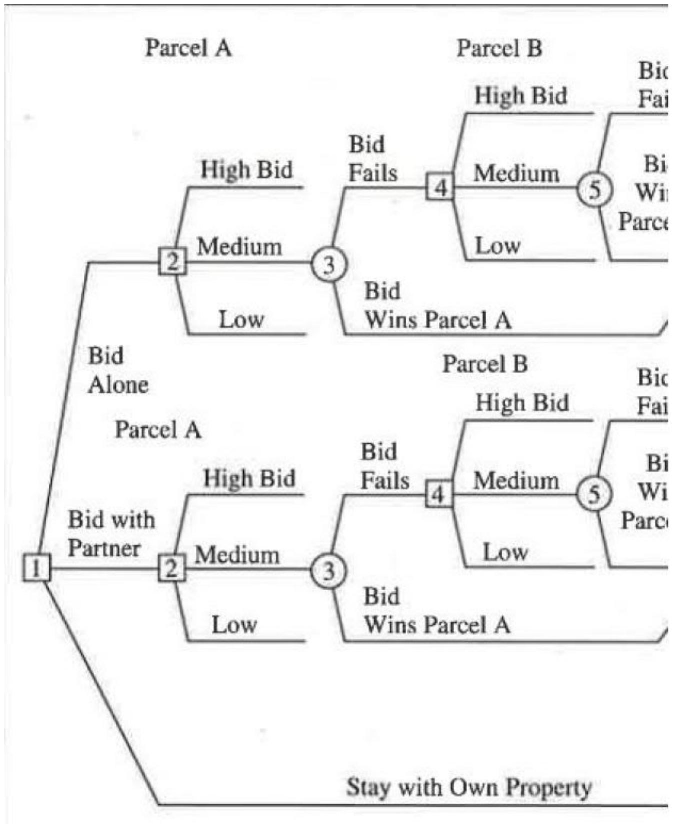


Neither alternative dominates the other in the sense of FSD

Implication: If an alternative is strictly dominated in the sense of FSD, then any DM who prefers more to less should not choose it.

FSD: Mining example

- A mining company has an opportunity to bid on two separate parcels of land
- Decisions to be made:
- Overall commitment of some $500 million
  – How much to bid?
  – Bid alone or with partner?
  – How to develop the site if the bid turns out successful?
- Large decision tree model built to obtain cumulative distribution functions of different strategies (= decision alternatives)

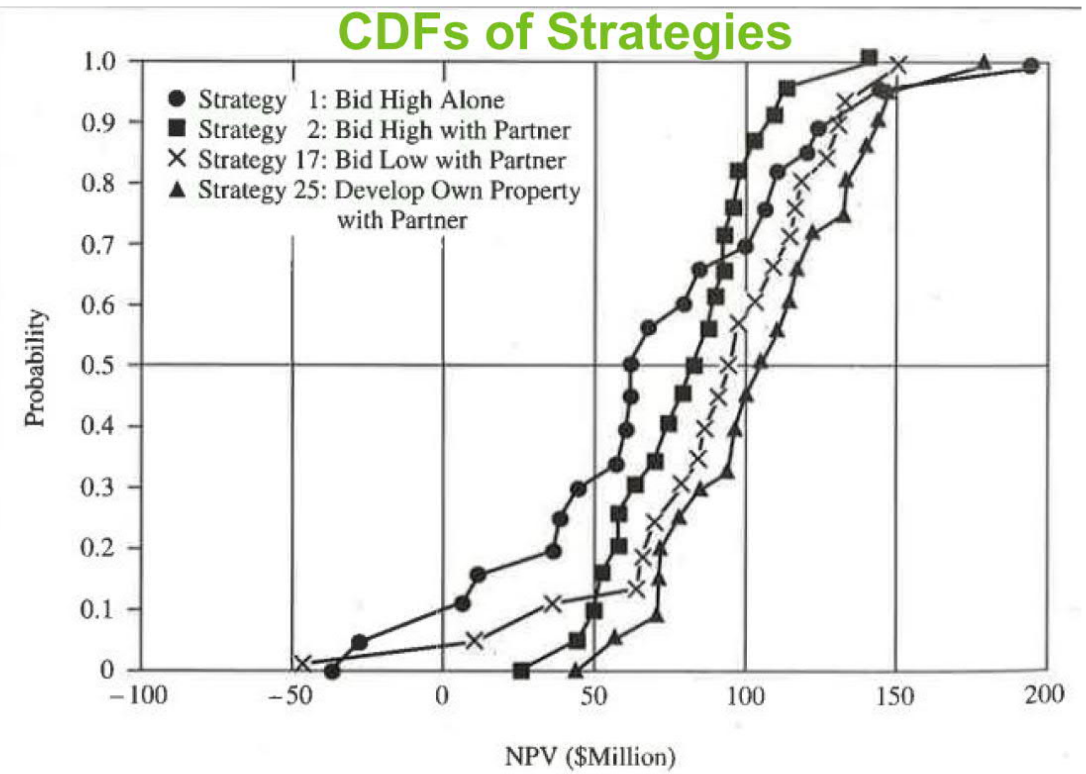


Source: Hax and Wing (1977): "The use of decision analysis in a capital investment probelm" In Bell, Keeney, and Raiffa (eds.): *Conflicting Objectives in Decisions*, Wiley.

FSD: Example (cont'd)

- ❑ Assume that the company prefers a larger net present value (NPV) to a smaller one

- ❑ Which strategies would you recommend?



Source: Hax and Wing (1977): "The use of decision analysis in a capital investment probelm" In Bell, Keeney, and Raiffa (eds.): *Conflicting Objectives in Decisions*, Wiley.

Second-degree Stochastic Dominance

- **Theorem:**

$$E[u(X)] \geq E[u(Y)] \;\; \forall u \in U^{\,ccv} \Leftrightarrow \int_{-\infty}^{z} [F_X(t) - F_Y(t)]dt \leq 0 \;\; \forall z \in T,$$

  where $U^{\,ccv} = \{u \in U^0 | u \text{ is concave}\}$.

- **Definition:** *X* dominates *Y* in the sense of Second-degree Stochastic Dominance (denoted $X \succcurlyeq_{\text{SSD}} Y$), if

$$\int_{-\infty}^{z} [F_X(t) - F_Y(t)]dt \leq 0 \;\; \forall z \in T.$$

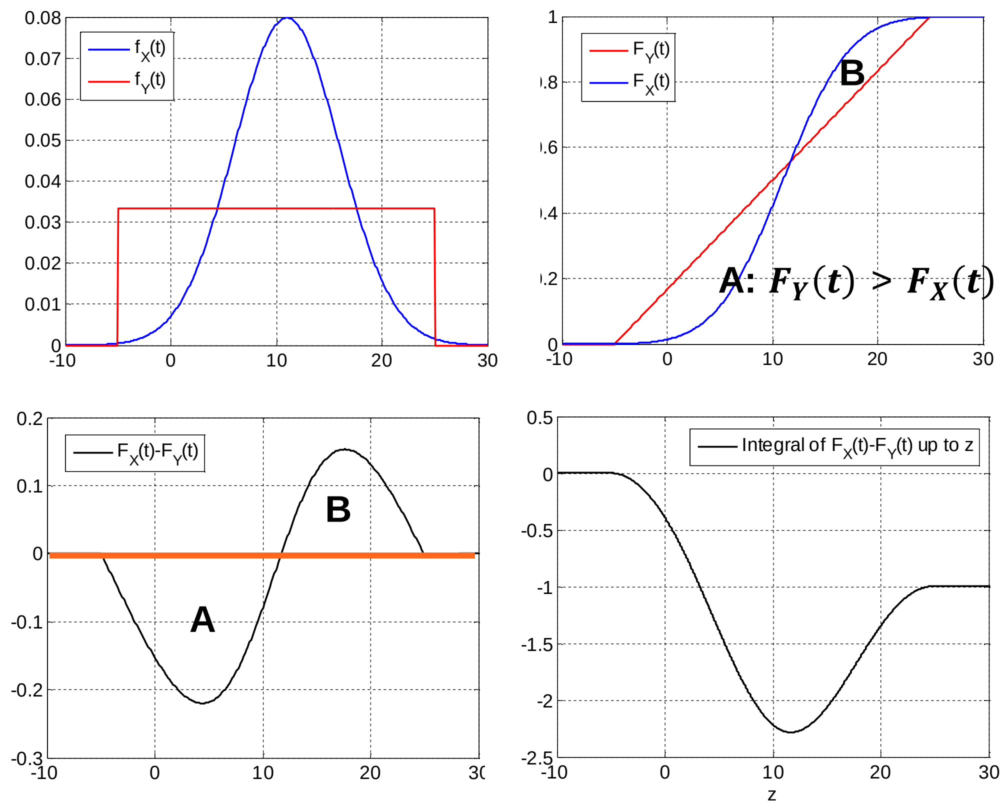
  with strict inequality for some *z.*

- Implication: If an alternative is strictly dominated in the sense of SSD, then any risk-averse or risk neutral DM who prefers more to less should not choose it.

SSD: graphical interpretation

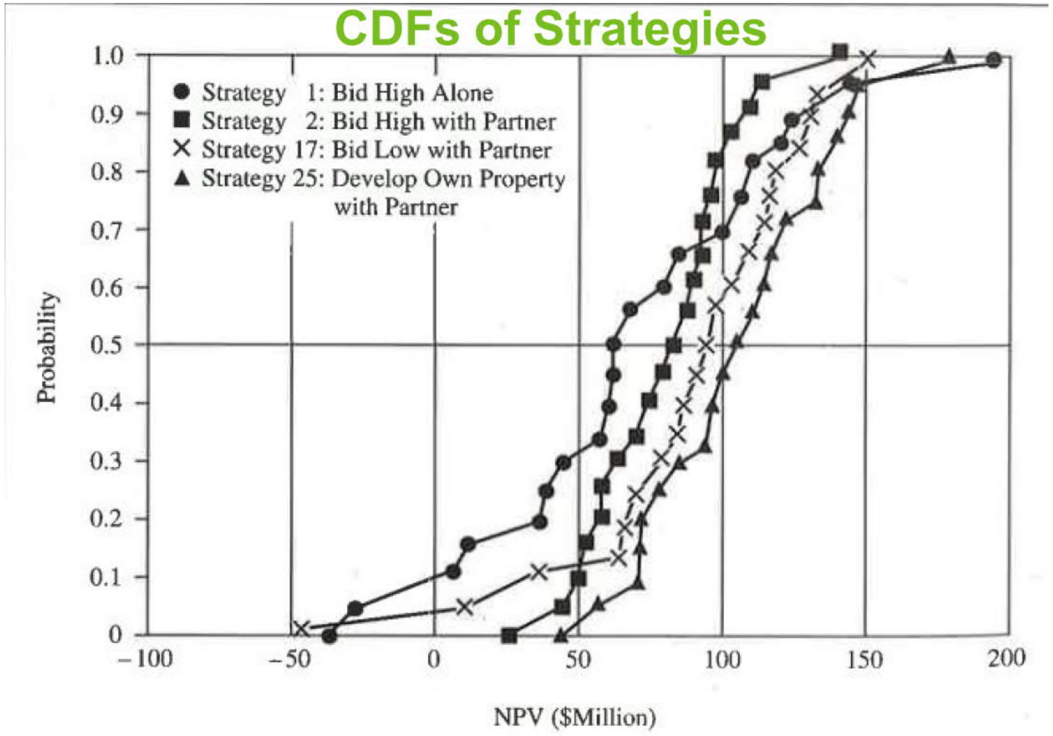
$$\int_{-\infty}^{z} [F_X(t) - F_Y(t)]dt \leq 0 \;\; \forall z \in T$$

- Integral
  = the area between $F_X(t)$ and $F_Y(t)$
  up to point z
  = the area between the $F_X(t)$-$F_Y(t)$
  and the horizontal axis up to point z
- If it is non-positive for all z, then
  $X \succcurlyeq_{SSD} Y$
- Here: $X \succcurlyeq_{SSD} Y$, because area **A** is bigger than area **B**, and **A** is left of **B**



SSD: Mining example revisited

- ❑ Assume that the mining company is either risk-averse or risk-neutral
- ❑ Which strategies would you recommend?



Properties of FSD and SSD

- Both FSD and SSD are transitive:
  - If X $\succcurlyeq_{FSD}$ Y and Y $\succcurlyeq_{FSD}$ Z, then X $\succcurlyeq_{FSD}$ Z
    - Why? Take any t. Then, $F_X(t) \leq F_Y(t) \leq F_Z(t)$.
  - If X $\succcurlyeq_{SSD}$ Y and Y $\succcurlyeq_{SSD}$ Z, then X $\succcurlyeq_{SSD}$ Z
    - Why? Take any $u \in U^{ccv}$. Then, $E[u(X)] - E[u(Z)] \geq E[u(Y)] - E[u(Z)] \geq 0$.
- FSD implies SSD:
  - If X $\succcurlyeq_{FSD}$ Y, then X $\succcurlyeq_{SSD}$ Y.
    - Why? Take any $u \in U^{ccv}$. Then, $u \in U^0$, and since X $\succcurlyeq_{FSD}$ Y, we have $E[u(X)] \geq E[u(Y)]$.
    - Or consider the definitions of FSD and SSD: If $F_X(t) \leq F_Y(t)\ \ \forall t \in T$, then

$$\int_{-\infty}^{z} [F_X(t) - F_Y(t)]dt \leq \int_{-\infty}^{z} 0dt \leq 0\ \ \forall z \in T$$

Summary

- **Utility function** is elicited **through** specification of equally **preferred lotteries**
  - Then: expected utilities equal

- The **shape** of the utility function determines the DM's **risk attitude**
  - Linear utility function = risk neutral
  - Concave utility function = risk averse
  - Convex utility function = risk seeking

- Even if the utility function is not completely specified, **decision recommendations** may be implied by **stochastic dominance**
  - If the DM prefers more to less, she should not choose an FSD dominated alternative
  - If the DM is also risk averse, she should not choose an SSD dominated alternative

Aalto University
School of Science

Decision making and problem solving – Lecture 4

- Risk measures
- Multiattribute value theory
- Axioms for preference relations
- Elicitation of attribute-specific value functions

Liesiö, Punkka, Salo, Vilkkumaa

Motivation

- **Last time we learned how :**
  - To model the DM's preferences over risk by eliciting her utility function
  - The shape (concave / linear / convex) of the utility function corresponds to the DM's risk attitude (risk averse / neutral / seeking)
  - Decision recommendations may be implied by stochastic dominance even if the utility function is not (completely) specified:
    - If the DM prefers more to less, she should not choose an FSD dominated alternative
    - If the DM is also risk averse, she should not choose an SSD dominated alternative
- **This time (Part A):**
  - We take a look at risk measures and examine how they can be used to describe alternatives' risks

Risk measures

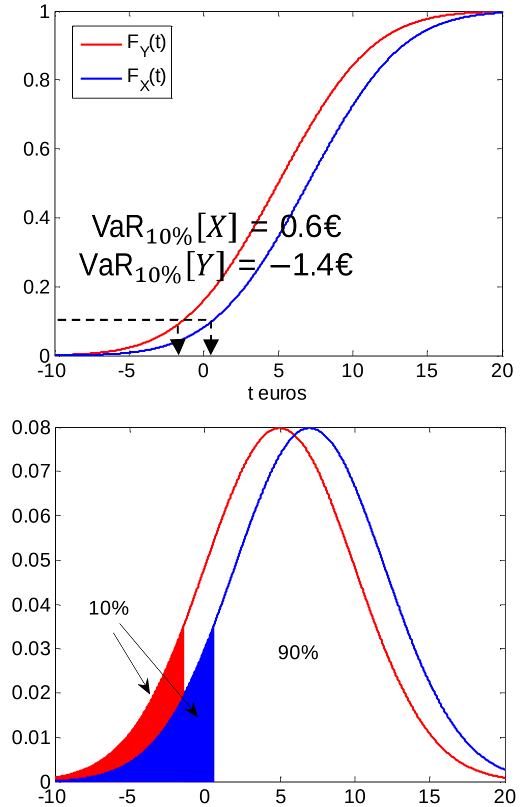
- Risk measure is a function that maps each decision alternative to a single number describing its risk
  - E.g., variance $Var[X] = E[(X - E[X])^2]$
    - The higher the variance, the higher the risk
- Risk measures are not based on EUT, but can be used together with expected values to produce decision recommendations
  - Risk constraint: Among alternatives whose risk is below some threshold, select the one with the highest expected value
  - Risk minimization: Among alternatives whose expected value is above some threshold, select the one with minimum risk
  - Efficient frontier: Select one of those alternative compared to which no other alternative yields higher expected value and smaller risk

Risk measures: Value-at-Risk (VaR)

- Value-at-Risk ($\text{VaR}_\alpha[X]$) is the outcome such that the probability of a worse or equal outcome is α:

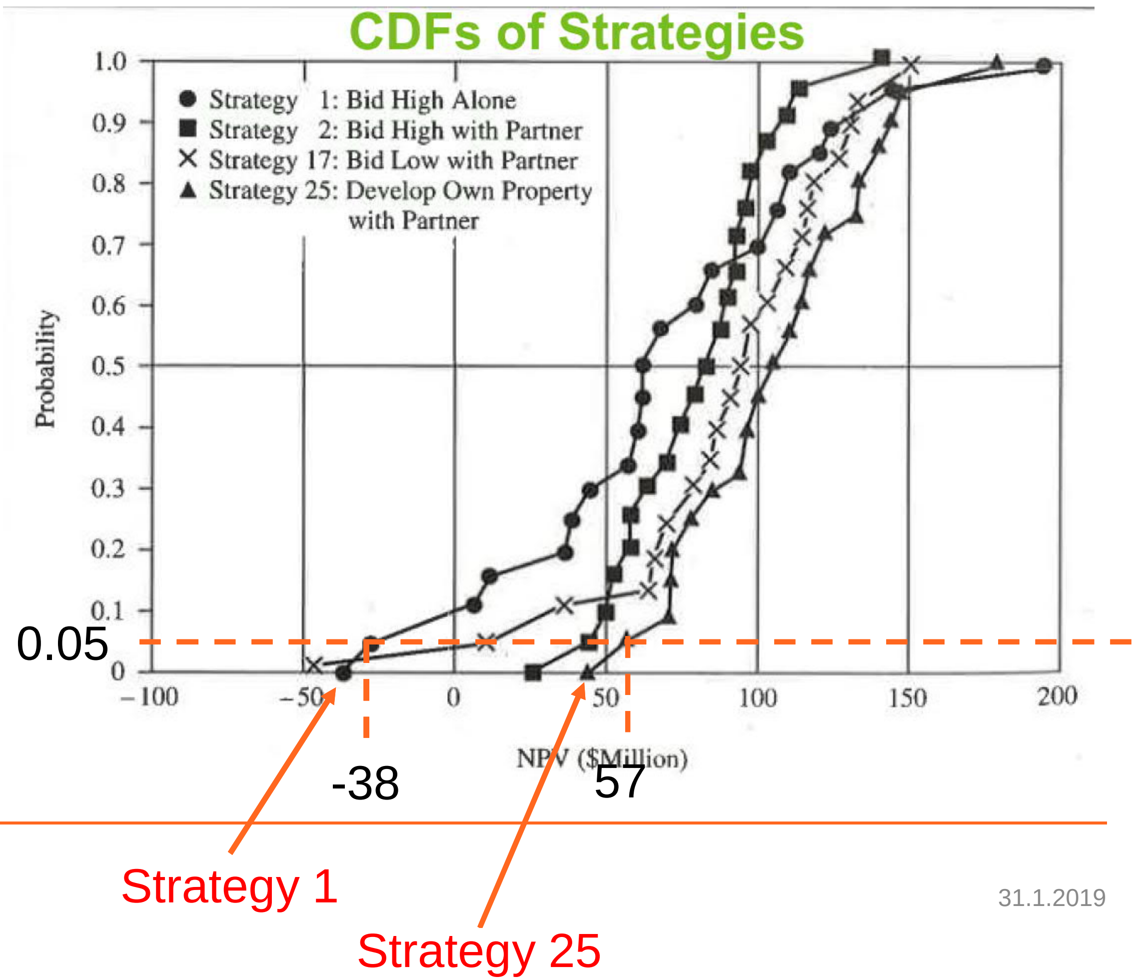
$$\int_{-\infty}^{\text{VaR}_\alpha[X]} f_X(t)dt = F_X(\text{VaR}_\alpha[X]) = \alpha.$$

- Higher VaR means smaller risk
  - Unless applied to a loss distribution
- Common values for α: 1%, 5%, and 10%
- Problem: the length/shape of the tail is not taken into account



Mining example revisited

- Assess $VaR_{5\%}$ for strategies 1 and 25

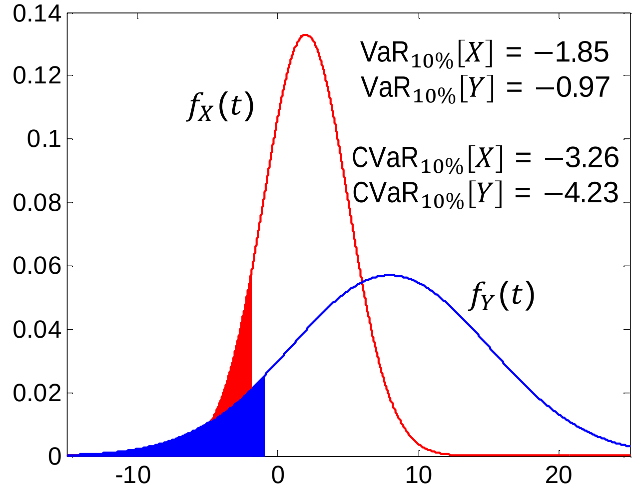


Risk measures: Conditional Value-at-Risk (CVaR)

- Conditional Value-at-Risk ($\text{CVaR}_\alpha[X]$) is the expected outcome given that the outcome is at most VaR_α:

$$\text{CVaR}[X] = E[X | X \le \text{VaR}_\alpha[X]]$$

- Higher CVaR means smaller risk (unless applied to losses)



- Computation of $\text{CVaR}[X]$ to discrete and continuous *X:*

$$E[X | X \le \text{VaR}_\alpha[X]] = \sum_{t \le \text{VaR}_\alpha[X]} t \frac{f_X(t)}{\alpha}, \qquad E[X | X \le \text{VaR}_\alpha[X]] = \int_{-\infty}^{\text{VaR}_\alpha[X]} t \frac{f_X(t)}{\alpha} dt.$$

  – Note: $\alpha = P(X \le \text{VaR}_\alpha[X])$; PMF/PDF $f_X(t)$ is scaled such that it sums/integrates up to 1.

Computation of VaR and CVaR

- If the inverse CDF of X is well-defined, VaR can be obtained from

$$\mathrm{VaR}_\alpha[X] = F_X^{-1}(\alpha)$$

  - In Excel: norm.inv, lognorm.inv, beta.inv, binom.inv etc
  - In Matlab: norminv, logninv, betainv, binoinv etc

- CVaR can then be computed using the formulas on the previous slide
  - Sometimes an analytic solution can be obtained; if, e.g., $X \sim N(\mu, \sigma^2)$ and $\mathrm{VaR}_\alpha[X] = \beta$, then

$$\mathrm{CVaR}_\alpha[X] = \mu - \sigma \frac{\phi\left(\frac{\beta-\mu}{\sigma}\right)}{\Phi\left(\frac{\beta-\mu}{\sigma}\right)},$$

    where ϕ and Φ are the standard normal PDF and CDF, respectively.
  - Sometimes numerical integration is needed

Computation of VaR and CVaR

- With discrete random variables VaR and CVaR are not always well defined for small values of α
  - Example:

t	-10	-5	1	10	20
$f_X(t)$	0.06	0.02	0.02	0.5	0.4

  - $\text{VaR}_{10\%}[X]$=1
  - $\text{CVaR}_{10\%}[X] = \frac{0.06(-10)+0.02(-5)+0.02(1)}{0.06+0.02+0.02}$=-6.8
  - But what are $\text{VaR}_{5\%}[X]$, $\text{CVaR}_{5\%}[X]$?

VaR and CVaR with Monte Carlo - Excel

=AVERAGE(D12:D211)

	A	B	C	D	E	F
1						
8		Col.mean	Col.mean	CVaR-10%		
9		0.507501	1008.35	147.4443		VaR-10%
10						410.5591
11	Sample	u	x	Below VaR		
12	1	0.691314	1249.789	above		
13	2	0.603076	1130.659	above		
14	3	0.548331	1060.723	above		
15	4	0.058081	214.4534	214.4534		
16	5	0.442469	927.6436	above		
17	6	0.628886	1164.452	above		
18	7	0.157181	496.9445	above		
19	8	0.355657	814.9539	above		
20	9	0.545768	1057.488	above		
21	10	0.416183	894.1666	above		
22	11	0.879097	1585.243	above		
23	12	0.022042	-6.64468	-6.64468		
24	13	0.000927	-556.359	-556.359		
25	14	0.071391	267.2461	267.2461		

=PERCENTILE.INC(C12:C211;0.1)

=IF(C12<=F10;C12;"above")

Note! 200 samples is very little, because only 1/10=20 are used to estimate CVaR

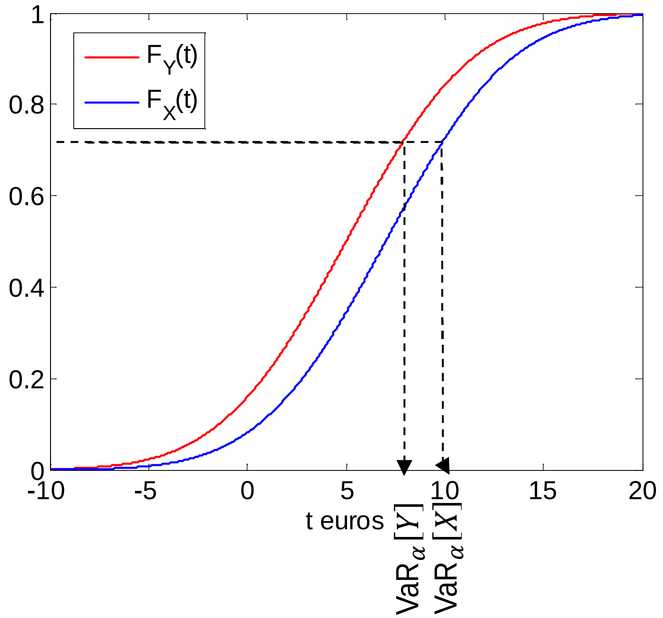
VaR and CVaR with Monte Carlo - Matlab

```
S=10^5;                              %Sample size 10,000
mu=1000;
sigma=500;
Sample=normrnd(mu,sigma,S,1);        %Generates 10^5 observations from N(mu,sigma)
VaR=prctile(Sample,10)               %Returns the 10% percentile of the sample
TailIndices=find(Sample<=VaR);       %Returns the indices of those elements
                                     %in the sample below or equal to VaR
CVaR=mean(Sample(TailIndices))       %Computes the arithmetic mean among those
                                     %elements in the sample belor or equal to VaR
```

Risk measures and stochastic dominance

- **Theorem:** $X \succcurlyeq_{\text{FSD}} Y$ if and only if $\text{VaR}_\alpha[X] \geq \text{VaR}_\alpha[Y]\ \forall\alpha \in [0,1]$

- **Theorem:** $X \succcurlyeq_{\text{SSD}} Y$ if and only if $\text{CVaR}_\alpha[X] \geq \text{CVaR}_\alpha[Y]\ \forall\alpha \in [0,1]$



EUT vs. Risk measures

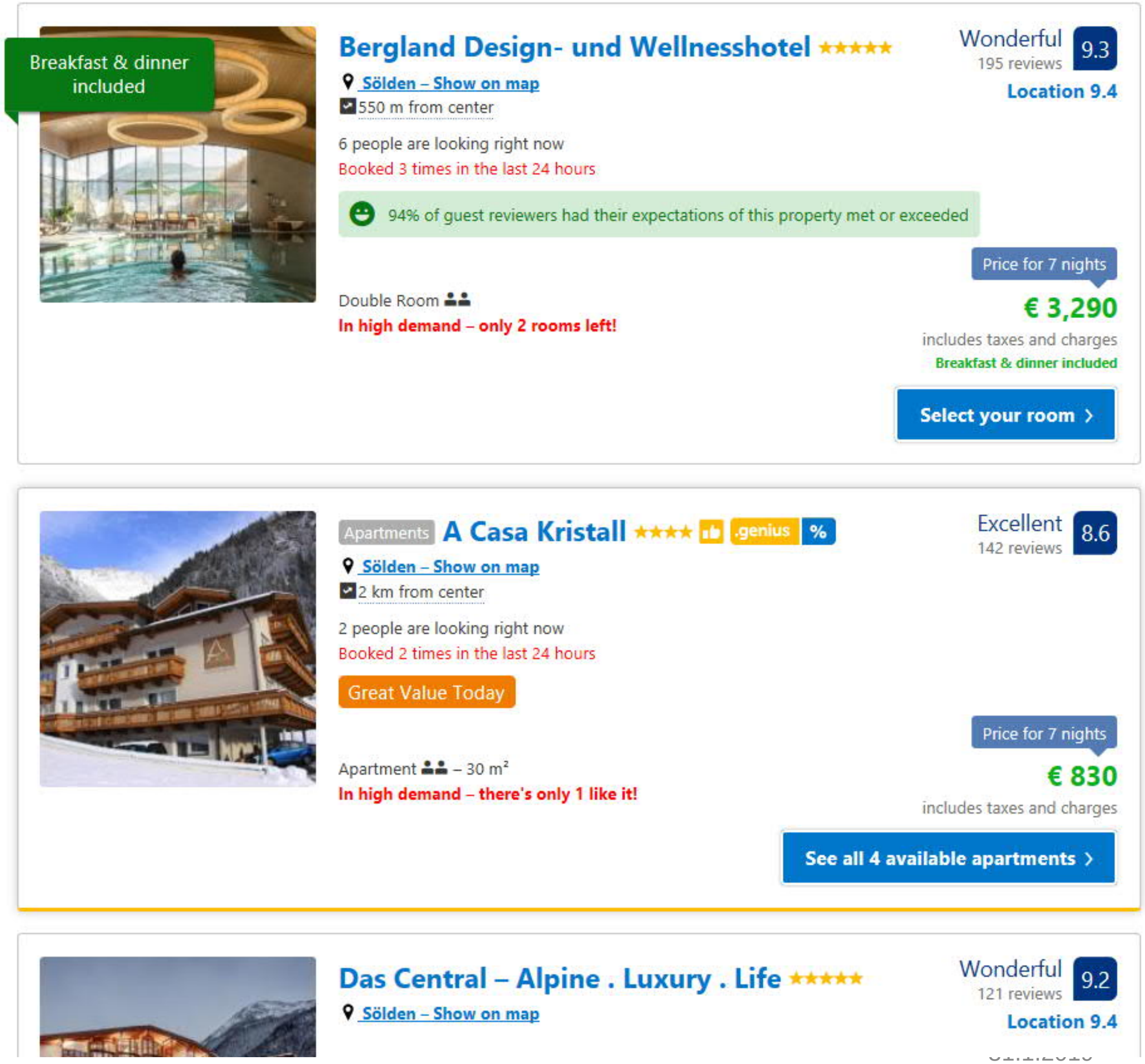
- EUT provides a more comprehensive way to capture the DM's preferences over uncertain outcomes
- With risk measures, one must answer questions such as
  - Which measure to use?
  - Which α to use in VaR and CVaR?
  - How to combine EV and the value of a risk measure into an overall performance measure?

- Yet, if answers to such questions are exogenously imposed, the use of risk measures can be easy
  - E.g., laws, regulations, industry standard etc.

Motivation

- ❑ **Consider yourself choosing accommodation for a (downhill) skiing vacation trip**

- ❑ **How do the accommodation alternatives differ from each other?**
  - ❑ What are the attributes that influence your decision?



Motivation

- **So far:**
  - We have considered decision-making situations in which the DM has one objective (e.g., maximize the expected value/utility of a monetary payoff)

- **This time:**
  - We consider decision-making situations in which the DM has multiple objectives or, more precisely…
  - Multiple attributes with regard to which the achievement of some fundamental objective is measured

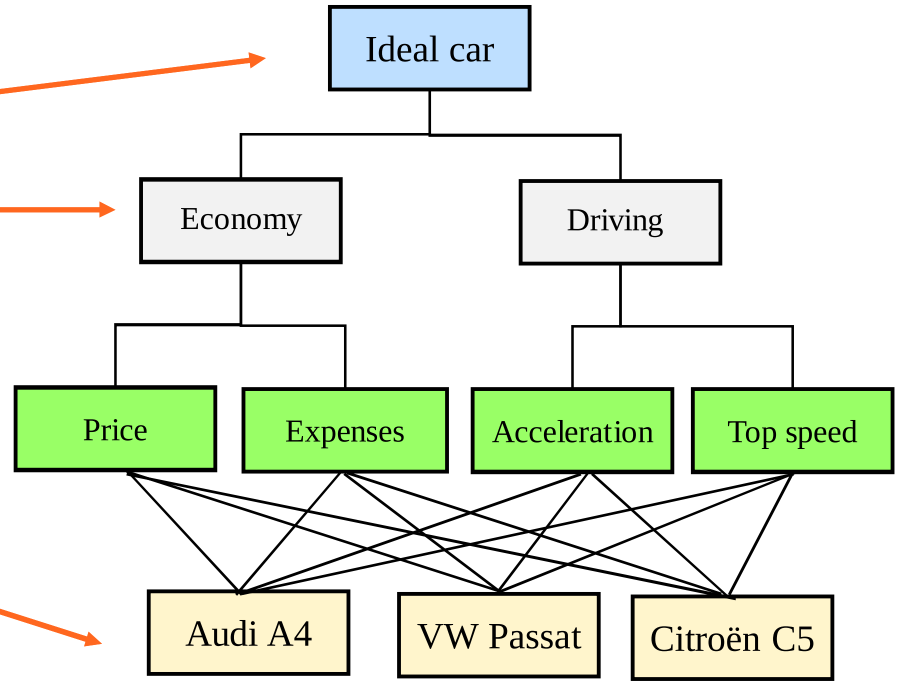
Multiattribute value theory

- Ralph Keeney and Howard Raiffa (1976): Decisions with Multiple Objectives: Preferences and Value Tradeoffs
- James Dyer and Rakesh Sarin (1979): Measurable multiattribute value functions, *Operations Research* Vol. 27, pp. 810-822

- Elements of MAVT
  - A value tree consisting of objectives, attributes, and alternatives
  - Preference relation over the alternatives' attribute-specific performances and differences thereof & their representation with an <u>attribute-specific value function</u>
  - Preference relation over the alternatives' overall performances and differences thereof & their representation with a <u>multiattribute value function</u>

Value tree: objectives, attributes, and alternatives

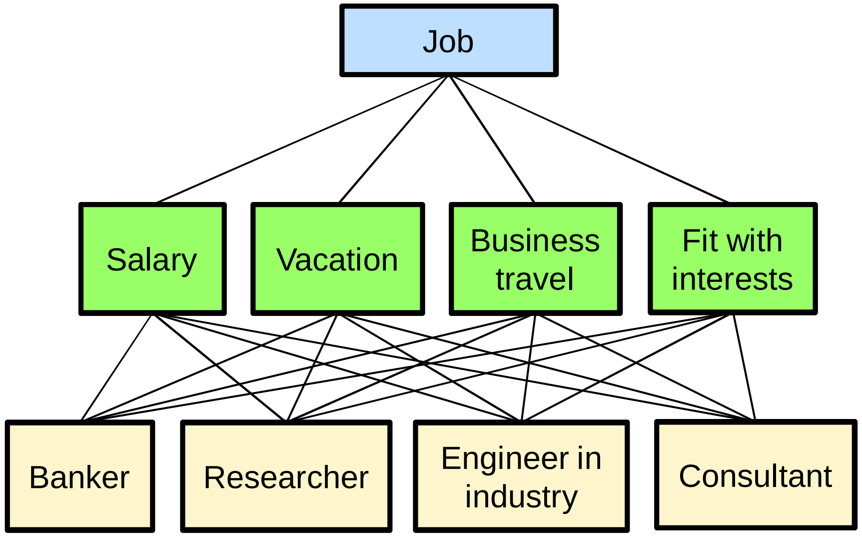
- A value tree consists of
  - A fundamental objective
  - Possible lower-level objectives
  - Attributes that measure the achievement of the objectives
  - Alternatives whose attribute-specific performances are being measured



Value tree: objectives, attributes and alternatives

- The attributes $a_1, \ldots, a_n$ have *measurement scales* X_i, i=1,…,n; e.g.,
  - X_1=[1000€/month, 6000€/month]
  - X_2 =[2 weeks/year, 8 weeks/year]
  - X_3 =[0 days/year, 200 days/year]
  - X_4 ={poor, fair, good, excellent}

- Alternatives $x = (x_1, x_2, \ldots x_n)$ are characterized by their performance w.r.t. the attributes; e.g.,
  - Banker=(6000€/month, 5 weeks/year, 40 days/year, fair) $\in X_1 \times X_2 \times X_3 \times X_4$.



Preference relation: attribute-specific performance

- Let $\succcurlyeq$ be preference relation among performance levels *a* and *b* on a given attribute

Preference $a \succcurlyeq b$: a **at least as preferable as** b

Strict preference $a \succ b$ **defined as** $\neg(b \succcurlyeq a)$

Indifference $a \sim b$ **defined as** $a \succcurlyeq b \ \wedge b \succcurlyeq a$

Axioms for preference relation

- **A1:** $\succcurlyeq$ is complete
  - For any $a, b \in X$, either $a \succcurlyeq b$ or $b \succcurlyeq a$ or both
- **A2:** $\succcurlyeq$ is transitive
  - If $a \succcurlyeq b$ and $b \succcurlyeq c$, then $a \succcurlyeq c$

Ordinal value function

Theorem: Let axioms **A1-A2** hold. Then, there exists an ordinal value function $v_i(\cdot): X_i \rightarrow \mathbb{R}$ that represents preference relation $\succcurlyeq$ in the sense that

$$v_i(a) \geq v_i(b) \Leftrightarrow a \succcurlyeq b$$

- An ***ordinal*** value function does not describe strength of preference, i.e., it does not communicate much more an object is preferred to another

Ordinal value function

- Assume you have two mopeds A and B with top speeds of 30 and 35km/h, respectively
- You have two alternatives for upgrade
  - Increase top speed of moped A to 40
  - Increase top speed of moped B to 45
- Your prefer a higher top speed to a lower one
  - 45>40>35>30
  - v(45)=1, v(40)=0.8, v(35)=0.5, v(30)=0.4
  - w(45)=0.9, w(40)=0.8, w(35)=0.6, w(30)=0.4
- Both v and w are ordinal value functions representing your preferences but they do not describe your preferences between the two upgrade alternatives
  - v(45)-v(35)=0.5 > v(40)-v(30)=0.4, but w(45)-w(35)=0.3 < w(40)-w(30) =0.4

Ordinal value function

Theorem: Ordinal value functions $v_i(\cdot)$ and $w_i(\cdot)$ represent the same preference relation $\succcurlyeq$ **if and only if** there exists a strictly increasing function $\phi: \mathbb{R} \to \mathbb{R}$ such that $w_i(a) = \phi[v_i(\cdot)]\ \ \forall a \in A$**.**

Example: Let *consultant* $\succ$ *professor* $\succ$ *janitor* be Jim's preferences over these jobs and v(consultant) = 10 > v(professor) = 8 > v(janitor) = 7.

Then v' and v'' **both represent the same preferences as ordinal value function** v

	consultant	professor	janitor
v	10	8	7
v'	20	16	14
v''	20	16	8

The goal is to compare <u>multi-attribute</u> alternatives, wherefore ordinal value functions are not enough

- Let $\succcurlyeq_d$ be preference relation among <u>differences</u> in performance levels on a given attribute
  - Preference $(a \leftarrow b) \succcurlyeq_d (c \leftarrow d)$: a change from b to a is at least as preferable as a change from d to c
  - Strict preference $(a \leftarrow b) \succ_d (c \leftarrow d)$ defined as $\neg((c \leftarrow d) \succcurlyeq_d (a \leftarrow b))$
  - Indifference $(a \leftarrow b) \sim_d (c \leftarrow d)$ defined as $(a \leftarrow b) \succcurlyeq_d (c \leftarrow d) \wedge (c \leftarrow d) \succcurlyeq_d (a \leftarrow b)$

Axioms for preference relation (cont’d)

- **A3:** $\forall a, b, c \in X_i$: $a \succcurlyeq b \Leftrightarrow (a \leftarrow b) \succcurlyeq_d (c \leftarrow c)$
  – If a is preferred to b, then a change from b to a is preferred to no change
- **A4:** $\forall a, b, c, d \in X_i$: $(a \leftarrow b) \succcurlyeq_d (c \leftarrow d) \Leftrightarrow (d \leftarrow c) \succcurlyeq_d (b \leftarrow a)$
  – E.g., if an increase in salary from 1500€ to 2000€ is preferred to an increase from 2000€ to 2500€, then a decrease from 2500€ to 2000€ is preferred to a decrease from 2000€ to 1500€
- **A5:** $\forall a, b, c, d, e, f \in X_i$: $(a \leftarrow b) \succcurlyeq_d (d \leftarrow e) \wedge (b \leftarrow c) \succcurlyeq_d (e \leftarrow f) \Rightarrow (a \leftarrow c) \succcurlyeq_d (d \leftarrow f)$
  – If two incremental changes are both preferred to some other two, then the overall change resulting from the first two increments is also preferred.
- **A6:** $\forall b, c, d \in X_i$ $\exists a \in X_i$ such that $(a \leftarrow b) \sim_d (c \leftarrow d)$ and $\forall b, c \in X_i$ $\exists a \in X_i$ such that $(b \leftarrow a) \sim_d (a \leftarrow c)$
  – Equally preferred differences between attribute levels can always be constructed
  – There is always an attribute level a between b and c such that a change from c to a is equally preferred to a change from a to b.
- **A7:** The set (or sequence) $\{a_n | b \succ a_n \ where (a_n \leftarrow a_{n-1}) \sim_d (a_1 \leftarrow a_0)\}$ is finite for any *b* in X_i
  – The sequence of equally preferred differences over a fixed interval is finite
  – “No b can be infinitely better than other performance levels”

As French (1988) incorrectly puts it; the idea here is that it is possible to construct equally preferred changes in order to represent preferences

Cardinal value function

- **Theorem:** Let axioms **A1-A7** hold. Then, there exists a <u>cardinal</u> value function $v_i(\cdot)$: $X_i \rightarrow \mathbb{R}$ that represents preference relations $\succcurlyeq$ and $\succcurlyeq_d$ in the sense that

$$v_i(a) \geq v_i(b) \Leftrightarrow a \succcurlyeq b$$
$$v_i(a) - v_i(b) \geq v_i(c) - v_i(d) \Leftrightarrow (a \leftarrow b) \succcurlyeq_d (c \leftarrow d).$$

Note: A cardinal value function is unique up to positive affine transformations, i.e., $v_i(x)$ and $v_i'(x) = \alpha v_i(x) + \beta, \alpha > 0$ and represent the same preferences

Cardinal value function: positive affine transformations

Example: Let *consultant* $\succ$ *professor* $\succ$ *janitor* and $(consultant \leftarrow professor) \succcurlyeq_d (professor \leftarrow janitor)$ be Jim's preferences and v(consultant) = 10 > v(professor) = 8 > v(janitor) = 7.

Then v' and v'' **both represent same preferences as cardinal value function** v

	consultant	professor	janitor
v	10	8	7
$v' = 2v$	20	16	14
$v'' = v' - 10$	10	6	4

Attribute-specific value functions

- A value function maps the attribute-specific measurement scale onto a numerical scale in accordance with the DM's preferences

- Value and utility:
  – Value is a measure of preference under certainty
  – Utility is a measure of preference under uncertainty

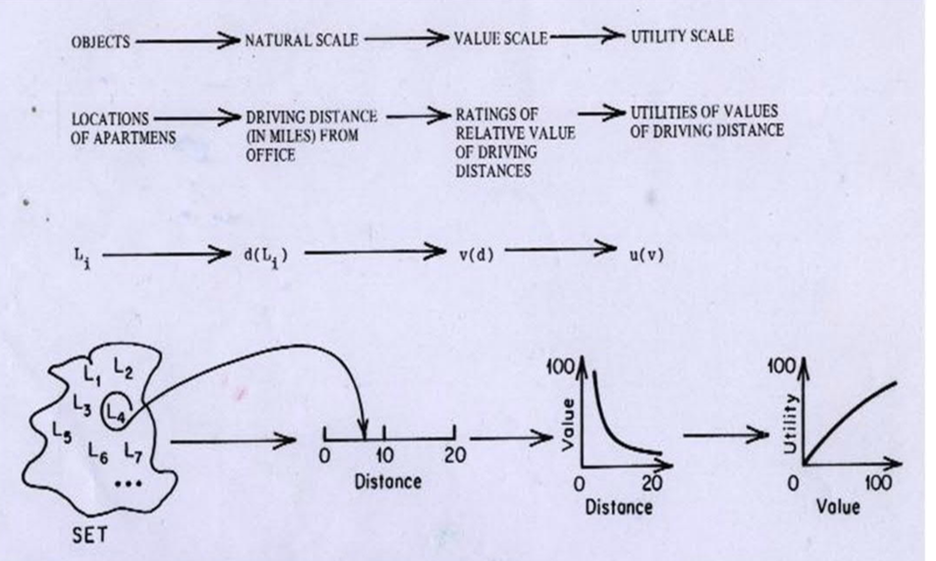


Figure 7.2. The four steps needed to construct value and utility functions.

Elicitation of value functions

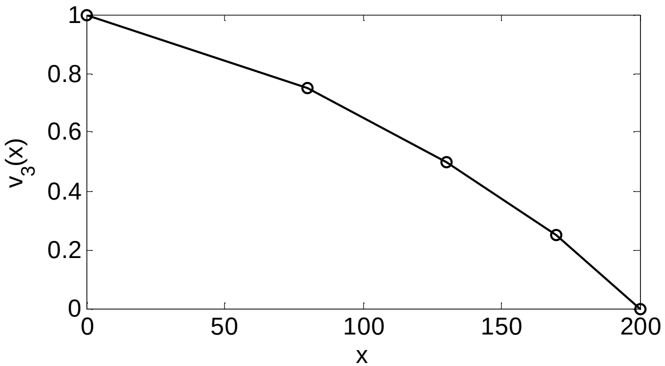
- **Phases:**
  - Define the measurement scale $X_i = [a_i^0, a_i^*]$ (or $[a_i^*, a_i^0]$)
  - Ask a series of eliciation questions
  - Check that the value function gives realistic results

Elicitation of value functions: Indifference methods

- **Bisection method:**
  - Ask the DM to assess level $x_{0.5} \in [a_i^0, a_i^*]$ such that she is indifferent between change $x_{0.5} \leftarrow a^0$ and change $a^* \leftarrow x_{0.5}$.
  - Then, ask her to assess levels $x_{0.25}$ and $x_{0.75}$ such that she is indifferent between
    - changes $x_{0.25} \leftarrow a^0$ and $x_{0.5} \leftarrow x_{0.25}$, and
    - changes $x_{0.75} \leftarrow x_{0.5}$ and $a^* \leftarrow x_{0.75}$.
  - Continue until sufficiently many points have been obtained
    - Use, e.g, linear interpolation between elicited points if needed
  - The value function can be obtained by fixing $v_i(a_i^0)$ and $v_i(a_i^*)$ at, e.g., 0 and 1

Elicitation of value functions: Indifference methods

- Example of the bisection method
  - Attribute a_3 : Traveling days per year
  - Measurement scale $[a_3^*, a_3^0]$, where a_3^* = 0 and a_3^0 = 200; fix $v_3(a_3^0)$ =0 and $v_3(a_3^*)$ =1
    - ”What would be the number $x_{0.5}$ of traveling days such that you would be indifferent between a decrease from 200 to $x_{0.5}$ days a year and a decrease from $x_{0.5}$ to zero days a year?” (Answer e.g., ”130”)
    - ”What would be the number $x_{0.25}$ of traveling days such that you would be indifferent between a decrease from 200 to $x_{0.25}$ days a year and a decrease from $x_{0.25}$ to 130 days a year?” (Answer e.g., ”170”)
    - ”What would be the number $x_{0.75}$ of traveling days such that you would be indifferent between a decrease from 130 to $x_{0.75}$ days a year and a decrease from $x_{0.75}$ to zero days a year?” (Answer e.g., ”80”)



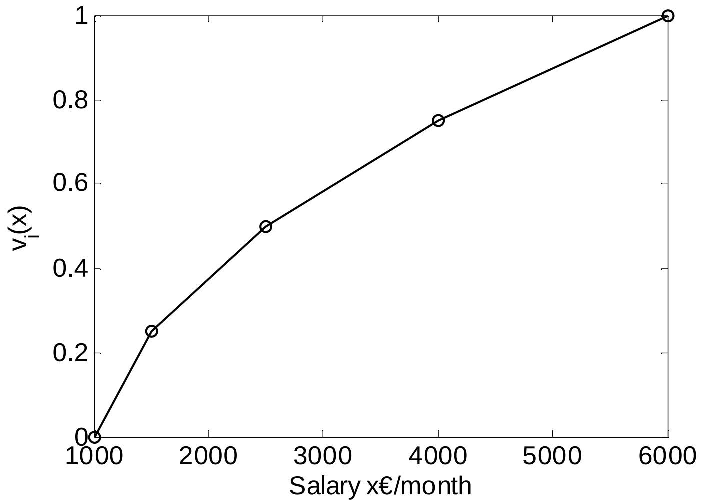
$$v_3(130) - v_3(200) = v_3(0) - v_3(130) \Rightarrow$$
$$v_3(130) = \frac{v_3(0) + v_3(200)}{2} = 0.5$$

$$v_3(170) - v_3(200) = v_3(130) - v_3(170) \Rightarrow$$
$$v_3(170) = \frac{v_3(130) + v_3(200)}{2} = 0.25$$

$$v_3(80) - v_3(130) = v_3(0) - v_3(80) \Rightarrow$$
$$v_3(80) = \frac{v_3(0) + v_3(130)}{2} = 0.75$$

Elicitation of value functions: Indifference methods

- **Sequence of equally preferred differences:**
  - Set $x_0 \in (a_i^0, a_i^*)$
  - Ask the DM to assess level $x_1 \in (x_0, a_i^*]$ such that he is indifferent between changes $x_0 \leftarrow a_i^0$ and $x_1 \leftarrow x_0$
    - $v_i(x_0) - v_i(a_i^0) = v_i(x_1) - v_i(x_0) \Rightarrow v_i(x_1){=}2v_i(x_0)$
  - Then, ask him to assess level $x_2 \in (x_1, a_i^*]$ such that he is indifferent between change $x_1 \leftarrow x_0$ and $x_2 \leftarrow x_1$
    - $v_i(x_1) - v_i(x_0) = v_i(x_2) - v_i(x_1) \Rightarrow v_i(x_2){=}3v_i(x_0)$
  - Continue until $x_N{=}a_i^*$ and solve the system of linear equations
    - $v_i(x_0) = \frac{v_i(x_N)}{N+1} = \frac{1}{N+1} \Rightarrow v_i(x_1){=}\frac{2}{N+1}$ etc.
  - If $x_N{>}a_i^*$ (see the exercises!)
    - Change a_i^* to x_N and interpolate, or
    - Interpolate to get $v_i(a_i^*) - v_i(a_i^0)$



Example:

$[\,a_i^0, a_i^*] = [1000, 6000]$, $x_0 = 1500$

$x_1 = 2500, x_2 = 4000, x_3 = 6000 = a_i^* \Rightarrow$

$v_i(1500) = \frac{1}{4}, v_i(2500) = \frac{1}{2}, v_i(4000) = \frac{3}{4}.$

Elicitation of value functions: Indifference methods

- Indifference methods are likely to result in a cardinal value function that captures the DM's preferences
- Therefore, they should be used whenever possible

- Yet: indifference methods cannot be used when the measurement scale is discrete
  - E.g., Fit with interest: X_4 ={poor, fair, good, excellent}
  - Cf. Axiom A6

Elicitation of value functions: direct methods

- Direct rating
  - Ask the DM to directly attach a value to each attribute level
  - E.g. ”Assume that the value of poor fit with interests is 0 and the value of excellent fit with interests is 1. What is the value of fair fit with interests? How about good fit?”
- Class rating
  - Divide the measurement scale into classes and ask the DM to attach a value to these classes
- Ratio evaluation
  - Take one attribute level as a reference point and ask the DM to compare the other levels to this
  - E.g., ”How many times more valuable is 1000€ than 900€?”

- Direct methods should be avoided whenever possible
  - Usually do not result in a cardinal value function

Next time: Aggregation of values

- **Problem:** How to measure the overall value of alternative $x = (x_1, x_2, \dots x_n)$?

$$V(x_1, x_2, \dots x_n) = ?$$

- **Question:** Could the overall value be obtained by aggregating attribute-specific values?

$$V(x_1, x_2, \dots x_n) = f\big(v(x_1), \dots, v(x_n)\big)?$$

- **Answer:** Yes, if the attributes are
  - Mutually preferentially independent and
  - Difference independent

Summary

- Under certain axioms, the DM’s preferences over changes on a measurement scale can be captured by a cardinal (measurable) value function
  - “I prefer a change from 0 euros to 10 euros to a change from 10 euros to 22 euros”

- Elicitation of the attribute-specific value functions
  - Use indifference methods if possible

Aalto University
School of Science

Decision making and problem solving – Lecture 5

- Preferential and difference independence
- Aggregation of values with an additive value function
- Interpretation and elicitation of attribute weights
- Trade-off methods
- SWING, SMART(S)

Liesiö, Punkka, Salo, Vilkkumaa

Last time

- Given certain axioms, a DM's preferences about a single attribute can be represented by a cardinal value function $v_i(x_i)$ such that

$$v_i(x_i) \geq v_i(y_i) \Leftrightarrow x_i \succcurlyeq y_i$$
$$v_i(x_i) - v_i(x_i') \geq v_i(y_i) - v_i(y_i') \Leftrightarrow (x_i \leftarrow x_i') \succcurlyeq_d (y_i \leftarrow y_i').$$

- Attribute-specific value functions are obtained by
  - Defining measurement scales $[x_i^0, x_i^*]$
  - Asking a series of elicitation questions through, e.g.,
    1. Bisection method
    2. Equally preferred differences
    3. Giving a functional form; e.g., $v_i(x_i)$ is linear and increasing
- Result: **<u>shape</u>** of the value function

- Value functions can be normalized such that $v_i\left(x_i^0\right)$ = 0 and $v_i(x_i^*)$ = 1.

This time

- How to measure the ***overall value*** of multi-attribute alternative $x = (x_1, x_2, \ldots x_n)$?

$$V(x_1, x_2, \ldots x_n) = ?$$

- Could the overall value be obtained by aggregating attribute-specific values?

$$V(x_1, x_2, \ldots x_n) = f\big(v(x_1), \ldots, v(x_n)\big) = \sum_{i=1}^{n} w_i v_i^N(x_i)\,?$$

- **Answer:** Yes, if the attributes are
  - Mutually preferentially independent and
  - Difference independent

- … But how to interpret and elicit ***attribute weights*** w_i?

Preferential independence

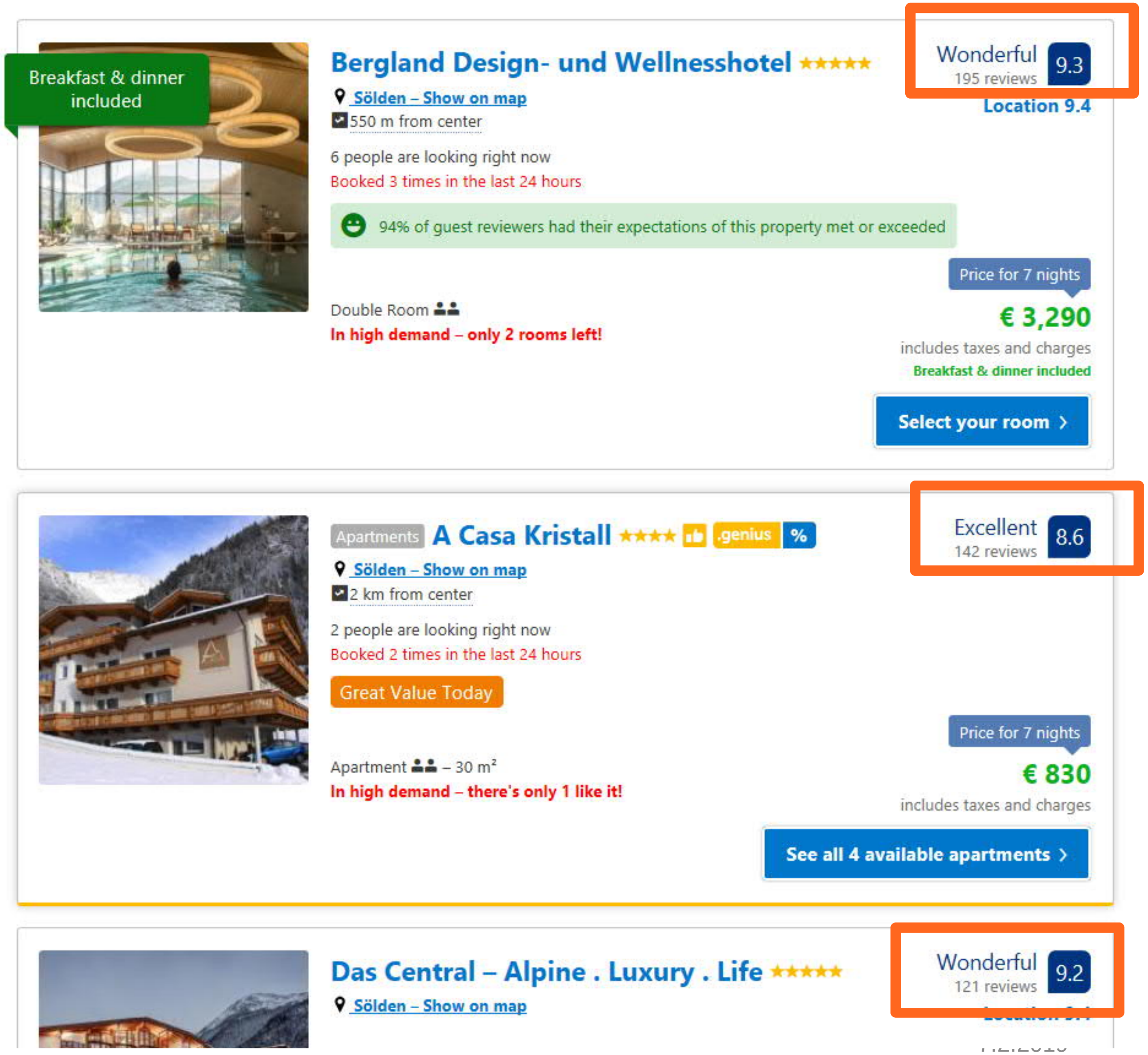
- **Definition:** Attribute *X* is preferentially independent of the other attributes ***Y***, if for all $x,x' \in X$

$$(x, \boldsymbol{y}') \succcurlyeq (x', \boldsymbol{y}') \Rightarrow (x, \boldsymbol{y}) \succcurlyeq (x', \boldsymbol{y}) \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}$$

- Interpretation: Preference over the level of attribute *X* does not depend on the levels of the other attributes, as long as they stay the same
  - "All other things Y being equal (no matter what they are), an alternative with performance level x w.r.t. X is preferred to an alternative with level x' ∈ X"

Last time

- ❑ **Consider yourself choosing accommodation for a (downhill) skiing vacation trip**
- ❑ **How do the accommodation alternatives differ from each other?**
  - ❑ What are the attributes that influence your decision?



Preferential independence: example 1

- Attribute *X* is preferentially independent of the other attributes ***Y***, if for all $x,x' \in X$

$$(x,\boldsymbol{y}') \succcurlyeq (x',\boldsymbol{y}') \Rightarrow (x,\boldsymbol{y}) \succcurlyeq (x',\boldsymbol{y}) \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}$$

- 2 Attributes
  - X={1,...,500} number of reviews
  - Y=[1,10] average of reviews
- Is *X* preferentially independent of *Y*?
  - No: $(500,10) \succcurlyeq (5,10)$, but $(500,1) \prec (5,1)$
- Is *Y* preferentially independent of *X*?
  - Yes (if higher average is preferred independently of #reviews, as long there are equally many reviews): $(500,10) \succcurlyeq (500,9) \Rightarrow (x,10) \succcurlyeq (x,9)$ for any x

Preferential independence: example 2

- Consider choosing a meal using two attributes:
  1. Food ∈ {beef, fish}
  2. Wine ∈ {red, white}
- Preferences:
  1. Beef is preferred to fish (no matter what the wine is):
     - (beef, red) ≽ (fish, red)
     - (beef, white) ≽ (fish, white)
  2. White wine is preferred with fish and red wine with beef
     - (fish, white) ≽ (fish, red)
     - (beef, red) ≽ (beef, white)
- Food **is** preferentially independent of wine
  - Beef is preferred to fish, no matter what the wine is: $(x, \boldsymbol{y'}) \succcurlyeq (x', \boldsymbol{y'}) \Rightarrow (x, \boldsymbol{y}) \succcurlyeq (x', \boldsymbol{y})$ for all y ∈ Y
- Wine **is not** preferentially independent of food
  - Attribute-specific valuation of wine is not meaningful from the meal's perspective

Mutual preferential independence

- **Definition:** Attributes A are mutually perferentially independent, if any subset of attributes **X**⊂A is preferentially independent of the other attributes **Y**=A**X**. I.e., for any **X**⊂A**, Y**=A**X:**

$$(\boldsymbol{x}, \boldsymbol{y}') \succcurlyeq (\boldsymbol{x}', \boldsymbol{y}') \Rightarrow (\boldsymbol{x}, \boldsymbol{y}) \succcurlyeq (\boldsymbol{x}', \boldsymbol{y}) \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}$$

- Interpretation: Preference over the levels of attributes ***X*** does not depend on the levels of the other attributes, as long as they stay the same

Mutual preferential independence: example

- Consider choosing a meal using three attributes:
  1. Food ∈ {beef, fish}
  2. Side dish ∈ {potato, rice}
  3. Wine ∈ {red, white}
- Preferences:
  1. All other things being equal, red ≽ white, beef ≽ fish, potato ≽ rice
  2. Full meals:
     - (beef, rice, red)≽(beef, potato, white)
     - (fish, potato, white) ≽ (fish, rice, red)

  Each attribute is preferentially independent of the other two, but the attributes are not mutually preferentially independent:

  $(\boldsymbol{y}', \boldsymbol{potato}, \boldsymbol{white}) \succcurlyeq (\boldsymbol{y}', \boldsymbol{rice}, \boldsymbol{red}) \not\Rightarrow (y, \boldsymbol{potato}, \boldsymbol{white}) \succcurlyeq (\boldsymbol{y}, \boldsymbol{rice}, \boldsymbol{red})$

Mutual pref. independence: example 2

- Choosing a car w.r.t. attributes *A*={*top speed, price, CO_2 emissions*}
  - Attributes defined on continuous scales
- Are all *A*'s subsets (X) preferentially independent of the other attributes (Y=A\X)?
- Each single attribute is preferentially independent of the other attributes, because
  - Lower price is preferred to higher price independent of other attributes (if other attributes are equal)
  - Higher top speed is preferred to lower
  - Smaller emissions are preferred to bigger ones

Mutual pref. independence: example 2

- Is X={*price*, *CO_2 emissions*} pref. independent of Y={*top speed*}?
  - Consider two cars which differ in price (e.g., 30000 e, 25000 e) and emissions (150 g/km, 200 g/km) so that one of the alternatives is better in emissions and the other in price. Set the same top speed for the alternatives (e.g. 230 km/h). Which one is better?
    - DM says (230 km/h, 30000 e, 150 g/km) $\succ$ (230 km/h, 25000 e, 200 g/km)
    - = when top speed is 230 km/h, she is willing to pay extra 5000 € on top of 25000 € for this emission reduction
  - Change the top speed. Is the first car still preferred to the second? e.g. does (150 km/h, 30000 e, 150 g/km) $\succ$ (150 km/h, 25000 e, 200 g/km) hold?
    - "No matter what the top speed is, (30000 e, 150 g/km) $\succ$ (25000 e, 200 g/km)"
  - Consider other prices and emissions; does your preference hold for all top speeds?
  - If varying the top speed does not influence preference between alternatives, then {price, CO_2 emissions} is preference independent of {top speed}

Difference independence

- **Definition**: Attribute *X* is difference independent of the other attributes ***Y*** if for all *x*,*x*' ∈ X

$$(x, \boldsymbol{y}') \leftarrow (x', \boldsymbol{y}') \sim_d (x, \boldsymbol{y}) \leftarrow (x', \boldsymbol{y}) \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}$$

- Interpretation: The preference over a <u>change</u> in attribute X does not depend on the levels of the other attributes **Y**, as long as they stay the same

Difference independence: example

- Is {*top speed*} difference independent of the other attributes {*price, CO_2 emissions*}?
  - Construct y and y' from any two levels of price and CO_2 emissions; y=(25000 e, 150 g/km) and y'=(30000 e, 200 g/km)
  - Consider any two levels of top speed; x'=200 km/h, x=250 km/h
  - Does (250 km/h, 30000 e, 200 g/km) ← (200 km/h, 30000 e, 200 g/km) $\sim_d$ (250 km/h, 25000 e, 150 g/km) ← (200 km/h, 25000 e, 150 g/km) hold?
    - If yes (for all *x*,*x*',***y***,***y'***), then difference independence holds
    - That is, does the value of increased top speed depend on the levels of other attributes or not?
    - Is the "amount of" value added by a fixed change in top speed independent of the other attributes?

Difference independence: example of implication

- We are choosing downhill skiing accommodation with regard to 6 attributes, which include *cost per night* (in €) and *possibility to go to sauna* (binary)
  - We think that (170 e, sauna, x_3, x_4, ...)~(145 e, no sauna, x_3, x_4, ...) with some $x_3,\ldots,x_6$ = we would pay an additional 25 € on top of 145 € for the sauna, with some $x_3,\ldots,x_6$
  - Then, if difference independence holds (for each attribute):

    (145 e, no sauna, x_3, x_4, ...) ← (170e, no sauna, x_3, x_4, ...) $\sim_d$

    (170 e, sauna, x_3, x_4, ...) ← (170 e, no sauna, x_3, x_4, ...) for any $x_3,\ldots,x_6$
  - For any $x_3,\ldots,x_6$ = ”No matter how close to nearest ski lifts , no matter how fancy the breakfast, how bad the reviews, etc.”

Implication: “the improvement needed in an attribute to compensate a loss in another attribute does not depend on the levels of other attributes”

Additive value function

Theorem: If all attributes are <u>mutually preferentially independent</u> and each attribute is <u>difference independent</u> of the others, then there exists an additive value function

$$V(x) = V(x_1, \dots, x_n) = \sum_{i=1}^{n} v_i(x_i)$$

which represents preference relations $\succcurlyeq$, $\succcurlyeq_d$ in the sense that

$$V(x) \geq V(y) \Leftrightarrow x \succcurlyeq y$$

$$V(x) - V(x') \geq V(y) - V(y') \Leftrightarrow (x \leftarrow x') \succcurlyeq_d (y \leftarrow y')$$

Note: The additive value function is unique up to positive affine transformations, i.e., V(x) and V'(x)=αV(x)+β, α>0 represent the same preferences

… But where are the attribute weights w_i?

Theorem: If all attributes are (…) , then there exists an additive value function

$$V(x) = V(x_1, \dots, x_n) = \sum_{i=1}^{n} v_i(x_i)$$

- **<u>Slide 3:</u>** Could the overall value be obtained by aggregating attribute-specific values?

$$V(x_1, x_2, \dots x_n) = f\big(v(x_1), \dots, v(x_n)\big) = \sum_{i=1}^{n} w_i v_i^N(x_i)\,?$$

Normalized form of the additive value function $V(x) = V(x_1, \dots, x_n) = \sum_{i=1}^{n} v_i(x_i)$

- **Denote**
  - x_i^0 = Least preferred level w.r.t to attribute i
  - x_i^* = Most preferred level w.r.t to attribute i
- **Then,**

$$V(x) = V(x) - V(x^0) + V(x^0)$$

$$= \sum_{i=1}^{n} v_i(x_i) - \sum_{i=1}^{n} v_i(x_i^0) + V(x^0) = \sum_{i=1}^{n} [v_i(x_i) - v_i(x_i^0)] + V(x^0)$$

$$= \sum_{i=1}^{n} \underbrace{[v_i(x_i^*) - v_i(x_i^0)]}_{W_i > 0} \frac{v_i(x_i) - v_i(x_i^0)}{v_i(x_i^*) - v_i(x_i^0)} + V(x^0)$$

$$= \sum_{i=1}^{n} W_i \left[\underbrace{\frac{1}{v_i(x_i^*) - v_i(x_i^0)}}_{\alpha_i > 0} v_i(x_i) + \underbrace{\frac{-v_i(x_i^0)}{v_i(x_i^*) - v_i(x_i^0)}}_{\beta_i} \right] + V(x^0) \dots$$

Normalized form of the additive value function (cont'd)

$$\ldots = \sum_{i=1}^{n} W_i \underbrace{[\alpha_i v_i(x_i) + \beta_i]}_{v_i^N \in [0,1]} + V(x^0)$$

Normalized attribute-specific value function $v_i^N(x_i) \in [0,1]$

$$= \sum_{i=1}^{n} \left[\left(\sum_{i=1}^{n} W_i\right) \cdot \underbrace{\frac{W_i}{\sum_{i=1}^{n} W_i}}_{= w_i > 0, \sum_{i=1}^{n} w_i = 1} \cdot v_i^N(x_i) \right] + V(x^0)$$

$$= \underbrace{\left(\sum_{i=1}^{n} W_i\right)}_{\chi > 0} \underbrace{\sum_{i=1}^{n} w_i v_i^N(x_i)}_{V^N(x)} + \underbrace{V(x^0)}_{\delta}$$

$$= \chi V^N(x) + \delta$$

Normalized additive value function $V^N(x) = \sum_{i=1}^{n} w_i v_i^N(x_i) \in [0,1]$

$V(x) = \chi V^N(x) + \delta$ is a positive affine transformation of $V^N(x)$; they represent the same preferences!

Interpretation of attribute weights

- By definition, $w_i = \frac{W_i}{\sum_{i=1}^{n} W_i} = \frac{v_i(x_i^*) - v_i(x_i^0)}{\sum_{i=1}^{n}(v_i(x_i^*) - v_i(x_i^0))} \propto v_i(x_i^*) - v_i(x_i^0)$

- Attribute weight w_i reflects the increase in overall value when the performance level on attribute a_i is changed from the worst level to the best – <u>relative to similar changes in other attributes</u>

- Weights thus reflect *trade-offs* between attributes; not their absolute ”importance”

- Elicitation of attribute weights without this interpretation is not meaningful
  - Do not ask: ”What is more important: environment or economy?”
  - Do ask: ”How much is society willing to pay to save an insect species?”

Interpretation of attribute weights

- Correct interpretation and hence application of the weights may lead to 'resistance'
  - Let the least preferred and the most preferred levels in
    - cost savings be 0 € and 1 B€ ("money")
    - the number of insect species saved from extinction in Finland be 0 and 1 ("environmental aspects")
    - Environmental aspects are likely to receive a small weight, as for example weighting (0.5, 0.5) would mean that we equally prefer saving 1 B€ and saving 1 species
  - Cf. .... Let the least preferred and the most preferred levels in
    - cost savings be 0 € and 1 B€
    - the number of insect species saved from extinction in Finland be 0 and 100

Conditions

- What if the conditions (mutual preferential independence and difference independence) do not hold?
  - Reconsider the attribute ranges $[a_i^0, a_i^*]$; conditions are more likely fulfilled when the ranges are small
  - Reconsider the attributes; are you using the right measures?

- Even if the conditions do not hold, additive value function is often used to obtain approximate results

Example (Ewing et al. 2006*): military value of an installation

- “How to realign US Army units and which bases to close in order to operate more cost-efficiently?”

- Many attributes, including ”total heavy maneuver area” (x_1) and ”largest contiguous area” (x_2; a measure of heavy maneuver area quality)

  - ”Total heavy maneuver area” is not difference independent of the other attributes $x_2 \cup \boldsymbol{y}''$ because (1000 ha, 100 ha, y’’) $\leftarrow$ (100 ha, 100 ha, y’’) $\sim_d$ (1000 ha, 10 ha, y’’) $\leftarrow$ (100 ha, 10 ha, y’’) as the ncrease from 100 to 1000 ha in total area is found quite useless, if total area consists of over 100 small isolated pieces of land

* Ewing, Tarantino, Parnell (2006): *Use of Decision Analysis in the Army Base Realignment and Closure (BRAC) 2005 Military Value Analysis*. Decision Analysis 3, 33-49

Example (Ewing et al. 2006*): military value of an installation

- Solution: unite the two attributes x_1 and x_2 into one attribute ”*heavy maneuver area*”
  - Then (1000 ha, 100 ha, Y) ← (100 ha, 100 ha, Y) $\succ_d$ (1000 ha, 10 ha, Y) ← (100 ha, 10 ha, Y) does not violate required difference independence conditions $(x, \boldsymbol{y}') \leftarrow (x', \boldsymbol{y}') \sim_d (x, \boldsymbol{y}) \leftarrow (x', \boldsymbol{y})$ for all $\boldsymbol{y} \in \boldsymbol{Y}$, because x_2 is no longer an element of y or y'
  - BUT we need to elicit preferences between different 'pairs' (x_1, x_2)

Largest contiguous area (1,000s acres)	Total heavy maneuver area (1,000s acres)			
	≤10	>10 and ≤50	>50 and ≤100	>100
≤10	0.1	0.2	1.4	2.0
>10 and ≤50		3.2	4.3	5.2
>50 and ≤100			6.1	7.6
>100				10.0

Elicitation of attribute weights

- Attribute weights are derived from the DM's preference statements

- Approaches to eliciting attribute weights:
  – Trade-off weighting
  – "Lighter" techniques: SWING, SMART(S), and ordinal methods

Trade-off weighting

- The DM is asked to
  1. Set the performance levels of two imaginary alternatives x and y such that they are equally preferred (x~y):

$$w_1 v_1^N(x_1) + \cdots + w_n v_n^N(x_n) = w_1 v_1^N(y_1) + \cdots + w_n v_n^N(y_n), \text{ or}$$

  2. Set the performance levels of four imaginary alternatives x, x', y, and y' such that changes x ← x' and y ← y' are equally preferred ($x \leftarrow x' \sim_d y \leftarrow y'$):

$$w_1(v_1^N(x_1) - v_1^N(x_1')) + \cdots + w_n(v_n^N(x_n) - v_n^N(x_n')) = w_1(v_1^N(y_1) - v_1^N(y_1')) + \cdots + w_n(v_n^N(y_n) - v_n^N(y_n'))$$

Trade-off weighting

- n-1 pairs of equally preferred alternatives/changes → n-1 linear constraints + 1 normalization constraint

- If the pairs are suitably selected (no linear dependencies), the system of n linear constraints has a unique solution
  - E.g., select a reference attribute and compare the other attributes against it
  - E.g., compare the ”most important” attribute to the second most important, the second most important to the third most important etc

Trade-off weighting: example (1/7)

- Consider two magazines A and B reporting a comparison of cars x^1 , x^2 , and x^3 , **based on the same expert appraisal**, using the **same attributes**:

	a_1: Top speed km/h	a_2: Acceleration 0-100 km/h	a_3: CO_2 emissions g/km	a_4: Maintenance costs €/year
x^1	192 km/h	12.0 s	120 g/km	400 €/year
x^2	200 km/h	10.4 s	140 g/km	500 €/year
x^3	220 km/h	8.2 s	150 g/km	600 €/year

Trade-off weighting: example (2/7)

- Consider changing top speed (reference attribute) **from 150 to 250 km/h**. All other things being equal, what would be an equally preferred change in
  - Acceleration time? Expert's answer: from 14 to 7 s ⇒

$$w_1\left(v_1^N(250)-v_1^N(150)\right)=w_2\left(v_2^N(7)-v_2^N(14)\right)\Rightarrow\frac{w_1}{w_2}=\frac{v_2^N(7)-v_2^N(14)}{v_1^N(250)-v_1^N(150)}$$

  - CO_2 emissions? Expert's answer: from 100 to 0 g/km ⇒

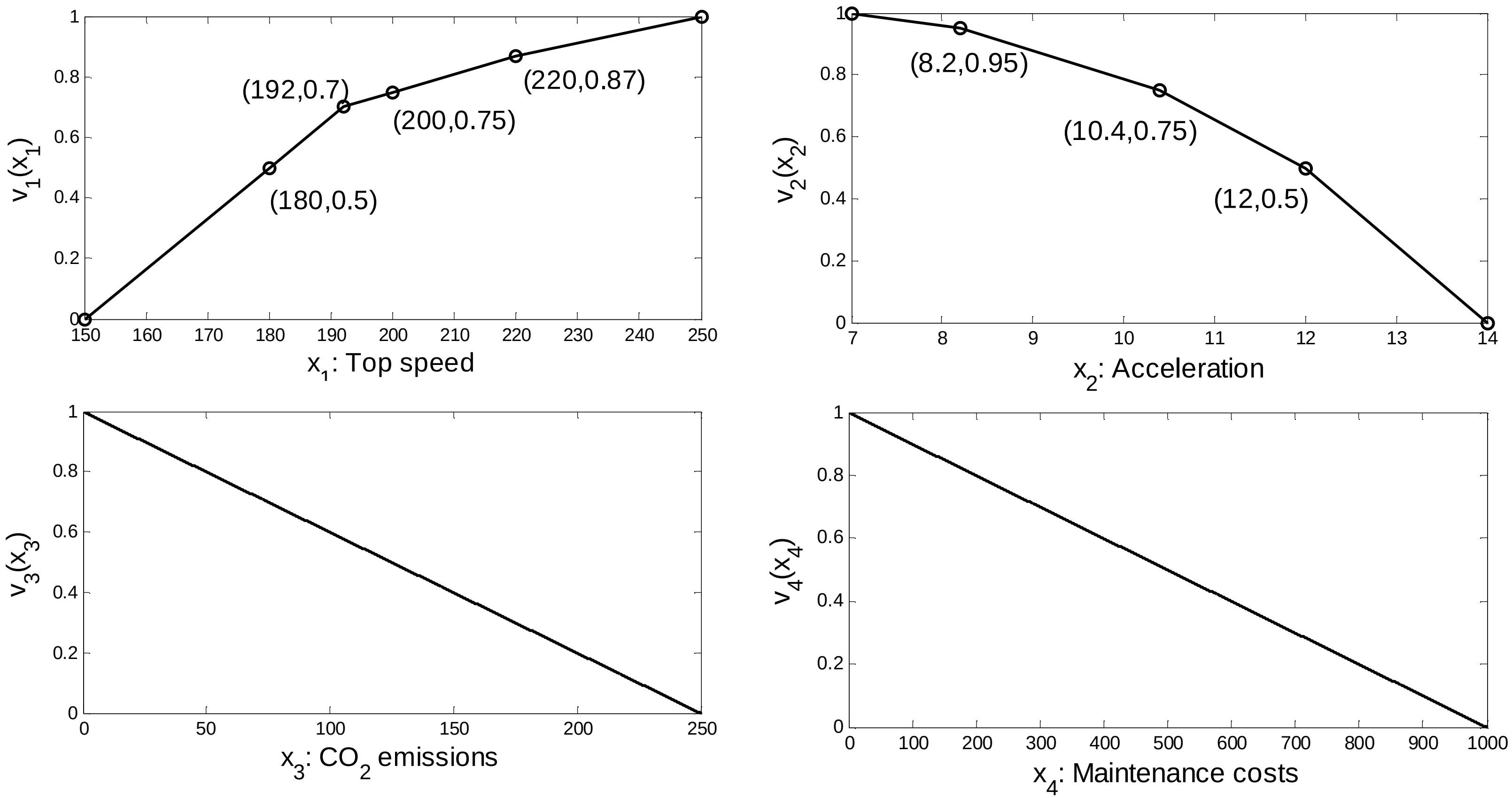
$$w_1\left(v_1^N(250)-v_1^N(150)\right)=w_3\left(v_3^N(0)-v_3^N(100)\right)\Rightarrow\frac{w_1}{w_3}=\frac{v_3^N(0)-v_3^N(100)}{v_1^N(250)-v_1^N(150)}$$

  - Maintenance costs? Expert's answer: from 800 to o €/year ⇒

$$w_1\left(v_1^N(250)-v_1^N(150)\right)=w_4\left(v_4^N(0)-v_4^N(800)\right)\Rightarrow\frac{w_1}{w_4}=\frac{v_4^N(0)-v_4^N(800)}{v_1^N(250)-v_1^N(150)}$$

Trade-off weighting: example (3/7)

- Attribute-specific value functions according to the expert:



Trade-off weighting: example (4/7)

- **Magazine A** uses the following measurement scales:

Attribute	Measurement scale	v_i^N
a_1: Top speed (km/h)	[150, 250]	$v_1^N(180) = 0.5$, $v_1^N(192) = 0.7$, $v_1^N(200) = 0.75$, $v_1^N(220) = 0.87$
a_2: Acceleration time (s)	[7, 14]	$v_2^N(12) = 0.5$, $v_2^N(10.4) = 0.75$, $v_2^N(8.2) = 0.95$
a_3: CO_2 emissions (g/km)	[120, 150]	$5 - x_3/30$
a_4: Maintenance costs (€/year)	[400,600]	$3 - x_4/200$

- $\frac{w_1}{w_2} = \frac{v_2^N(7) - v_2^N(14)}{v_1^N(250) - v_1^N(150)} = 1$
- $\frac{w_1}{w_3} = \frac{v_3^N(0) - v_3^N(100)}{v_1^N(250) - v_1^N(150)} = \frac{\frac{100}{30}(v_3^N(120) - v_3^N(150))}{1} = \frac{10}{3}$
- $\frac{w_1}{w_4} = \frac{v_4^N(0) - v_4^N(800)}{v_1^N(250) - v_1^N(150)} = \frac{\frac{800}{200}(v_3^N(400) - v_3^N(600))}{1} = 4$

- The three equalities and $\sum_{i=1}^{4} w_i = 1$ give $w_1 = w_2 = 0.39$, $w_3 = 0.12$, $w_4 = 0.10$.

Trade-off weighting: example (5/7)

- **Magazine A** reports the alternatives' attribute-specific values multiplied by 10 (i.e., scaled to interval [0,10]) and the attribute weights:

	v_1 : Top speed	v_2 : Acceleration	v_3 : CO_2	v_4 : Maintenance	Overall value
x^1	7	5	10	10	6.86
x^2	7.5	7.5	3.3	5	6.76
$\boldsymbol{x^3}$	**8.7**	**9.5**	**0**	**0**	**7.14**
Weights w_i	39%	39%	12%	10%	

- Possible (mis)interpretations / ”headlines”:
  - ”Only power matters – minor emphasis on costs and environment”
  - ”Car x^3 terrible w.r.t. CO_2 emissions and maintenance costs – yet, it's the expert's choice!”
  - ”No significant differences in top speed – differences are in CO_2 emissions and maintenance costs”

Trade-off weighting: example (6/7)

- **Magazine B** uses the following measurement scales:

Attribute	M. scale	v_i^N
a_1: Top speed	[192, 220]	$v_1^N(150) = -4.12,\ v_1^N(180) = -1.18,\ v_1^N(192) = 0,\ v_1^N(200) = 0.29,\ v_1^N(220) = 1,\ v_1^N(250) = 1.76$
a_2: Acceleration	[8.2, 12]	$v_2^N(14) = -1.11, v_2^N(12) = 0,\ v_2^N(10.4) = 0.56,\ v_2^N(8.2) = 1,\ v_2^N(7) = 1.11$
a_3: CO_2 emissions	[0, 250]	$1 - x_3/250$
a_4: Maintenance	[0,1000]	$1 - x_4/1000$

– $w_1\left(v_1^N(250) - v_1^N(150)\right) = w_2\left(v_2^N(7) - v_2^N(14)\right) \Rightarrow \frac{w_1}{w_2} = \frac{v_2^N(7)-v_2^N(14)}{v_1^N(250)-v_1^N(150)} = \frac{1.11+1.11}{1.76+4.12} = 0.378$

– $\frac{w_1}{w_3} = \frac{v_3^N(0)-v_3^N(100)}{v_1^N(250)-v_1^N(150)} = \frac{1-\frac{150}{250}}{1.76+4.12} = 0.068$

– $\frac{w_1}{w_4} = \frac{v_4^N(0)-v_4^N(800)}{v_1^N(250)-v_1^N(150)} = \frac{1-\frac{200}{1000}}{1.76+4.12} = 0.136$

- The three equalities and $\sum_{i=1}^4 w_i = 1$ give $w_1 = 0.039, w_2 = 0.103,\ w_3 = 0.572, w_4 = 0.286$.

Trade-off weighting: example (7/7)

- **Magazine B** reports the alternatives' attribute-specific values multiplied by 10 (i.e., scaled to interval [0,10]) and the attribute weights:

	v_1 : Top speed	v_2 : Acceleration	v_3 : CO_2	v_4 : Maintenance	Overall value
x^1	0	0	5.2	6	4.7
x^2	2.9	5.6	4.4	5	4.6
$\boldsymbol{x^3}$	**10**	**10**	**4**	**4**	**4.9**
Weights w_i	3.9%	10.3%	57.2%	28.6%	

- Possible (mis)interpretations:
  - ”Emphasis on costs and environmental issues”
  - ”x^3 wins only on the least important attributes – yet, it's the expert's choice!”
  - ”Car x^1 terrible w.r.t. top speed and acceleration time”

Trade-off weighting

- Weights reflect value differences over the measurement scales → changing the measurement scales changes the weights

- The attribute-specific values used in trade-off weighting take the measurement scales explicitly into account → weights represent the DM's preferences regardless of the measurement scales

- Trade-off weighting has a solid theoretical foundation and requires thinking; use whenever possible

SWING

Swing-weighting process:

1. Consider alternative $x^0 = (x_1^0, \dots, x_n^0)$ (each attribute on the worst level).
2. Choose the attribute a_j that you would first like to change to its most preferred level x_j^* (i.e., the attribute for which such a change is the most valuable). Give that attribute a (non-normalized) weight $W_j = 100$.
3. Consider x^0 again. Choose the next attribute a_k that you would like to change to its most preferred level. Give it weight $W_j \in (0,100]$ that reflects this improvement relative to the first one.
4. Repeat step 3 until all attributes have been weighted.
5. Obtain weights w_j by normalizing W_j.

SWING: example

- **Magazine A**'s measurement scales
  - Alternative $x^0 = \left(150\frac{km}{h}, 14s, 150\frac{g}{km}, 600\frac{€}{year}\right)$
  - The first attribute to be changed from the worst to the best level: $a_1 \rightarrow W_1 = 100$
  - The second attribute: $a_2 \rightarrow W_2 = 100$
  - The third attribute: $a_3 \rightarrow W_3 = 30$
  - The fourth attribute: $a_4 \rightarrow W_4 = 20$
  - Normalized weights: $w_1 = w_2 = 40\%$ $w_3 = 12\%$, $w_4 = 8\%$.

Attribute	Measurement scale
a_1: Top speed	[150, 250]
a_2: Acceleration	[7, 14]
a_3: CO_2 emissions	[120, 150]
a_4: Maintenance	[400,600]

About SWING weighting

- **The mode of questioning explicitly (but only) considers the least and most preferred levels of the attributes**
  - Assumes that the DM can directly numerically assess the strength of preference of changes between these levels

- **NOTE that we only have two preference relations: $\succcurlyeq$ and $\succcurlyeq_d$**
  - For example preference statement $W_1 = 100$, $W_4 = 20$ is equal to $v_1(x_1^*) - v_1(x_1^0) = 5[v_4(x_4^*) - v_4(x_4^0)]$, which assumes that there exist levels $x_1^{0.2}$, $x_1^{0.4}$, $x_1^{0.6}$, $x_1^{0.8}$ so that $(x_1^{0.2} \leftarrow x_1^0) \sim_d (x_1^{0.4} \leftarrow x_1^{0.2}) \sim_d \ldots \sim_d (x_1^* \leftarrow x_1^{0.8})$
    - Then $v_1(x_1^*) - v_1(x_1^0) = 5[v_1(x_1^{0.2}) - v_1(x_1^0)] = 5[v_4(x_4^*) - v_4(x_4^0)]$ if $(x_1^{0.2}, x_2, x_3, x_4\ ) \leftarrow (x_1^0, x_2, x_3, x_4\ ) \sim_d (x_1\ , x_2, x_3, x_4^*) \leftarrow (x_1\ , x_2, x_3, x_4^0)$

SMART

- Simple Multi-Attribute Rating Technique process:
  1. Select the least important attribute and give it a weight of 10 points.
  2. Select the second least important attribute and give it a weight (≥10 points) that reflects its importance compared to the least important attribute.
  3. Go through the remaining attributes in ascending order of importance and give them weights that reflect their importance compared to the less important attributes.
  4. Normalize the weights.

- This process does not consider the measurement scales at all → interpretation of weights is questionable

SMARTS

SMARTS = SMART using Swings

1. Select the attribute corresponding to the least preferred change from worst to best level and give it a weight of 10 points.
2. Go through the remaining attributes in ascending order of preference over changing the attribute from the worst to the best level, and give them weights that reflect their importance compared to the less preferred changes.
3. Normalize the weights.

SMARTS: example

- **Magazine A**'s measurement scales
  - Alternative $x^0 = \left(150\frac{km}{h}, 14s, 150\frac{g}{km}, 600\frac{€}{year}\right)$
  - Least preferred change from the worst to the best level: $a_4 \rightarrow W_4 = 10$
  - The second least preferred change: $a_3 \rightarrow W_3 = 20$
  - The third least preferred change : $a_2 \rightarrow W_2 = 40$
  - The fourth least preferred change: $a_1 \rightarrow W_1 = 40$
  - Normalized weights: $w_1 = w_2 = 36\%$, $w_3 = 18\%$, $w_4 = 9\%$.

Attribute	Measurement scale
a_1: Top speed	[150, 250]
a_2: Acceleration	[7, 14]
a_3: CO_2 emissions	[120, 150]
a_4: Maintenance	[400,600]

Empirical problems related to SWING & SMARTS

- People tend to use only multiples of 10 when assessing the weights, e.g.,
  - SWING: $W_1 = W_2 = 100, W_3 = 30, W_4 = 20 \rightarrow w_1 = w_2 = 0.40, w_3 = 0.12, w_4 = 0.08$
  - SMARTS: $W_1 = W_2 = 40, W_3 = 20, W_4 = 10 \rightarrow w_1 = w_2 = 0.36, w_3 = 0.18, w_4 = 0.09$
  - SWING and SMARTS typically produce different weights

- Assessments may reflect only ordinal, not cardinal information about the weights
  - E.g., SMARTS weights $W_4 = 10$ and $W_3 = 20$ only imply that $W_4 < W_3$, not that $W_3 / W_4 = 2$

Summary

- Additive value function describes the DM's preferences if and only if the attributes are mutually preferentially independent and each attribute is difference independent of the others
- The only meaningful interpretation for attribute weight w_i:

The improvement in overall value when attribute a_i is changed from its worst level to its best **relative to** similar changes in other attributes

- In trade-off weighting, attribute weights are elicited by specifying equally preferred alternatives (or changes in alternatives), which differ from each other on at least two attributes
  - Use trade-off weighting whenever possible

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Decision making and problem solving – Lecture 6

- Ordinal weighting methods
- Incomplete preference statements
- Modeling incomplete information
- Dominance and non-dominated alternatives
- Computing dominance relations
- Decision rules

Liesiö, Punkka, Salo, Vilkkumaa

Last time

- If the attributes are mutually preferentially independent and each attribute is difference independent of the others, then there exists an additive value function

$$V(x)=\sum_{i=1}^{n} w_i v_i^N(x_i)$$

such that

$$V(x) \geq V(y) \Leftrightarrow x \succcurlyeq y$$
$$V(x) - V(x') \geq V(y) - V(y') \Leftrightarrow (x \leftarrow x') \succcurlyeq_d (y \leftarrow y').$$

- Decision recommendation: choose the alternative with the highest overall value $V(x)$

Last time

- The <u>only</u> meaningful interpretation for attribute weight w_i:

The improvement in overall value when attribute a_i is changed from its worst level to its best **<u>relative to</u>** similar changes in other attributes

- Attribute weights cannot be interpreted without this interpretation
  - Changing the measurement scale changes the weights

- In trade-off weighting, attribute weights are elicited by specifying equally preferred alternatives (or changes in alternatives), which differ from each other on at least two attributes
  - Use trade-off weighting whenever possible

This time

- Specifying equally preferred alternatives requires quite an attempt. *Do we need such an exhaustive representation of preferences to produce defensible decision recommendations*?
  - Answer: Typically not, we can for example derive decision recommendations based only on ordinal information– like SWING without giving the points to the attributes
    - But... the simplest of such methods have severe problems
  - Answer2: Typically not, we learn how to
    - Accommodate <u>incomplete preference statements</u> in the decision model
    - Generate <u>robust decision recommendations</u> that are compatible with such statements

Ordinal weighting methods

- The DM is only asked to rank the attributes in terms of their importance (i.e., preferences over changing the attributes from the worst to the best level, cf. SWING)
  - $R_j = 1$ for the most important attribute
  - $R_j = n$ for the least important attribute

- This ranking is then converted into numerical weights such that these weights are compatible with the ranking
  - $w_i > w_j \Leftrightarrow R_i < R_j$

Ordinal weighting methods

- **Rank sum** weights are proportional to the opposite number of the ranks

$$w_i \propto (n - R_i + 1)$$

- **Rank exponent** weights are relative to some power of $(n - R_i + 1)$

$$w_i \propto (n - R_i + 1)^z$$

  – If z > 1 (z < 1), the power increases (decreases) the weights of the most important attributes compared to Rank sum weights.

e.g. attribute 1 more important

$$W_1 = 2 - 1 + 1 = 2$$
$$W_2 = 2 - 2 + 1 = 1$$

Normalize to get

$$w_1 = \frac{2}{3}, w_2 = \frac{1}{3}$$

Ordinal weighting methods

- **Rank reciprocal** weights are proportional to the inverse of the ranks

$$w_i \propto \frac{1}{R_i}$$

- **Centroid** weights are in the center of the set of weights that are compatible with the rank ordering
  - Order the attributes such that $w_1 \geq w_2 \geq \cdots \geq w_n$.
  - Then, the extreme points of the compatible weight set are (1,0,0,0...), (½, ½,0,0,...), (1/3, 1/3, 1/3,0,...),... (1/n,...,1/n).
  - The average of these extreme points is

$$w_i = \frac{1}{n}\sum_{j=i}^{n} \frac{1}{R_i}$$

Example: centroid weights

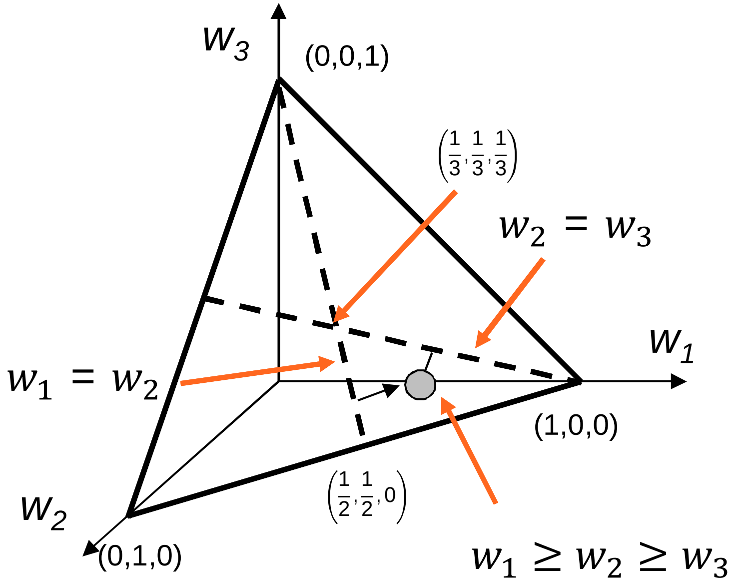
$$w_i = \frac{1}{n}\sum_{j=i}^{n}\frac{1}{R_i}$$

- Rank ordering $w_1 \geq w_2 \geq w_3$:

$$w_1 = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18} \approx 0.61$$

$$w_2 = \frac{1}{3}\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{5}{18} \approx 0.28$$

$$w_3 = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} \approx 0.11$$



Ordinal weighting methods: example

- Four attributes $\{a_1, a_2, a_3, a_4\}$ in descending order of importance $\rightarrow R_1 = 1, R_2 = 2, R_3 = 3, R_4 = 4$.

	a_1	a_2	a_3	a_4	$\sum$
Rank sum	4	3	2	1	10
weights	**0.4**	**0.3**	**0.2**	**0.1**	1
Rank exp(z=2)	16	9	4	1	30
weights	**0.53**	**0.30**	**0.13**	**0.03**	1
Rank reciprocal	1	1/2	1/3	1/4	25/12
weights	**0.48**	**0.24**	**0.16**	**0.12**	1
Centroid	25/48	13/48	7/48	3/48	1
weights	**0.52**	**0.27**	**0.15**	**0.06**	1

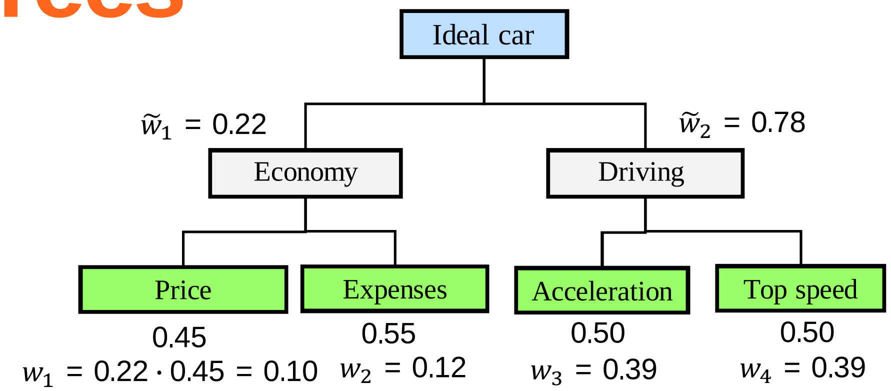
- Different methods produce different weights!

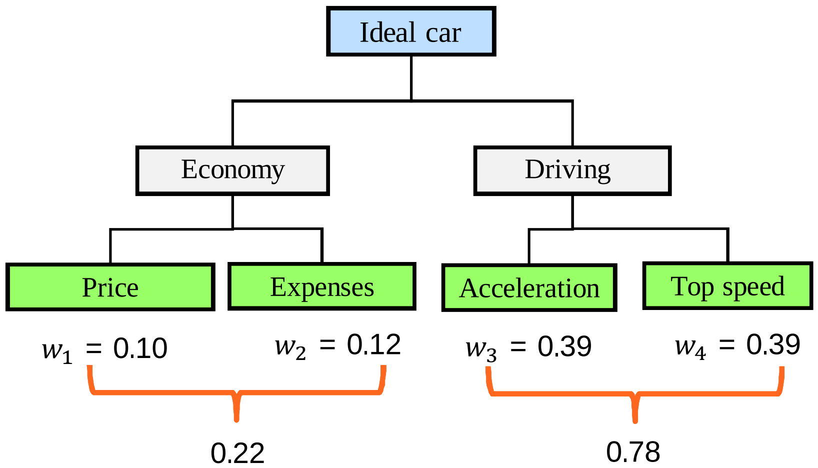
Ordinal weighting methods: example (cont'd)

- Assume that the measurement scale of the most important attribute a_1 is changed from [0€,1000€] to [0€,2000€].
- Because $w_1 \propto v_1(x_1^*) - v_1(x_1^0)$, the weight of attribute a_1 should be even larger.
- Yet,
  - Ranking among the attributes remains the same → rank-based weights remain the same
  - The alternatives' normalized scores on attribute a_1 become smaller → attribute a_1 has a smaller impact on the decision recommendation
- <u>Avoid using ordinal methods, which produce a "point estimate" weight</u>

Weighting in value trees

- Two modes of weighting
  - Hierarchical: all weights are elicited and then multiplied vertically
    - Problem: elicitation questions for the higher-level attributes are difficult to interpret:
      $\tilde{w}_1 = w_1 + w_2 \propto (v_1(x_1^*) - v_1(x_1^0)) + (v_2(x_2^*) - v_2(x_2^0))$
      $\rightarrow$ Avoid!
  - Non-hierarchical: weights are only elicited for the twig-level attributes





Recap: elements of MAVT

- Elements of MAVT:
  - Alternatives $X = \{x^1, \dots, x^m\}$
  - Attributes $A = \{a_1, \dots, a_n\}$
  - Attribute weights $w = [w_1, \dots, w_n] \in \mathbb{R}^n$
  - Attribute-specific (normalized) values $v \in \mathbb{R}^{m \times n}$, $v_{ji} = v_i^N\left(x_i^j\right) \in [0,1]$
  - Overall values of alternatives $V\left(x^j, w, v\right) = \sum_{i=1}^n w_i v_{ji}$, $j = 1, \dots, m$

Recap: Elicitation of attribute weights

- Defining **equally preferred** alternatives / changes between alternatives leads on a **linear equation** on the weights
  - E.g., ”All else being equal, a change 150 → 250 km/h in top speed is equally preferred to a change 14 → 7 s in acceleration time” ⇒

$$w_1 v_1^N(250) + w_2 v_2^N(14) + w_3 v_3^N(x_3) + w_4 v_4^N(x_4) - V\left(150, 14, x_3, x_4\right) =$$
$$w_1 v_1^N(150) + w_2 v_2^N(7) + w_3 v_3^N(x_3) + w_4 v_4^N(x_4) - V\left(150, 14, x_3, x_4\right) \Leftrightarrow$$
$$w_1 v_1^N(250) - w_1 v_1^N(150) = \mathbf{w_2} \mathbf{v_2^N}(7) - \mathbf{w_2} \mathbf{v_2^N}(14)$$

- **Question:** What if the DM finds it difficult or even impossible to define such alternatives / changes?
  - E.g., she can only state that a change 150 → 250 km/h in top speed is preferred to a change 14 → 7 s in acceleration time?

Incomplete preference statements

- Set the performance levels of two imaginary alternatives x and y such that
  $x \succcurlyeq y \Rightarrow$
  $$w_1 v_1^N(x_1) + \cdots + w_n v_n^N(x_n) \geq w_1 v_1^N(y_1) + \cdots + w_n v_n^N(y_n).$$

Attribute	Measurement scale
a_1: Top speed (km/h)	[150, 250]
a_2: Acceleration time (s)	[7, 14]
a_3: CO_2 emissions (g/km)	[120, 150]
a_4: Maintenance costs (€/year)	[400,600]

- For instance, a change 150 → 250 km/h in top speed is preferred to a change 14 → 7 s in acceleration time:
  $$w_1 v_1^N(250) + w_2 v_2^N(14) + w_3 v_3^N(x_3) + w_4 v_4^N(x_4) - V(150,14,x_3,x_4) \geq$$
  $$w_1 v_1^N(150) + w_2 v_2^N(7) + w_3 v_3^N(x_3) + w_4 v_4^N(x_4) - V(150,14,x_3,x_4)$$
  $$\Leftrightarrow w_1 \geq w_2$$
- Incomplete preference statements result in **linear inequalities** between the weights

Incomplete preference statements: example

- Consider attributes
  - CO_2 emissions $a_3 \in [120g, 150g]$
  - Maintenance costs $a_4 \in$ [400€ , 600€]

- Preferences are elicited with SMARTS:
  - Q: ”If the change 600€ → 400€ in maintenance costs is worth 10 points, how valuable is change 150g → 120g in CO_2 emissions?”
  - A: ”Between 15 and 20 points”

$$1.5w_4[v_4^N(400) - v_4^N(600)] \leq w_3[v_3^N(120) - v_3^N(150)] \leq 2w_4[v_4^N(400) - v_4^N(600)]$$

$$\Rightarrow 1.5w_4 \leq w_3 \leq 2w_4$$

Incomplete preference statements: example

- Preferences are elicited with trade-off methods:
  - Q: "Define an interval for x such that 600€ → 400€ in maintenance costs is as valuable as 150 g → x g in CO_2 emissions."
  - A: "x is between 130 and 140 g"

Attribute	Measurement scale
a_1: Top speed (km/h)	[150, 250]
a_2: Acceleration time (s)	[7, 14]
a_3: CO_2 emissions (g/km)	[120, 150]
a_4: Maintenance costs (€/year)	[400,600]

For x>140, the change in maintenance costs is more valuable

For x<130, the change in CO2 emissions is more valuable

$$w_3[v_3^N(\boxed{140}) - v_3^N(150)] \leq w_4[v_4^N(400) - v_4^N(600)] \leq w_3[v_3^N(\boxed{130}) - v_3^N(150)]$$

$$\Rightarrow v_3^N(140)w_3 \leq w_4 \leq v_3^N(130)w_3$$

$$\Rightarrow \frac{1}{3}w_3 \leq w_4 \leq \frac{2}{3}w_3, \text{ if } v_3^N \text{ is linear and decreasing.}$$

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Highest and lowest x for which equality possible

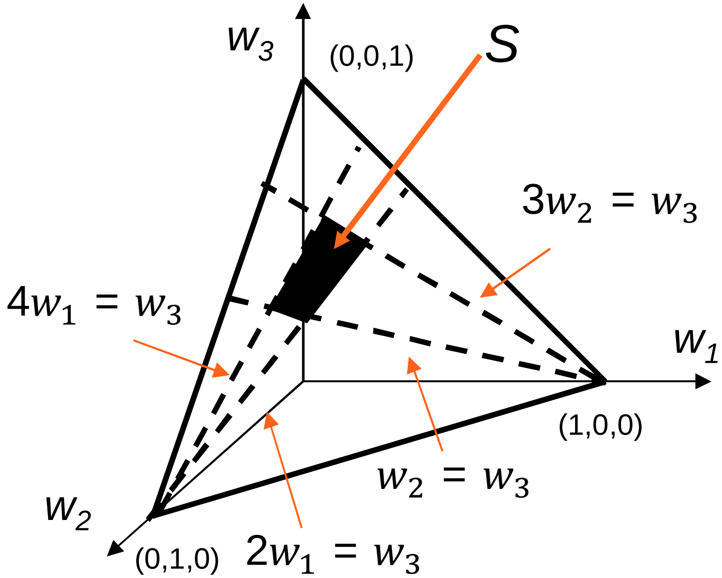
Modeling incomplete informaation

- Incomplete information about attribute weights is modeled as set S of feasible weights that are consistent with the DM's preference statements:

$$S \subseteq S^0 = \left\{ w \in \mathbb{R}^n | \sum_{i=1}^{n} w_i = 1, w_i \geq 0 \;\; \forall i \right\}$$

Modeling incomplete information

- Linear inequalities on weights can correspond to
  1. Weak ranking $w_i \geq w_j$
  2. Strict ranking $w_i - w_j \geq \alpha$
  3. Ranking with multiples $w_i \geq \alpha w_j$ (equivalent to incompletely defined weight ratios $w_i / w_j \geq \alpha$)
  4. Interval form $\alpha \leq w_i \leq \alpha + \varepsilon$
  5. Ranking of differences $w_i - w_j \geq w_k - w_l$



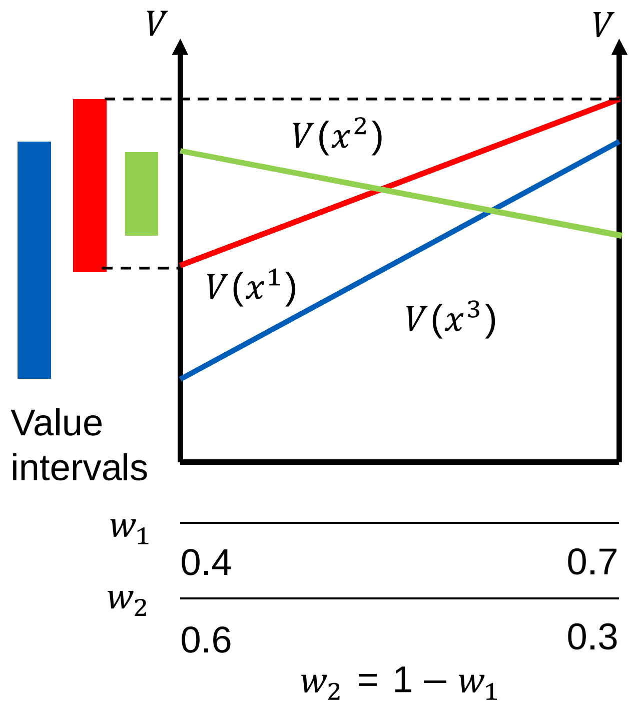
$w_2 \leq w_3 \leq 3w_2,$
$2w_1 \leq w_3 \leq 4w_1$

Overall value intervals

- Due to incompletely specified weights, the alternatives' overall values are <u>intervals:</u>

$$V(x,w,v) \in \left[\min_{w\in S} V(x,w,v)\,,\max_{w\in S} V(x,w,v)\right]$$

- Note: linear functions obtain their minima and maxima at an extreme point of S
  - E.g., $S = \{w \in S^0 \subseteq \mathbb{R}^2 | 0.4 \leq w_1 \leq 0.7\} \Rightarrow ext(S) = \{(0.4, 0.6), (0.7, 0.3)\}$



Dominance

- Preference over interval-valued alternatives can be established through a *dominance relation*

- **Definition:** x^k dominates x^j in S, denoted $x^k \succ_S x^j$, iff

$$\begin{cases} V(x^k, w, v) \geq V(x^j, w, v) \text{ for all } w \in S \\ V(x^k, w, v) > V(x^j, w, v) \text{ for some } w \in S \end{cases}$$

  i.e., iff the overall value of x^k is greater than or equal to that of x^j for all feasible weights and strictly greater for some.

Non-dominated alternatives

- An alternative is *non-dominated* if no other alternative dominates it

- The set of non-dominated alternatives is

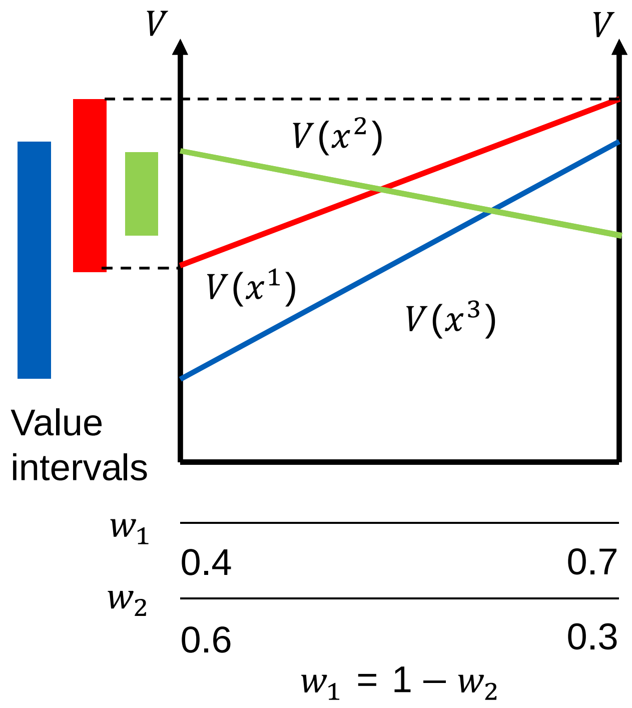
$$X_{ND} = \{x^k \in X \mid \nexists j \text{ such that } x^j \succ_S x^k\}$$

- X_{ND} contains all good decision recommendations
  – I.e., alternatives compared to which no other alternative has at least as high value for all feasible weights and strictly higher for some

Non-dominated alternatives

x^k is non-dominated if no other alternative has higher value than x^k for all feasible weights

- Alternative x^1 dominates x^3
- Alternatives x^1 and x^2 are non-dominated



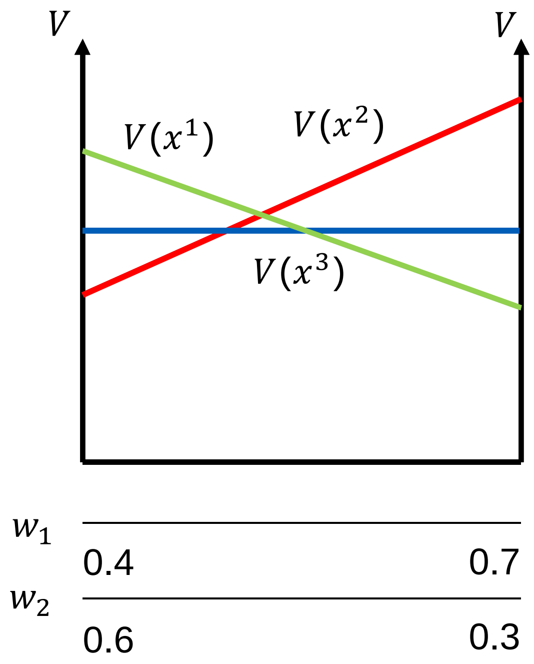
Non-dominated vs. potentially optimal alternatives

- A non-dominated alternative is not necessarily optimal for any $w \in S$

  – x^1, x^2 and x^3 are all non-dominated

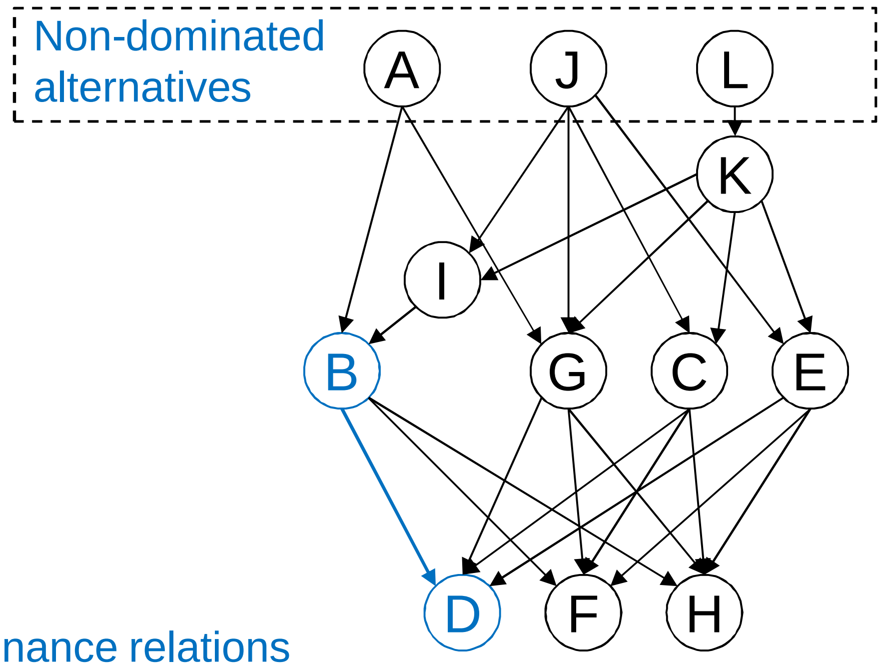
  – **Only** x^1 and x^2 are *potentially optimal* in that they maximize *V* for some $w \in S$

  – Still, neither of them can be guaranteed to be better than x^3



Properties of dominance relation

- Transitive
  - If A dominates B and B dominates C, then A dominates C
- Asymmetric
  - If A dominates B, then B does not dominate A
- Irreflexive
  - A does not dominate itself



Dominance relations expressed with a directed arc: B dominates D

Computing dominance relations

- If x^k dominates x^j:
  1. $V(x^k, w, v) \geq V(x^j, w, v)$ for all $w \in S$
     $\Leftrightarrow \min_{w \in S}[V(x^k, w, v) - V(x^j, w, v)] \geq 0 \Leftrightarrow \min_{w \in S}[\sum_{i=1}^{n} w_i (v_{ki} - v_{ji})] \geq 0$
  2. $V(x^k, w, v) > V(x^j, w, v)$ for some $w \in S$
     $\Leftrightarrow \max_{w \in S}[V(x^k, w, v) - V(x^j, w, v)] > 0 \Leftrightarrow \max_{w \in S}[\sum_{i=1}^{n} w_i (v_{ki} - v_{ji})] > 0$
- Dominance relations between two alternatives can thus be established by comparing their minimum and maximum value differences

Computing dominance relations: example

- Consider three cars with normalized attribute-specific values:

Car	v_1^N: Top speed	v_2^N: Acceleration	v_3^N: CO_2 emissions	v_4^N: Maintenance
x^1	0.7	0.5	1	1
x^2	0.75	0.75	0.33	0.5
x^3	0.87	0.95	0	0

- Incomplete preference statements have resulted in feasible set of weights *S*:

$$S = \{w \in S^0 \subseteq \mathbb{R}^4 | w_1 = w_2 \geq 3w_3,\ w_3 \geq w_4 \geq 0.1\}$$

Computing dominance relations: example

```
Values=[0.7 0.5 1 1; 0.75 0.75 0.33 0.5; 0.87 0.95 0 0];
A=[0 -1 3 0;0 0 -1 1;0 0 0 -1];
b=[0;0;-0.1];
Aeq=[1 -1 0 0;1 1 1 1];
beq=[0;1];
MinValueDiff=zeros(3,3);
MaxValueDiff=zeros(3,3);

for i=1:3
    for j=i+1:3
       [w,fval]=linprog((Values(i,:)-Values(j,:))',A,b,Aeq,beq);
       MinValueDiff(i,j)=fval;
       [w,fval]=linprog((Values(j,:)-Values(i,:))',A,b,Aeq,beq);
       MaxValueDiff(i,j)=-fval;
       MinValueDiff(j,i)=-MaxValueDiff(i,j);
       MaxValueDiff(j,i)=-MinValueDiff(i,j);
       if MinValueDiff(i,j)>=0 && MaxValueDiff(i,j)>0
          disp(['Alternative ' num2str(i) ' dominates ' num2str(j) '.'])
       elseif MinValueDiff(j,i)>=0 && MaxValueDiff(j,i)>0
          disp(['Alternative ' num2str(j) ' dominates ' num2str(i) '.'])
       end

    end
end
```

Matlab function linprog(f,A,b,Aeq,beq) solves the optimization problem:

$$\min_x f^T x \text{ such that}$$

$$\begin{cases} A \cdot x \leq b \\ Aeq \cdot x = beq \end{cases}$$

Computing dominance relations: example

- Minimum and maximum value differences

$$\min_{w\in S}[V(x^1,w,v) - V(x^2,w,v)] = -0.003 < 0$$
$$\max_{w\in S}[V(x^1,w,v) - V(x^2,w,v)] = 0.0338 > 0$$

→ Neither x^1 nor x^2 dominate the other

$$\min_{w\in S}[V(x^2,w,v) - V(x^3,w,v)] = -0.045 < 0$$
$$\max_{w\in S}[V(x^2,w,v) - V(x^3,w,v)] = -0.0163 < 0$$

→ x^3 dominates x^2

$$\min_{w\in S}[V(x^1,w,v) - V(x^3,w,v)] = -0.048 < 0$$
$$\max_{w\in S}[V(x^1,w,v) - V(x^3,w,v)] = 0.0175 > 0$$

→ Neither x^1 nor x^3 dominate the other

- $X_{ND} = \{x^1, x^3\}$

Computing dominance relations: example

- Note: because value differences are linear in *w*, minimum and maximum value differences are obtained at the extreme points of set *S*:

$$w^1 = (0.4\ 0.4\ 0.1\ 0.1)$$

$$w^2 = \left(\frac{27}{70}, \frac{27}{70}, \frac{9}{70}, \frac{1}{10}\right) \approx (0.386, 0.386, 0.129, 0.10)$$

$$w^3 = \left(\frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right) = (0.375, 0.375, 0.125, 0.125)$$

	w^1	w^2	w^3
$V(x^1)$-$V(x^2)$	-0.003	0.0204	0.0338
$V(x^2)-V(x^3)$	-0.045	-0.031	-0.0163
$V(x^1)-V(x^3)$	-0.048	-0.0106	0.0175

Additional information

- If information set *S* results in too many non-dominated alternatives, additional preference statements (i.e., linear constraints) can be elicited

- New information set $S' \subset S$ preserves all dominance relations and usually yields new ones $\rightarrow X_{ND}$ stays the same or becomes smaller

$$S' \subset S, ri(S) \cap S' \neq \emptyset: \begin{cases} x^k \succ_S x^j \Rightarrow x^k \succ_{S'} x^j \\ X_{ND}(S) \supseteq X_{ND}(S') \end{cases},$$

where $ri(S)$ is the relative interior of *S.*

– $ri(S) \cap S' \neq \emptyset$: S' is not entirely on the "border" of S

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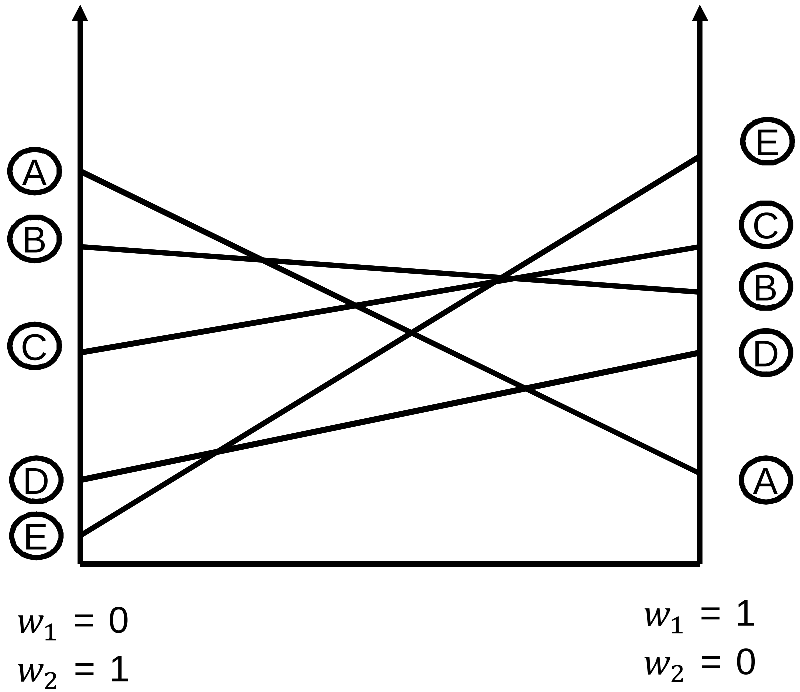
Additional information: example

- No weight information

$$S = S^0 = \left\{ w \in \mathbb{R}^2 \middle| \sum_{i=1}^{2} w_i = 1, w_i \geq 0 \right\}$$

- Dominance relations
  1. B dominates D
  2. C dominates D

- Non-dominated alternatives
  – A,B,C,E

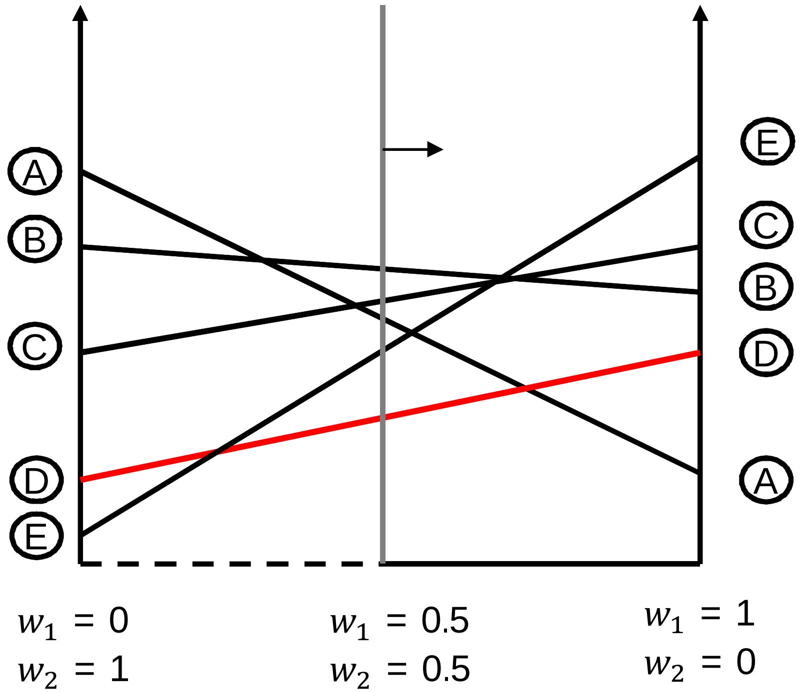


Additional information: example (2/3)

- Ordinal weight information
  $$S = \{w \in S^0 | w_1 \geq w_2\}$$

- Dominance relations
  1. B dominates D
  2. C dominates D
  3. E dominates D
  4. B dominates A
  5. C dominates A

- Non-dominated alternatives
  – B,C,E

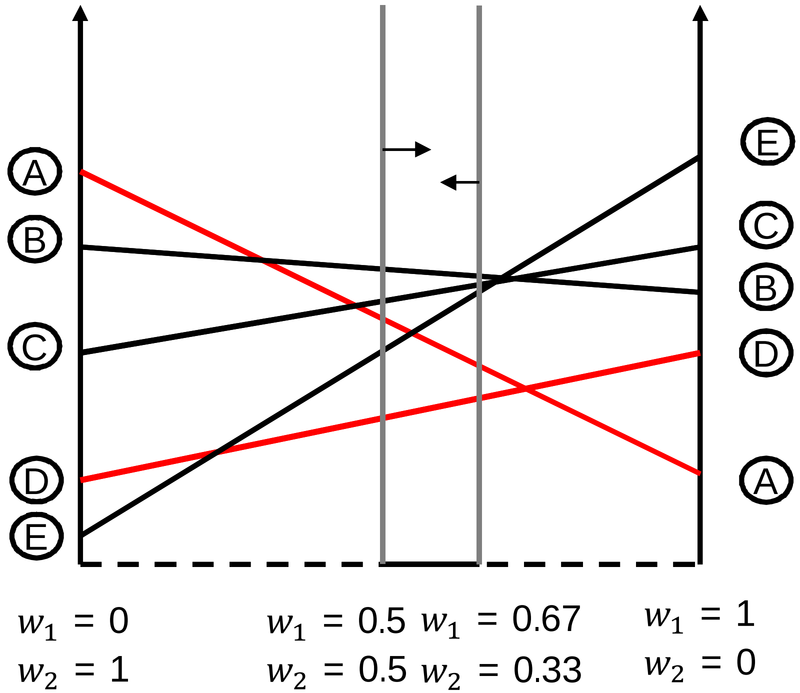


Additional information: example (3/3)

- More information

  $$S = \{w \in S^0 | w_2 \leq w_1 \leq 2w_2\}$$

- Dominance relations
  1. B dominates D
  2. C dominates D
  3. E dominates D
  4. B dominates A
  5. C dominates A
  6. B dominates C
  7. B dominates E



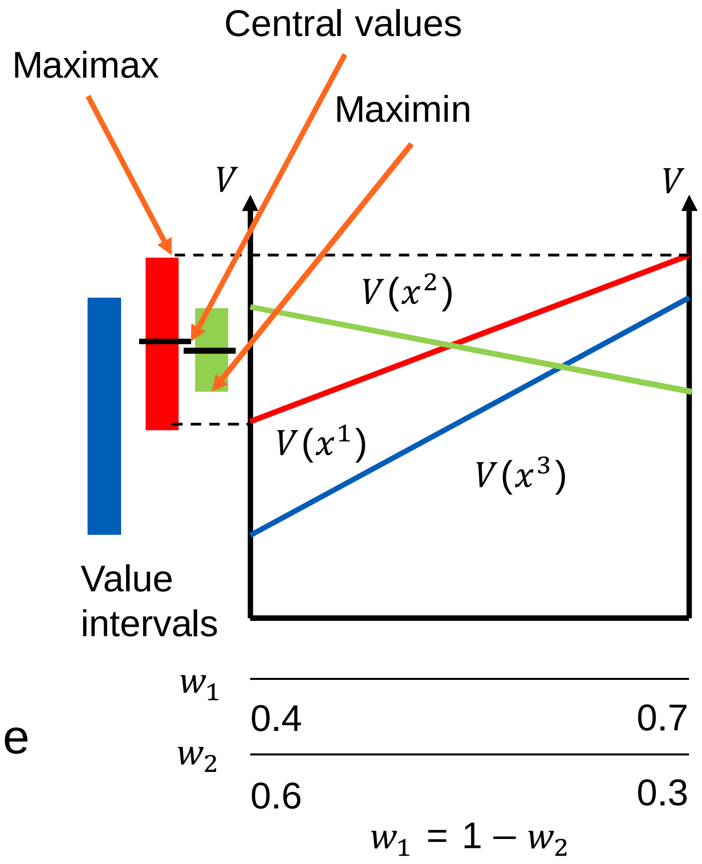
- Non-dominated alternatives: B

Value intervals

Can value intervals be used in deriving decision recommendations?

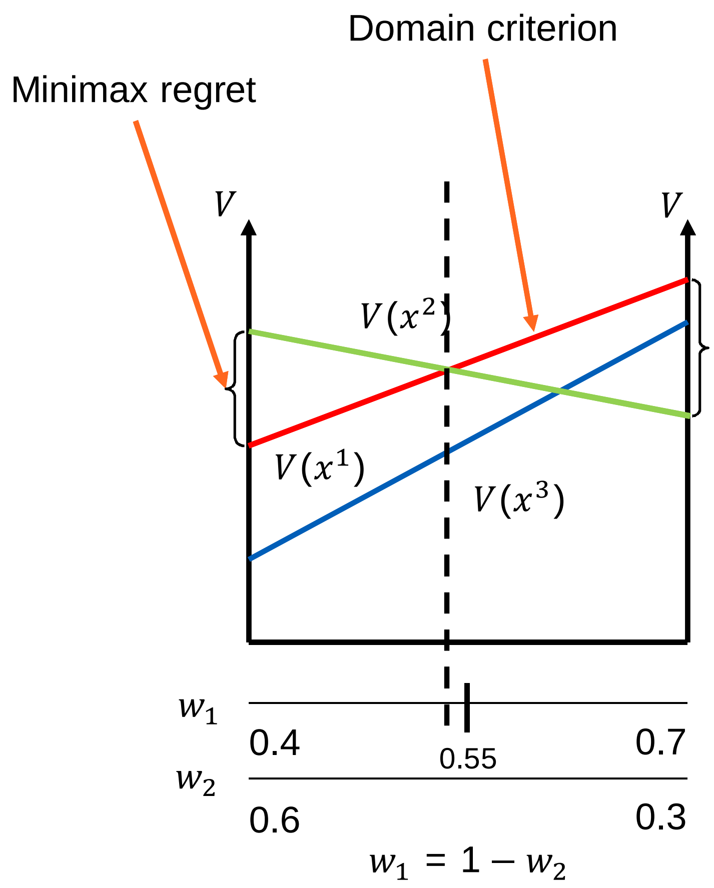
Some suggestions for "decision rules" from literature:

- **Maximax**: choose the alternative with the highest maximum overall value over the feasible weights
- **Maximin**: choose the alternative with the highest lowest overall value over the feasible weights
- **Central values**: choose the alternative with the highest sum of the maximum and minimum values



…more decision rules

- **Minimax regret**: choose the alternative with the smallest maximum regret (= value difference compared to any other alternative)
- **Domain criterion**: choose the alternative which is favored by the largest set of weights



Example

- DM asks 2 experts to compare fruit baskets (x_1,x_2) containing apples x_1 and oranges x_2
- Linear attribute-specific value functions v_1 and v_2
- DM: (2,0) >~ (0,1) and (0,2)>~(1,0)
  - One orange is not preferred to 2 apples, one apple is not preferred to 2 oranges
- Fruit baskets (1,2) and (2,1) do not dominate each other
- What do the decision rules recommend?

Expert 1: $x^0=(0,0)$, $x^*=(2,4)$

$$v_1^N(x_1)=\frac{x_1}{2}, v_2^N(x_2)=\frac{x_2}{4}$$

$$V(2,0)\geq V(0,1) \Leftrightarrow$$

$$\frac{2}{2}w_1+0w_2 \geq 0w_1+\frac{1}{4}w_2=\frac{1}{4}(1-w_1) \Leftrightarrow w_1 \geq \frac{1}{5}$$

$$V(0,2)\geq V(1,0) \Leftrightarrow$$

$$\frac{2}{4}w_2=\frac{1}{2}(1-w_1)\geq \frac{1}{2}w_1 \Leftrightarrow w_1 \leq \frac{1}{2}$$

$$V(x)=w_1\frac{x_1}{2}+w_2\frac{x_2}{4}=w_1\left(\frac{x_1}{2}-\frac{x_2}{4}\right)+\frac{x_2}{4}$$

Expert 2: $x^0=(0,0)$, $x^*=(4,2)$

$$v_1^N(x_1)=\frac{x_1}{4}, v_2^N(x_2)=\frac{x_2}{2}$$

$$V(2,0)\geq V(0,1) \Leftrightarrow$$

$$\frac{2}{4}w_1 \geq \frac{1}{2}w_2=\frac{1}{2}(1-w_1) \Leftrightarrow w_1 \geq \frac{1}{2}$$

$$V(0,2)\geq V(1,0) \Leftrightarrow$$

$$w_2=1-w_1 \geq \frac{1}{4}w_1 \Leftrightarrow w_1 \leq \frac{4}{5}$$

$$V(x)=w_1\left(\frac{x_1}{4}-\frac{x_2}{2}\right)+\frac{x_2}{2}$$

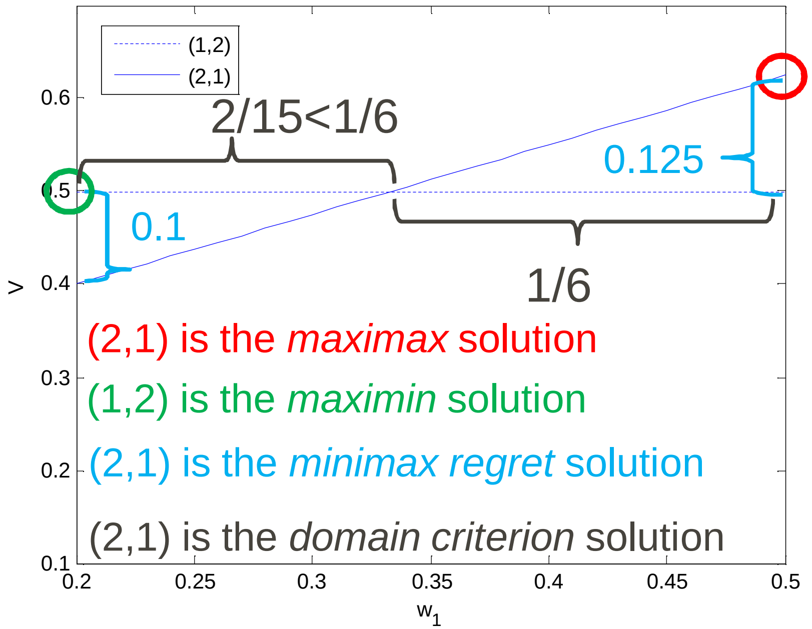
$$V(x) = w_1\left(\frac{x_1}{2} - \frac{x_2}{4}\right) + \frac{x_2}{4}$$

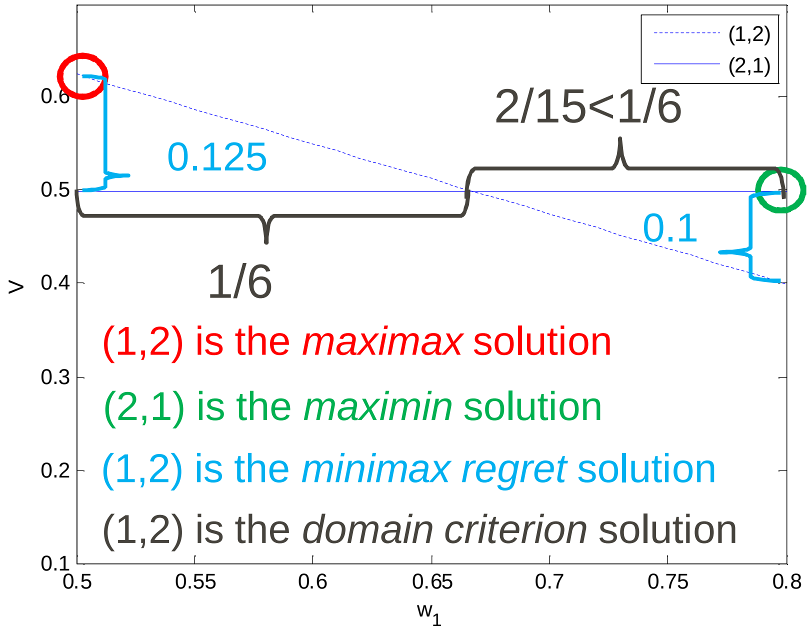
$$V(1,2) = w_1\left(\frac{1}{2} - \frac{2}{4}\right) + \frac{2}{4} \equiv \frac{1}{2}$$

$$V(2,1) = w_1\left(\frac{2}{2} - \frac{1}{4}\right) + \frac{1}{4} = \frac{3}{4}w_1 + \frac{1}{4}$$

$$V(x) = w_1\left(\frac{x_1}{4} - \frac{x_2}{2}\right) + \frac{x_2}{2}$$

$$V(1,2) = -\frac{3}{4}w_1 + 1$$





On decision rules

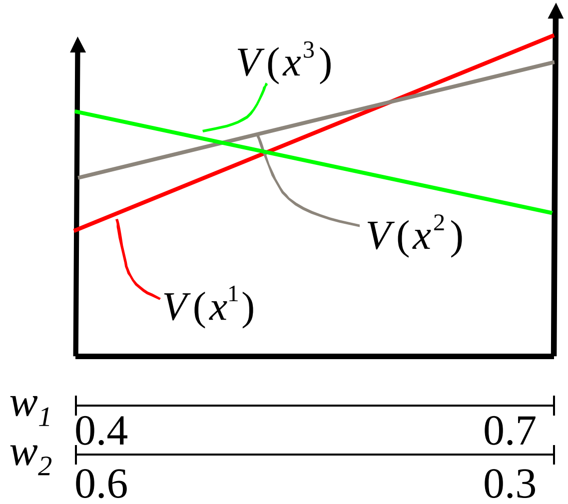
- A common problem for all of the above decision rules: changing the measurement scales $[x_i^0, x_i^*]$ can change the recommendations

- Different attribute weightings w and w* represent value functions V and V* – they cannot be compared
  - If V represents the DM's preferences, so do all its positive affine transformations, too
  - How to choose one of the value functions which all represent the same preferences?
- **Avoid using measures which compare overall values across different value functions (i.e. attribute weightings)**

Rank (sensitivity) analysis

- For any weights, the alternatives can be ranked based on their overall values
  - This ranking is <u>not</u> influenced by normalization (i.e., positive affine transformations of V)

- How do the rankings of alternatives change when attribute weights vary?



ranks	x^1	x^2	x^3
minimum	1	1	1
maximum	3	2	3

Computation of rank intervals

The minimum ranking of x^k is

$$r_S^-(x^k) = 1 + \min_{(w,v)\in S} |\{x^j \in X \mid V(x^j, w, v) > V(x^k, w, v)\}|$$

which is obtained as a solution to the mixed integer LP

$$\min_{\substack{(w,v)\in S \\ y^j \in \{0,1\}}} \sum_{j=1}^{m} y^j$$

$$V(x^j, w, v) \leq V(x^k, w, v) + y^j M \qquad j = 1, \ldots, m$$

$$y^k = 1$$

Maximum rankings with a similar model

Rank analysis – example (1/5)

- **Academic ranking of world universities 2007**
  - 508 universities

- **Additive multi-attribute model**
  - 6 attributes
  - Attribute weights (denoted by w^*) and scores
  - Universities ranked based on overall values

Rank analysis – example (2/5)

Criteria	Indicator	Code	Weight
Quality of Education	Alumni of an institution winning Nobel Prizes and Fields Medals	Alumni	10%
Quality of Faculty	Staff of an institution winning Nobel Prizes and Fields Medals	Award	20%
	Highly cited researchers in 21 broad subject categories	HiCi	20%
Research Output	Articles published in Nature and Science*	N&S	20%
	Articles in Science Citation Index-expanded, Social Science Citation Index	SCI	20%
Size of Institution	Academic performance with respect to the size of an institution	Size	10%
Total			100%

Rank analysis – example (3/5)

Scores (some of them)

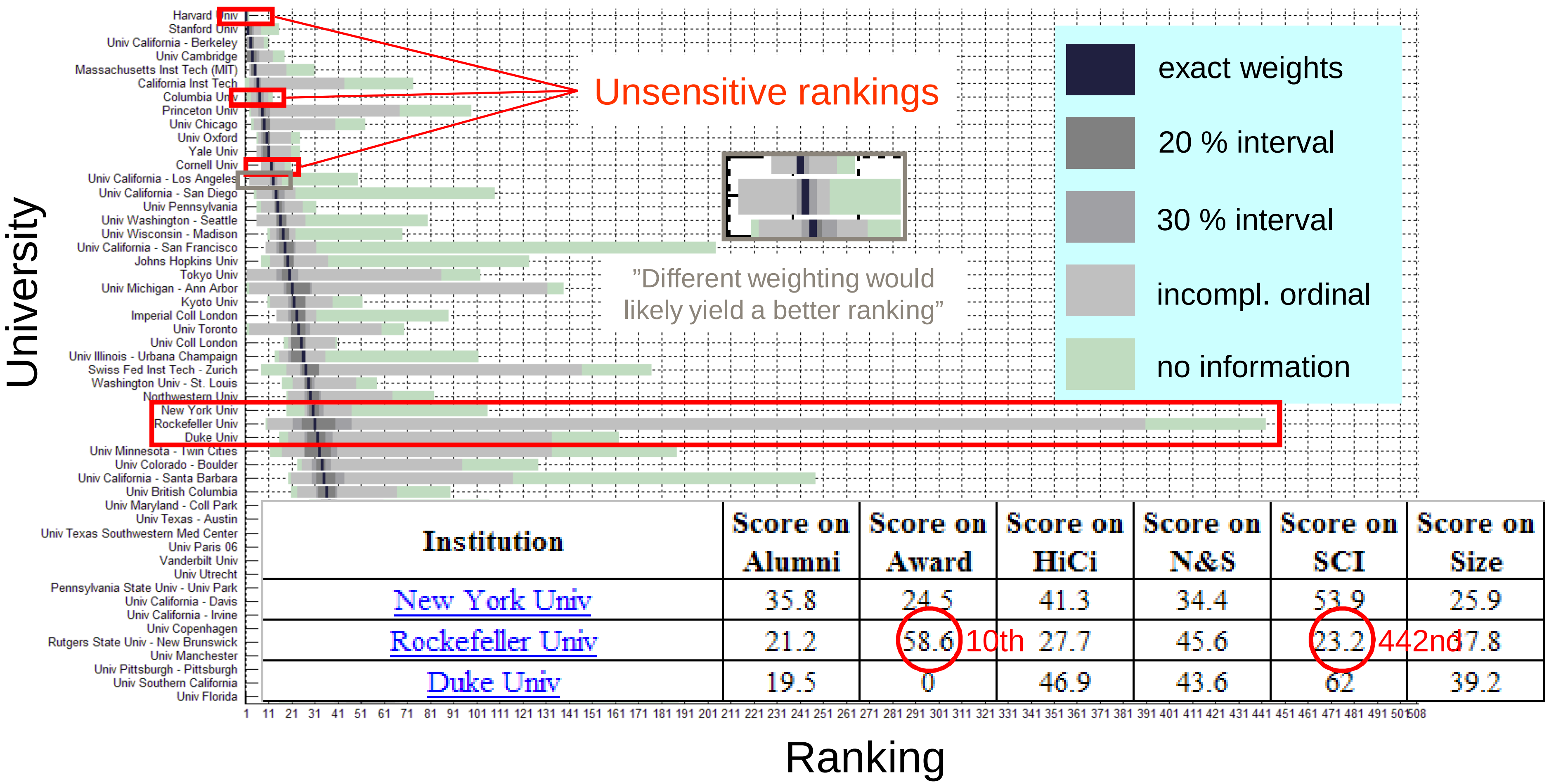
World Rank	Institution	Score on Alumni	Score on Award	Score on HiCi	Score on N&S	Score on SCI	Score on Size	Total Score
1	Harvard Univ	100	100	100	100	100	73	100
2	Stanford Univ	42	78.7	86.1	69.6	70.3	65.7	73.7
3	Univ California - Berkeley	72.5	77.1	67.9	72.9	69.2	52.6	71.9
4	Univ Cambridge	93.6	91.5	54	58.2	65.4	65.1	71.6
5	Massachusetts Inst Tech (MIT)	74.6	80.6	65.9	68.4	61.7	53.4	70.0
6	California Inst Tech	55.5	69.1	58.4	67.6	50.3	100	66.4
7	Columbia Univ	76	65.7	56.5	54.3	69.6	46.4	63.2
8	Princeton Univ	62.3	80.4	59.3	42.9	46.5	58.9	59.5
9	Univ Chicago	70.8	80.2	50.8	42.8	54.1	41.3	58.4
10	Univ Oxford	60.3	57.9	46.3	52.3	65.4	44.7	56.4
11	Yale Univ	50.9	43.6	57.9	57.2	63.2	48.9	55.9
12	Cornell Univ	43.6	51.3	54.5	51.4	65.1	39.9	54.3
13	Univ California - Los Angeles	25.6	42.8	57.4	49.1	75.9	35.5	52.6
14	Univ California - San Diego	16.6	34	59.3	55.5	64.6	46.6	50.4
15	Univ Pennsylvania	33.3	34.4	56.9	40.3	70.8	38.7	49.0
16	Univ Washington - Seattle	27	31.8	52.4	49	74.1	27.4	48.2
17	Univ Wiscon http://www.washingto	40.3	35.5	52.9	43.1	67.2	28.6	48.0
18	Univ California follow. Click and hold t	0	36.8	54	53.7	59.8	46.7	46.8
19	Johns Hopkins Univ	48.1	27.8	41.3	50.9	67.9	24.7	46.1
20	Tokyo Univ	33.8	14.1	41.9	52.7	80.9	34	45.9
21	Univ Michigan - Ann Arbor	40.3	0	60.7	40.8	77.1	30.7	44.0
22	Kyoto Univ	37.2	33.4	38.5	35.1	68.6	30.6	43.1
23	Imperial Coll London	19.5	37.4	40.6	39.7	62.2	39.4	43.0
23	Univ Toronto	26.3	19.3	39.2	37.7	77.6	44.4	43.0
25	Univ Coll London	28.8	32.2	38.5	42.9	63.2	33.8	42.8
26	Univ Illinois - Urbana Champaign	39	36.6	44.5	36.4	57.6	26.2	42.7
27	Swiss Fed Inst Tech - Zurich	37.7	36.3	35.5	39.9	38.4	50.5	39.9
28	Washington Univ - St. Louis	23.5	26	39.2	43.2	53.4	39.3	39.7
29	Northwestern Univ	20.4	18.9	46.9	34.2	57	36.9	38.2
30	New York Univ	35.8	24.5	41.3	34.4	53.9	25.9	38.0
30	Rockefeller Univ	21.2	58.6	27.7	45.6	23.2	37.8	38.0
32	Duke Univ	19.5	0	46.9	43.6	62	39.2	37.4
33	Univ Minnesota - Twin Cities	33.8	0	48.6	35.9	67	23.5	37.0
34	Univ Colorado - Boulder	15.6	30.8	39.9	38.8	45.7	30	36.6
35	Univ California - Santa Barbara	0	35.3	42.6	36.2	42.7	35.1	35.8
36	Univ British Columbia	19.5	18.9	31.4	31	63.1	36.3	35.4
37	Univ Maryland - Coll Park	24.3	20	40.6	31.2	53.3	25.9	35.0
38	Univ Texas - Austin	20.4	16.7	46.9	28	54.8	21.3	34.4
39	Univ Texas Southwestern Med Center	22.8	33.2	30.6	35.5	38	31.9	33.8

Rank analysis – example (4/5)

Incomplete weight information

- Relative intervals: $w \in \{w \in S_w^0 \mid (1-\alpha)w_i^* \le w_i \le (1+\alpha)w_i^*\}$
  - For α=0.1, 0.2, 0.3
  - e.g. α=0.2 , w_i*=0.20: $0.16 \le w_i \le 0.24$
- Incomplete ordinal: $w \in \{w \in S_w^0 \mid w_i \ge w_k \ge 0.02 \,\forall\, i \in \{2,3,4,5\}, k \in \{1,6\}\}$
  - Consistent with initial weights and lower bound b = 0.02
- Only lower bound: $w \in \{w \in S_w^0 \mid w_i \ge 0.02 \forall i = 1,\dots,6\}$
- No weight information: $w \in S_w^0$

Rank analysis – example (5/5)



Institution	Score on Alumni	Score on Award	Score on HiCi	Score on N&S	Score on SCI	Score on Size
New York Univ	35.8	24.5	41.3	34.4	53.9	25.9
Rockefeller Univ	21.2	58.6	27.7	45.6	23.2	[illegible]7.8
Duke Univ	19.5	0	46.9	43.6	62	39.2

Example: prioritization of innovation ideas*

- 28 ”innovation ideas” evaluated by several people on a scale from 1 – 7 with regard to *novelty*, *feasibility* and *relevance*
  - Innovation ideas described by the 3 averages of these evaluations
- No preference information about the relative values of the attributes
- ”Which 10 innovation ideas should be selected for further development?”
  - Sets of ideas called portfolios
- The value of a portfolio is the sum of its constituent projects

* Könnölä et al. (Technological Forecasting & Social Change, 2007)

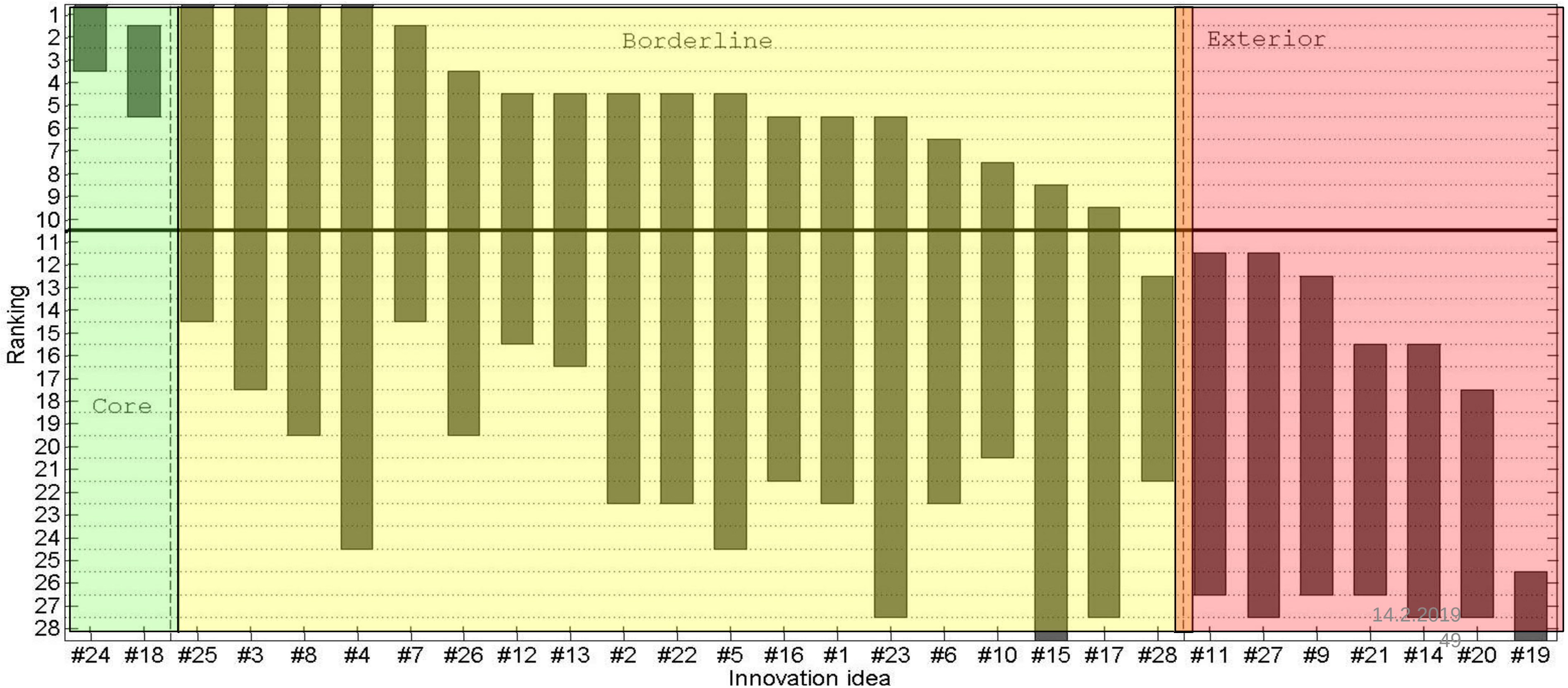
Example: prioritization of innovation ideas

- Robust Portfolio Modeling* method was used to compute *non-dominated portfolios* of 10 ideas and discriminate between
  - Core ideas that belong to all non-dominated portfolios
  - Borderline ideas that belong to some non-dominated portfolios
  - Exterior ideas that do not belong to any non-dominated portfolio

- How do ranking intervals compare with this division?
  - If the ranking of an idea cannot be worse than 10, is it a core project?
  - If the ranking of an idea cannot be better than 11, is it an exterior project?

* Liesiö, Mild, Salo (European Journal of Operational Research, 2007)

Ranking intervals vs. core, borderline and exterior ideas



Ranking intervals divide the innovation ideas into core, borderline and exterior ideas among *potentially optimal* portfolios

Rationales for using incomplete information

- Limited time and effort can usually be devoted to preference elicitation
- Complete preference specification may not even be needed to reach a decision
- DM's preferences may evolve during the analysis → iteration can be helpful
- Experts / stakeholders may have conflicting preferences
- Take-it-or-leave-it solutions may be resented in group decision settings → results based on incomplete information leave room for negotiation

Summary

- Complete specification of attribute weights is often difficult
  - Trade-off methods take time and effort
  - SWING and SMARTS are prone to biases

- Incomplete preference statements can be modeled by linear inequalities on the weights → alternatives' overall values become intervals

- Preference over interval-valued alternatives can be established through dominance relations
  - Non-dominated alternatives are good decision recommendations

- Avoid methods which compare numerical values of different value functions

Aalto University
School of Science

Decision making and problem solving – Lecture 7

- From EUT to MAUT
- Axioms for preference relations
- Assessing attribute-specific utility functions and attribute weights
- Decision recommendations
- MAVT vs. MAUT

Liesiö, Punkka, Salo, Vilkkumaa

Motivation

- ❑ Multiattribute value theory helps generate decision recommendations, when
  - Alternatives are evaluated w.r.t. multiple attributes
  - Alternatives' attribute-specific values are certain
- ❑ What if the attribute-specific performances are *uncertain*?
  - Planning a supply chain: minimize cost, minimize supply shortage, minimize storage costs
  - Building an investment portfolio: maximize return, minimize risk

→ Multiattribute utility theory

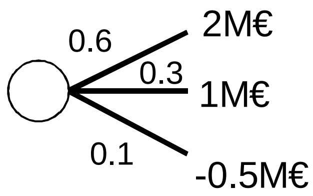
From EUT to MAUT

EUT

- ❑ Set of possible outcomes T:
  - – E.g., revenue $T = \mathbb{R}$ euros, demand $T = \mathbb{N}$
- ❑ Set of all possible lotteries L:
  - – A lottery $f \in L$ associates a probability $f(t) \in [0,1]$ with each possible outcome $t \in T$
- ❑ Deterministic outcomes are modeled as degenerate lotteries

Lottery

Decision tree

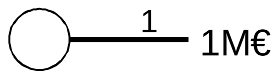


Probability mass function

$$f(t) = \begin{cases} 0.6, t = 2M€ \\ 0.3, t = 1M€ \\ 0.1, t = -0.5M€ \\ 0, elsewhere \end{cases}$$

Degenerate lottery

Decision tree



Probability distribution function

$$f(t) = \begin{cases} 1, t = 1M€ \\ 0, elsewhere \end{cases}$$

From EUT to MAUT

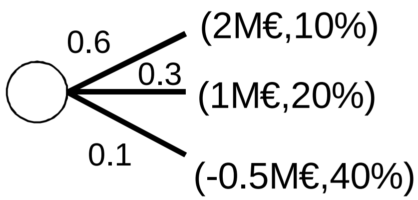
MAUT

- Multidimensional set of outcomes *X*:

$$X = X_1 \times \cdots \times X_n$$

  - E.g., X_1 = revenue (€), X_2 = market share
- Set of all possible lotteries *L*:
  - A lottery $f \in L$ associates a probability $f(t) \in [0,1]$ with each possible outcome $x = (x_1, \ldots, x_n) \in X$
- Deterministic outcomes are modelled as degenerate lotteries

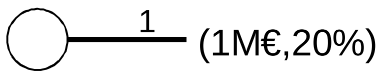
Lottery



$$f(t) = \begin{cases} 0.6, t = (2,0.1) \\ 0.3, t = (1,0.2) \\ 0.1, t = (-0.5,0.4) \\ 0, elsewhere \end{cases}$$

Degenerate lottery

Decision tree



PDF

$$f(t) = \begin{cases} 1, t = (1,0.2) \\ 0, elsewhere \end{cases}$$

Aggregation of utilities

- **Problem:** How to measure the overall utility of alternative $x = (x_1, x_2, \dots x_n)$?

$$U(x_1, x_2, \dots x_n) = ?$$

- **Question:** Could the overall utility be obtained by a weighted sum of the attribute-specific utilities?

$$U(x_1, x_2, \dots x_n) = \sum_{i=1}^{n} w_i \, u_i(x_i)?$$

- **Answer:** Yes, if the attributes are
  - Mutually preferentially independent and
  - Additive independent (new)

Preferential independence (old)

- **Definition:** Attribute X is **preferentially independent (PI)** of the other attributes $\boldsymbol{Y}$, if the preference order of degenerate lotteries that differ only in X does not depend on the levels of attributes $\boldsymbol{Y}$

$$(x, \boldsymbol{y}) \succcurlyeq (x', \boldsymbol{y}) \Rightarrow (x, \boldsymbol{y}') \succcurlyeq (x', \boldsymbol{y}') \text{ for all } \boldsymbol{y}' \in \boldsymbol{Y}$$

- Interpretation: Preference over the <u>certain</u> level of attribute X does not depend on the <u>certain</u> levels of the other attributes, as long as they stay the same

- Same as in MAVT

Mutual preferential independence (old)

- **Definition:** Attributes A are **mutually perferentially independent (MPI),** if any subset **X** of attributes A is preferentially independent of the other attributes **Y**=A**X**. I.e., for any degenerate lotteries:

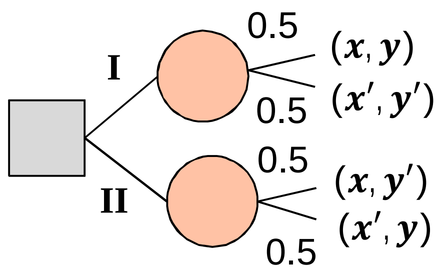
$$(\boldsymbol{x}, \boldsymbol{y}') \succcurlyeq (\boldsymbol{x}', \boldsymbol{y}') \Rightarrow (\boldsymbol{x}, \boldsymbol{y}) \succcurlyeq (\boldsymbol{x}', \boldsymbol{y}) \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}.$$

- Interpretation: Preference over the <u>certain</u> levels of attributes ***X*** does not depend on the <u>certain</u> levels of the other attributes, as long as they stay the same

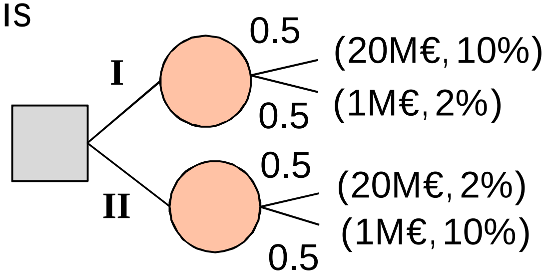
- Same as in MAVT

Additive independence (new)

- **Definition:** Subset of attributes **X**⊂A is **additive independent (AI),** if the DM is indifferent between lotteries **I** and **II** for any $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}') \in A$

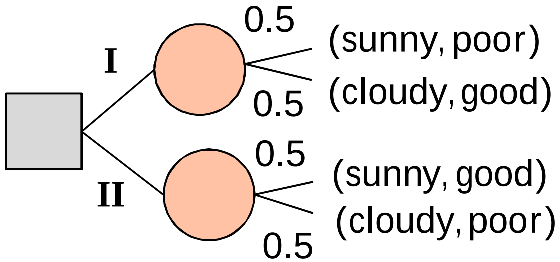


- Example:
  - Profit is AI if the DM is indifferent between I and II
  - However, she might prefer II, because it does not include an outcome where all attributes have very poor values. In this case profit is not AI.



Additive independence (new)

- Example:
  - A tourist is planning a downhill skiing weekend trip to the mountains
  - 2 attributes: sunshine ({sunny, cloudy}) and snow conditions ({good, poor})
  - Additive independence holds, if she is indifferent between I and II
    - In both, there is a 50 % probability of getting sunshine
    - In both, there is a 50 % probability of having good snow conditions
    - If the DM values sunshine and snow conditions independently of each other, then I and II can be equally preferred



Additive multiattribute utility function

- **Theorem:** The attributes are mutually preferentially independent and **single** attributes are additive independent iff preference relation $\succcurlyeq$ is represented by an **additive multi-attribute utility** function

$$U(x) = \sum_{i=1}^{n} w_i u_i^N(x_i),$$

where $u_i^N(x_i^0) = 0$, $u_i^N(x_i^*) = 1$, and $\sum_{i=1}^{n} w_i = 1$, $w_i \geq 0$.

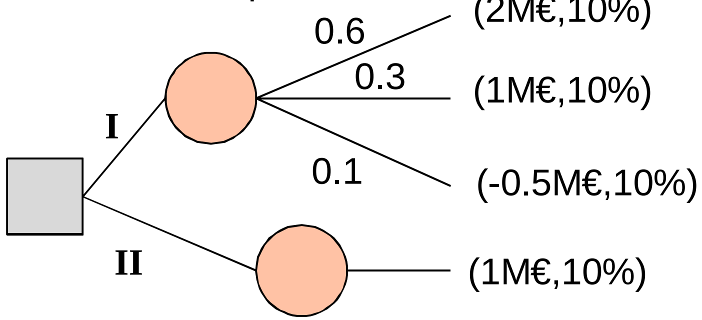
What if the MPI & AI do not hold?

- **Definition:** Attribute $X \in A$ is **utility independent (UI)** if the preference order between lotteries that have equal <u>certain</u> outcomes on attributes **Y**=$A\backslash X$ does not depend on the level of these outcomes, i.e.,

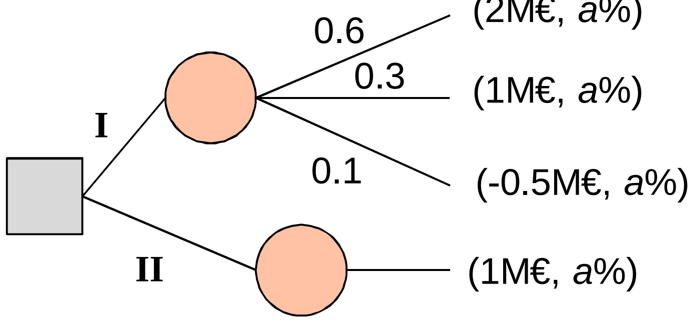
$$(\tilde{x}, \boldsymbol{y}) \succcurlyeq (\tilde{x}', \boldsymbol{y}) \Rightarrow (\tilde{x}, \boldsymbol{y}') \succcurlyeq (\tilde{x}', \boldsymbol{y}') \forall \boldsymbol{y}'$$

- Example:

Assume DM prefers **I**



If profit is UI, then the DM should prefer **I** for any *a*

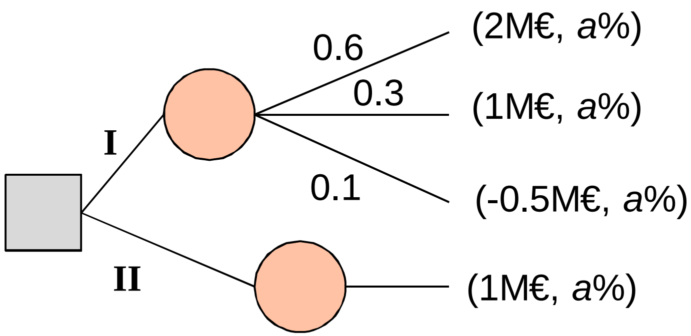


However, for a small market share (a), the DM may be more risk averse and choose **II** → profit is not UI

Mutual utility independence

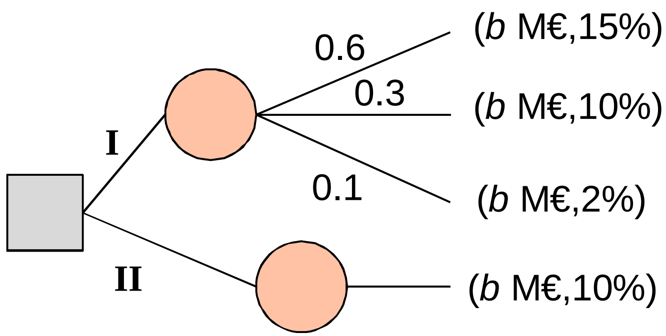
- **Definition:** Attributes A are **mutually utility independent (MPI),** if every subset $\boldsymbol{X} \subset A$ is the utility independent of the other attributes **Y**=A\X i.e.,

$$(\tilde{\boldsymbol{x}},\boldsymbol{y}) \succcurlyeq (\tilde{\boldsymbol{x}}',\boldsymbol{y}) \Rightarrow (\tilde{\boldsymbol{x}},\boldsymbol{y}') \succcurlyeq (\tilde{\boldsymbol{x}}',\boldsymbol{y}')\ \forall \boldsymbol{y}'$$



If DM prefers **I** for some *a*, she should prefer **I** for all *a*

AND



If DM prefers **I** for some *b*, she should prefer **I** for all *b*

Other multi-attribute utility functions

- If attributes are **mutually utility independent**, then preferences are represented by a multiplicative utility function:

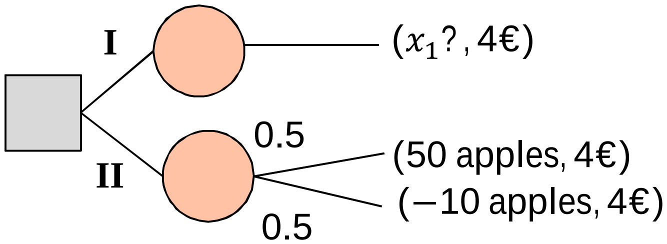
$$U(x) = \frac{\prod_{i=1}^{n}[1 + kw_i u_i(x_i)]}{k} - \frac{1}{k}$$

- If each single attribute is **utility independent**, then preferences are represented by a so-called multilinear utility function

- AI is the strongest of the three preference assumptions
  - Let X ⊂ A. Then, X is AI ⇒ X is UI ⇒ X is PI

Assessing attribute-specific utility functions

- Use the same techniques as with a unidimensional utility function
  - Certainty equivalent, probability equivalent, etc. & scale such that $u_i^N(x_i^0) = 0$, $u_i^N(x_i^*) = 1$.
  - Also direct rating often applied in practice
- What about the other attributes?
  - Fix them at the same level in every outcome
  - Do not matter! → Usually not even explicitly shown to the DM



$$U(x_1, 4) = 0.5U(50,4) + 0.5U(-10,4)$$
$$\Leftrightarrow w_1u_1(x_1) + w_2u_2(4) = 0.5w_1u_1(50) + 0.5w_2u_2(4) + 0.5w_1u_1(-10) + 0.5w_2u_2(4)$$
$$\Leftrightarrow w_1u_1(x_1) = 0.5w_1u_1(50) + 0.5w_1u_1(-10)$$
$$\Leftrightarrow u_1(x_1) = 0.5u_1(50) + 0.5u_1(-10)$$

Example: Choosing a software supplier

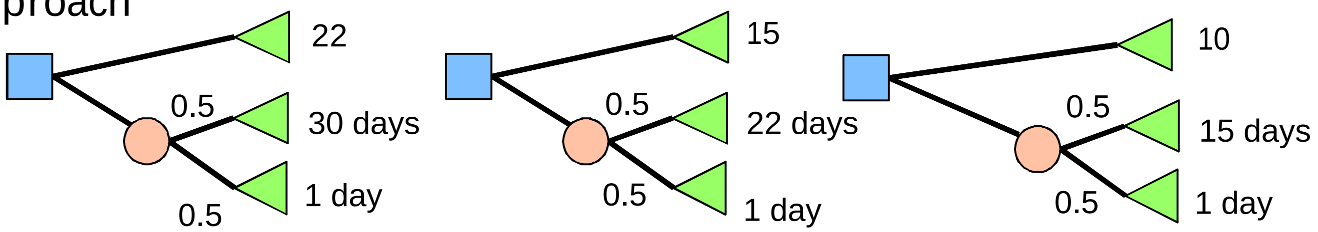
- Three attributes: cost, delay, quality

i	Name	X_i	x_i^0	x_i^*
1	Cost	[10,40] k€	40	10
2	Delay	{1,2,…,30} days	30	1
3	Quality	{fair, good, excellent}	fair	excellent

Example: Choosing a software supplier

i	Name	X_i	x_i^0	x_i^*
1	Cost	[10,40] k€	40	10
2	Delay	{1,2,…,30} days	30	1
3	Quality	{fair, good, exc.}	fair	exc.

- Assessment of the attribute-specific utility functions
  - Quality: Direct assessment
    - u_3(fair)=0, u_3(good)=0.4, u_3(excellent)=1
  - Cost: Linear decreasing utility function
    - $u_1(x_1) = \frac{40-x_1}{30}$
  - Delay: Assessment with certainty equivalent (CE) approach



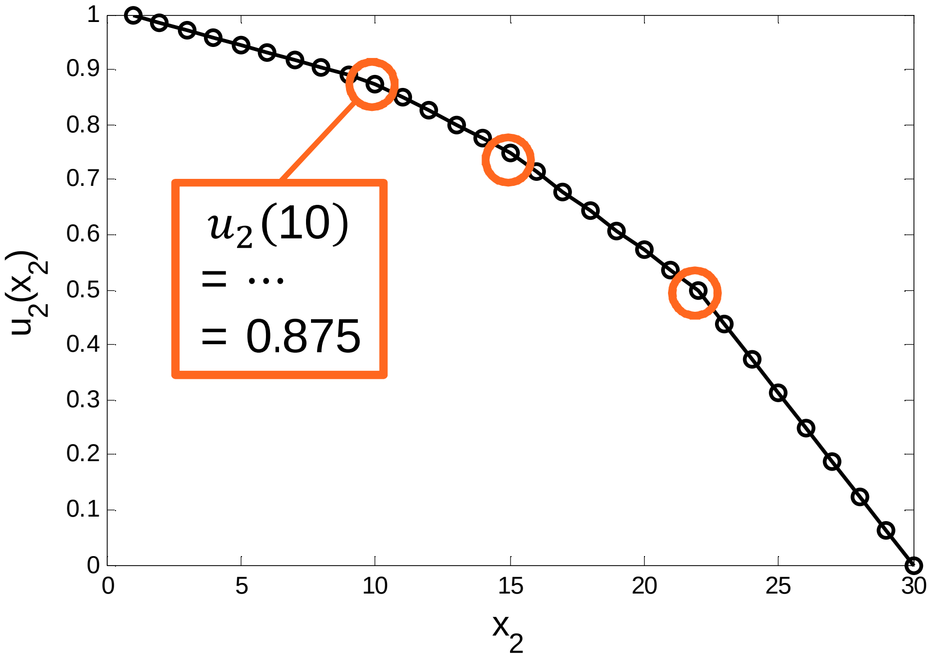
$$u_2(22) = 0.5u_2(1) + 0.5u_2(30) = 0.5 * 1 + 0.5 * 0 = 0.5$$

$$u_2(15) = 0.5u_2(1) + 0.5u_2(22) = 0.5 * 1 + 0.5 * 0.5 = 0.75$$

$$u_2(10) = 0.5u_2(1) + 0.5u_2(22) = 0.5 * 1 + 0.5 * 0.75 = 0.875$$

Example: Choosing a software supplier

For *delay*, linear interpolation between specified values



x_2	$u_2(x_2)$	x_2	$u_2(x_2)$
1	1	16	0.7143
2	0.9861	17	0.6786
3	0.9722	18	0.6429
4	0.9583	19	0.6071
5	0.9444	20	0.5714
6	0.9306	21	0.5357
7	0.9167	22	0.5
8	0.9028	23	0.4375
9	0.8889	24	0.375
10	0.875	25	0.3125
11	0.85	26	0.25
12	0.825	27	0.1875
13	0.8	28	0.125
14	0.775	29	0.0625
15	0.75	30	0

Assessing attribute weights

- Attribute weights are elicited by constructing two equally preferred degenerate lotteries
  - E.g., ask the DM to establish a preference order for n hypothetical alternatives specified so that $(x_1^0, \dots, x_i^*, \dots, x_n^0),\ i = 1, \dots, n.$
  - Assume that $(x_1^*, x_2^0, \dots, x_n^0) \succcurlyeq (x_1^0, x_2^*, \dots, x_n^0) \succcurlyeq \cdots \succcurlyeq (x_1^0, x_2^0, \dots, x_n^*)$
  - Then, for each i=1,...,n-1 ask the DM to define $x_i \in X_i$ such that
$$(\dots x_i, x_{i+1}^0, \dots) \sim (\dots x_i^0, x_{i+1}^*, \dots)$$
$$\Rightarrow U(\dots x_i, x_{i+1}^0, \dots) = U(\dots x_i^0, x_{i+1}^*, \dots)$$
$$\Rightarrow w_i u_i(x_i) = w_{i+1}$$
  - n-1 such comparisons + 1 normalization constraint ⇒ unique set of weights

Example: Choosing a software supplier

- Assessment of the attribute weights
  - Assume preferences (40k€, 1 day, fair)$\succcurlyeq$(10k€, 30 days, fair)$\succcurlyeq$(40k€, 30 days, exc.)
  - Choose delay $x_2 \in \{1, \dots, 30\}$ such that $(40, x_2, x_3) \sim (10{,}30, x_3)$
  - Answer $x_2 = 8$ gives

$$w_1 u_1(40) + w_2 u_2(8) + w_3 u_3(x_3) = w_1 u_1(10) + w_2 u_2(30) + w_3 u_3(x_3)$$
$$w_2 u_2(8) = w_1$$
$$\Leftrightarrow w_2 \cdot 0.9028 = w_1$$

  - Choose cost $x_1 \in [10{,}40]$ such that $(x_1, x_2, \text{fair}) \sim (40, x_2, \text{excellent})$
  - Answer $x_1 = 20$ gives

$$w_1 u_1(20) + w_2 u_2(x_2) + w_3 u_3(\text{fair}) = w_1 u_1(40) + w_2 u_2(x_2) + w_3 u_3(\text{excellent})$$
$$w_1 u_1(20) = w_3$$
$$\Leftrightarrow w_1 \cdot \frac{2}{3} = w_3$$

  - Attribute weights: $w \approx \left(\frac{9}{25}, \frac{10}{25}, \frac{6}{25}\right)$

MAUT: Decision recommendations

- Consider *m* decision alternatives $x^j = \left(x_1^j, \dots, x_n^j\right)$, $j = 1, \dots, m$, where x^j is a random variable with PDF $f_{x^j}(x)$
- Alternatives are ranked by their expected (multiattribute) utilities

$$E\left[U\left(x^j\right)\right] = \sum_{x \in A} f_{x^j}(x)\, U(x) = \sum_{x \in A} f_{x^j}(x) \sum_i w_i u_i(x)$$

  – Integral for continuous random variables

- In a decision tree, MAU is used just like unidimensional utility

Example: Choosing a software supplier

- Consider three suppliers:
  - Supplier 1: Expensive, fair quality, can deliver without delay

    $$x^1 = (35k€, 1 \text{ day}, fair)$$

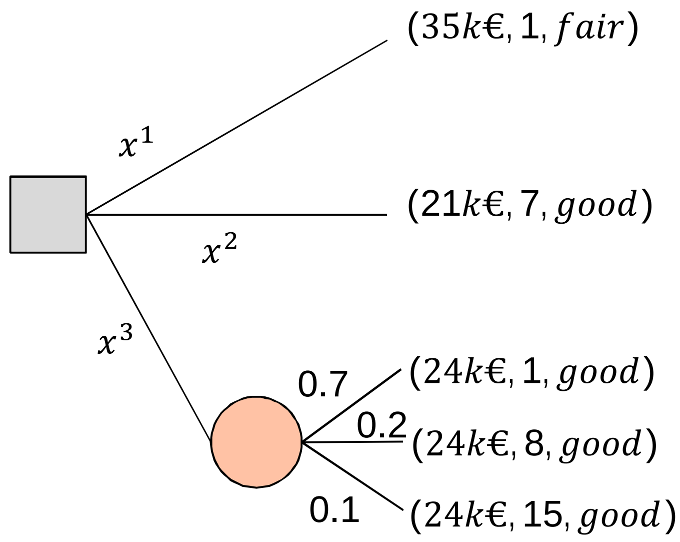
  - Supplier 2: Cheap, good quality, can deliver in 1 week

    $$x^2 = (21k€, 7 \text{ days}, good)$$

  - Supplier 3: Moderate price, good quality, 20% chance of 1-week delay and 10% chance of 2-week delay

    $$x^3 = (24k€, \tilde{x}_2^3, good),$$

    $$f_{\tilde{x}_2^3}(x) = \begin{cases} 0.7, x = (24k€, 1 \text{ day}, good) \\ 0.2, x = (24k€, 8 \text{ days}, good) \\ 0.1, x = (24k€, 15 \text{ days}, good) \end{cases}$$



Example: Choosing a software supplier

	$\boldsymbol{u_1^N}$	$\boldsymbol{u_2^N}$	$\boldsymbol{u_3^N}$	$\boldsymbol{U}$	$\boldsymbol{f_{x_k^j}}$	$\mathrm{E}[U]$
x^1	0.17	1.00	0.00	0.46	1	0.46
$\boldsymbol{x^2}$	**0.63**	**0.92**	**0.40**	**0.69**	**1**	**0.69**
$x^3\ (s_1)$	0.53	1.00	0.40	0.69	0.7	0.67
$x^3\ (s_2)$	0.53	0.90	0.40	0.65	0.2	
$x^3\ (s_3)$	0.53	0.75	0.40	0.59	0.1	
w	0.36	0.40	0.24			

MAVT vs. MAUT

- **MAVT:** Preference between <u>alternatives with certain outcomes</u> can be represented by an additive multiattribute value function, iff the attributes are
  - Mutually preferentially independent
  - Difference independent

- **MAUT**: Preference between <u>lotteries with uncertain outcomes</u> can be represented by additive multiattribute utility function, iff the attributes are
  - Mutually preferentially independent
  - Additive independent

MAVT vs. MAUT

- **Attribute-specific <u>value</u> functions** are elicited by asking the DM to specify equally preferred differences in attribute levels
  - E.g., ”Specify salary x such that you would be indifferent between change 1500€ → x€ and x€ → 2000€”

- **Attribute-specific <u>utility</u> functions** are elicited by asking the DM to specify equally preferred lotteries
  - E.g., ”Specify salary x such that you would be indifferent between getting x€ for certain and a 50-50 gamble between getting 1500€ or 2000€”

- **Attribute weights** are elicited similarly in MAVT and MAUT

MAVT vs. MAUT

- In principal, the natural / measurement scale is first mapped to value scale and then (if needed) to utility scale

- Yet, in practice the value function is ”hidden” in the utility function
  - E.g, if certainty equivalent of 50-50 gamble between 3k€ and 5k€ salary is 3.9k€, is this a sign of risk aversion or decreasing marginal value of salary?

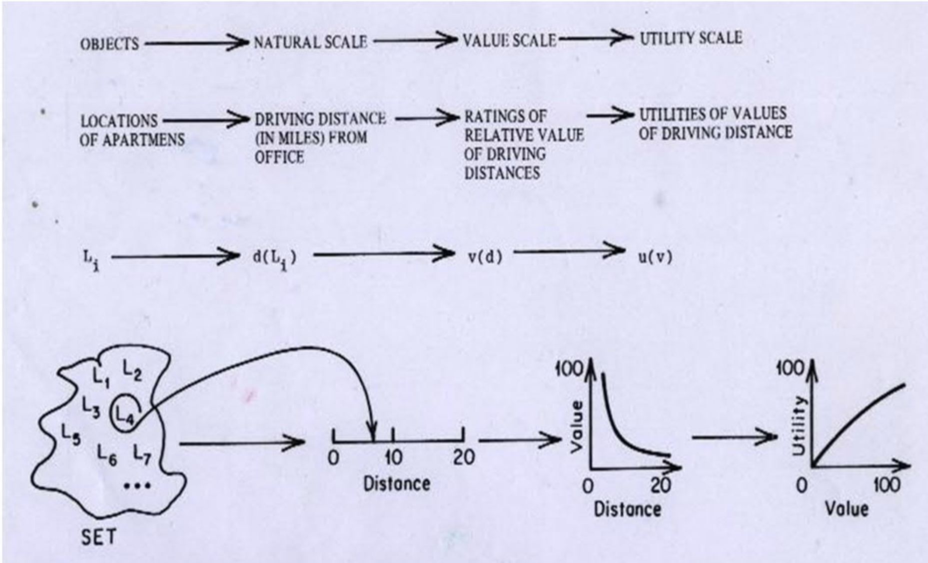


Figure 7.2. The four steps needed to construct value and utility functions.

Summary

- Multiattribute utility theory helps establish a preference relation between alternatives with uncertain outcomes on multiple attributes
- Preference relation is represented by an additive utility function, iff the attributes are mutually preferentially independent and additive independent
- Attribute-specific utility functions are elicited as in the unidimensional case
- Attribute weights are elicited as in MAVT
- Decision recommendation: the alternative with highest expected utility
- Robust methods can also be used with MAUT

Aalto University
School of Science

Decision making and problem solving – Lecture 8

- Multiple objective optimization (MOO)
- Pareto optimality (PO)
- Approaches to solving PO-solutions: weighted sum, weighted max-norm, and value function methods

Liesiö, Punkka, Salo, Vilkkumaa

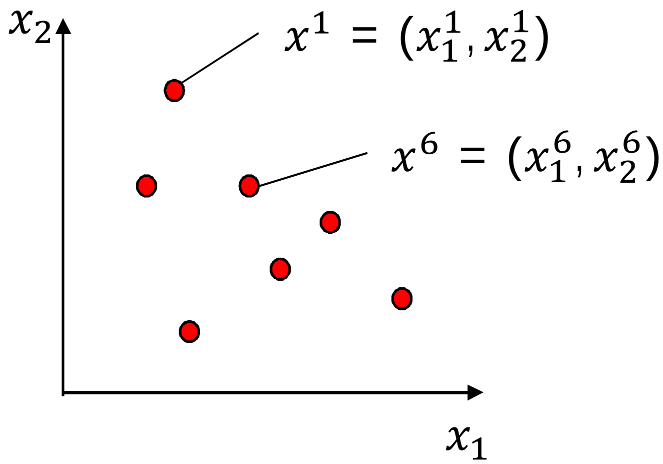
Until this lecture

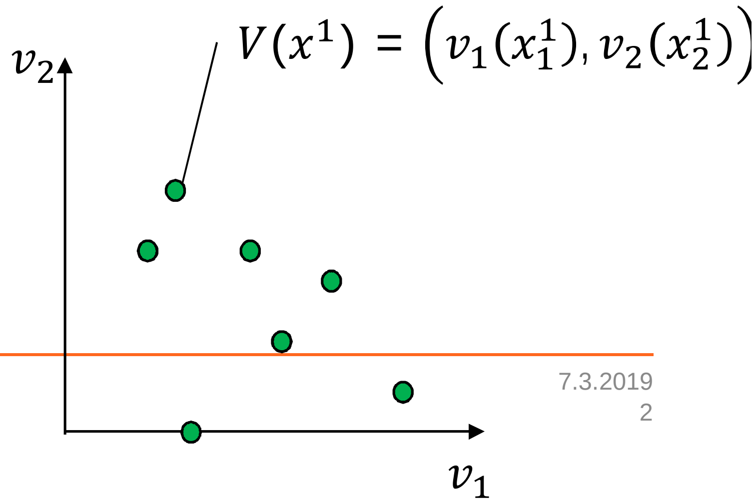
- Explicit set of alternatives $X = \{x^1, \dots, xm\}$, which are evaluated with regard to n criteria

- Evaluations $x_i^j : X \rightarrow \mathbb{R}^n$

- Preference modeling
  - Value functions

    $$\max_{x^j \in X} V\left(x^j\right) = V(x_1^j, \dots, x_n^j)$$





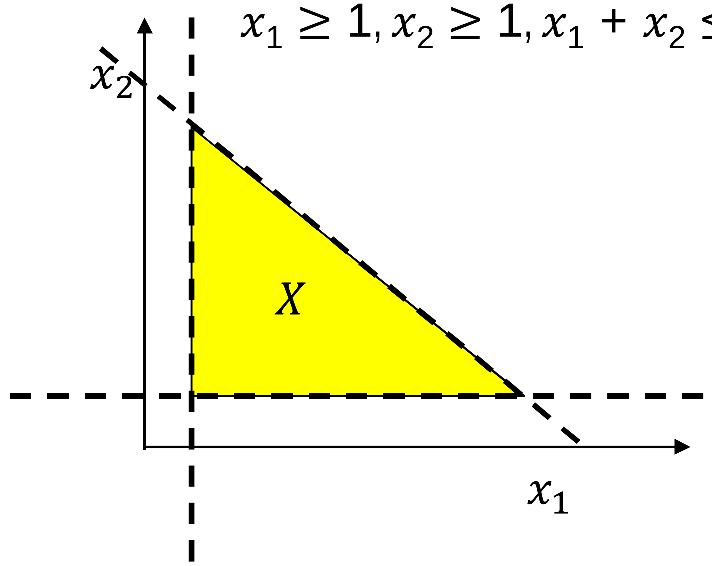
Need for other kind of approaches

- The decision alternatives cannot necessarily be listed
- Preference modeling can be time-consuming and difficult at the early stages of the analysis
- Conditions required for the additive value function to represent preferences do not necessarily hold or are difficult to validate
- We might want to see some results quickly to get a better understanding of the problem at hand

Multi-objective optimization: concepts

- Set of feasible solutions
  $$X = \{x \in \mathbb{R}^n | g(x) \leq 0\}$$
- Objective functions
  $$f = (f_1, \dots, f_n): X \to \mathbb{R}^n$$
- Preference modeling on trade-offs between objectives
  - Value functions
    $$\max_{x \in X} V(f(x)) = V(f_1(x), \dots, f_n(x))$$
  - Pareto approaches
    $$\text{v}-\max_{x \in X} V(f(x)) = (f_1(x), \dots, f_n(x))$$
  - Interactive approaches (not covered)

$$X = \{(x_1, x_2) \in \mathbb{R}^2 | \\ x_1 \geq 1, x_2 \geq 1, x_1 + x_2 \leq 7\}$$



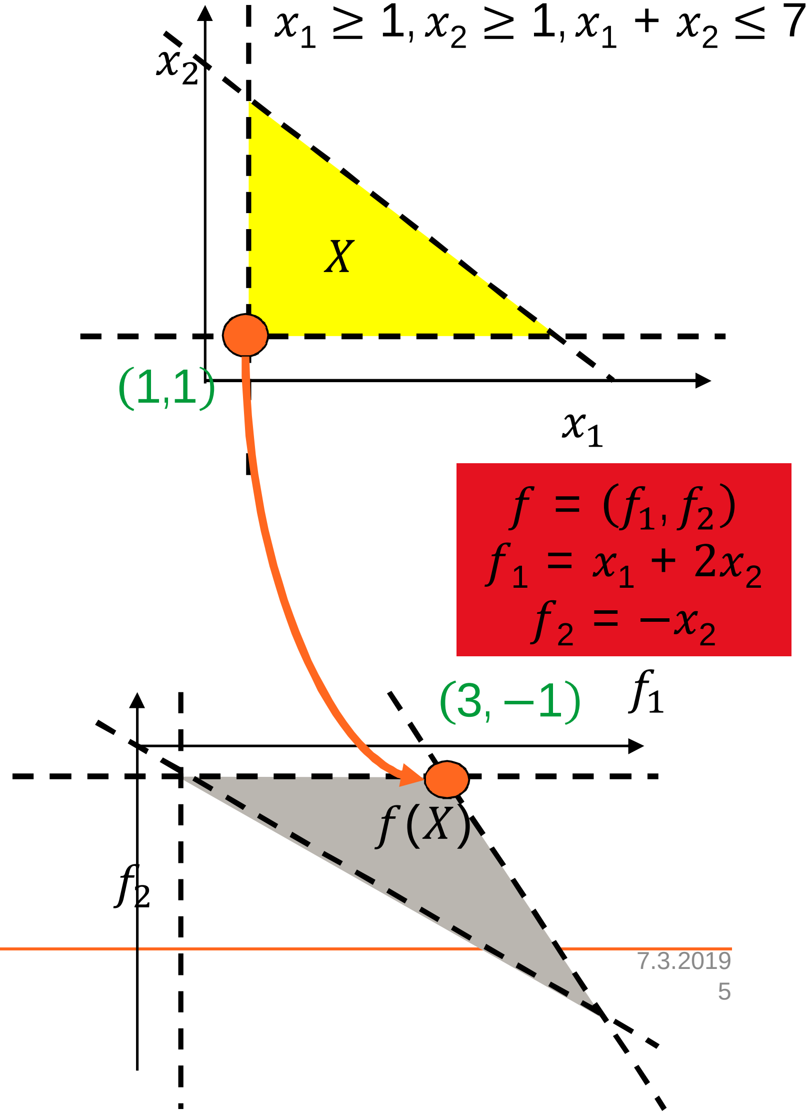
$$f = (f_1, f_2) = (x_1 + 2x_2, -x_2)$$

Multi-objective optimization: concepts

- Objective functions f map the feasible solutions X to $f(X)$ in the solution space:

$$f(X) = \{y \in \mathbb{R}^n | \exists x \in X \text{ so that } y = f(x)\}$$

$$X = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 1, x_2 \geq 1, x_1 + x_2 \leq 7\}$$



$$f(X) = \{(f_1, f_2) \in \mathbb{R}^2 | f_2 \leq -1, f_2 \leq 7 - f_1, 2f_2 \geq 1 - f_1\}$$

Preferential independence

- In multi-objective optimization (MOO), each objective is assumed preferentially independent of the others

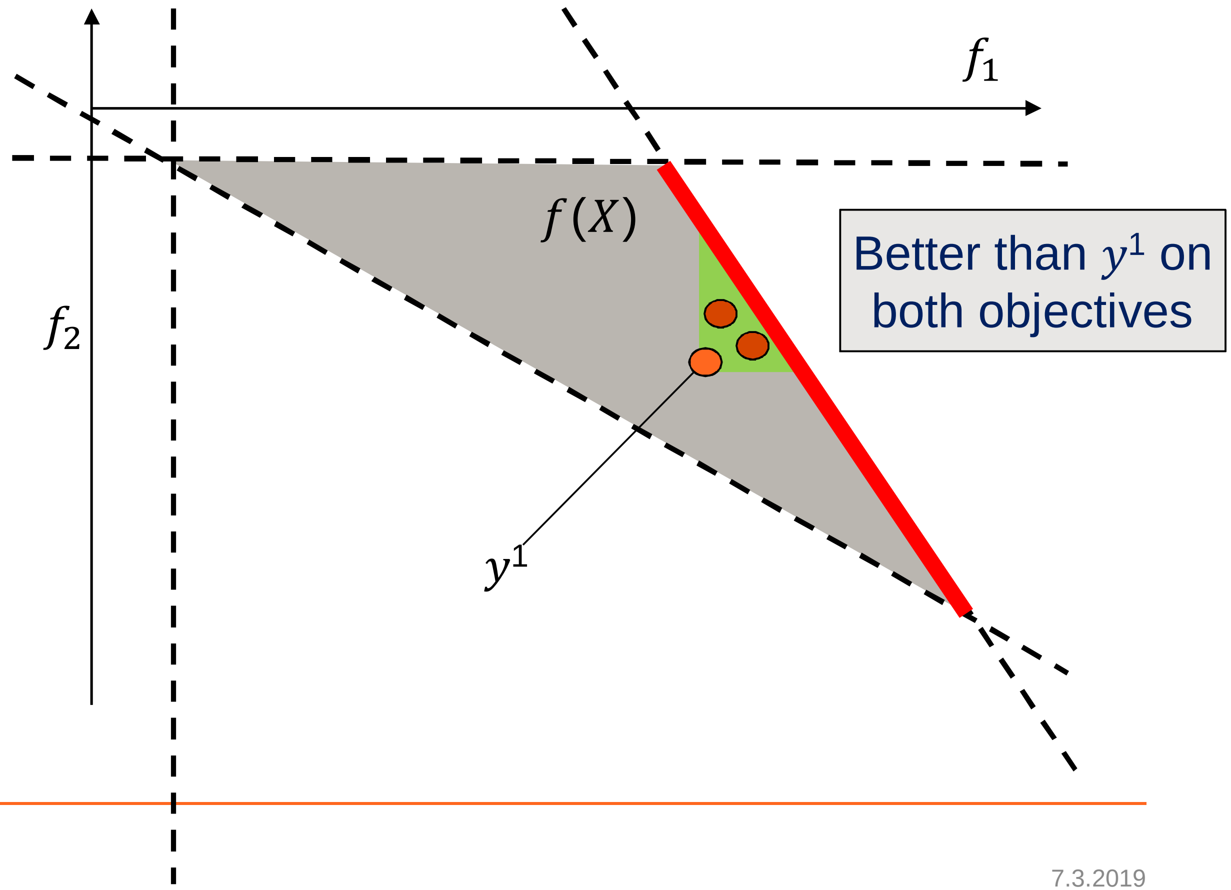
- **Definition** (cf. Lecture 5): Preference between two values of objective function *i* does not depend on the values of the other objective functions

→ Without loss of generality, we can assume all objectives to be maximized

– MIN can be transformed to MAX: $\min_{x \in X} f_i(x) = -\max_{x \in X}[-f_i(x)]$

Which feasible solution(s) to prefer?

- Selection of y^1 cannot be supported because other solutions have higher f_1 and f_2
  → Focus on Pareto-optimal solutions

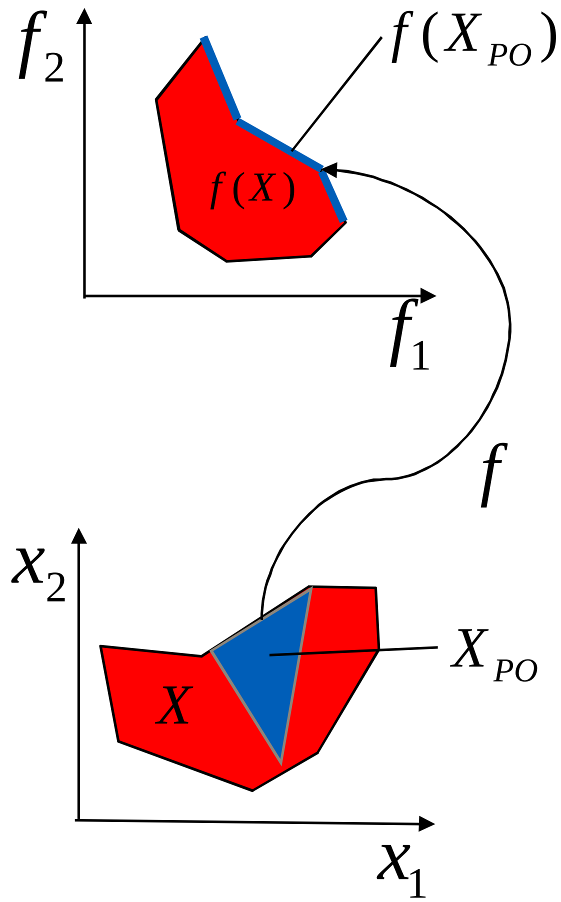


Pareto-optimality

Definition. $x^* \in X$ is Pareto-optimal if there does not exist $x \in X$ such that

$$\begin{cases} f_i(x) \geq f_i(x^*) & \text{for all } i \in \{1, \ldots, n\} \\ f_i(x) > f_i(x^*) & \text{for some } i \in \{1, \ldots, n\} \end{cases}$$

Set of all Pareto-optimal solutions: X_{PO}

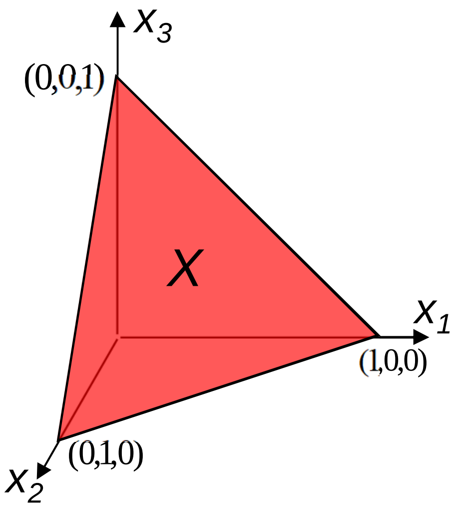


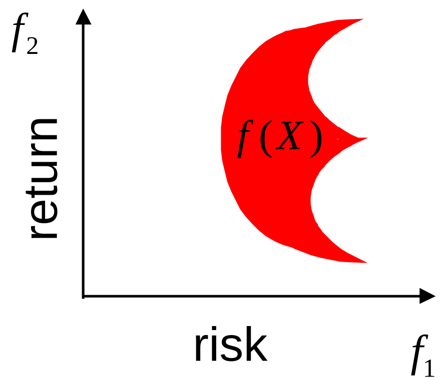
Definition. Objective vector $y \in f(X)$ is Pareto-optimal, if there exists a Pareto-optimal $x^* \in X$ s.t. $f(x^*)=y$

- Set of Pareto-optimal objective vectors: $f(X_{PO})$
- Notation $f(X_{PO}) = \text{v}-\max_{x \in X} f(x)$

Example: Markowitz model

- Optimal asset portfolio selection
  - How to allocate funds to m assets based on
    - Expected returns $\bar{r}_i$, i=1,...,m
    - Covariances of returns σ_{ij}, i,j=1,...,m
- Set of feasible solutions
  - Decision variables $x_1,\ldots,x_m$
    - Allocate x_j *100% of funds to j-th asset
  - Portfolio $x \in X = \{x \in \mathbb{R}^m | x_i \geq 0, \sum_{i=1}^m x_i = 1\}$
- Objective functions
  1. Maximize expected return of portfolio $f_2(x) = \sum_{i=1}^n \bar{r}_i x_i$
  2. Minimize variance (risk) of portfolio $f_1(x) = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^m \sigma_{ij} x_i x_j$



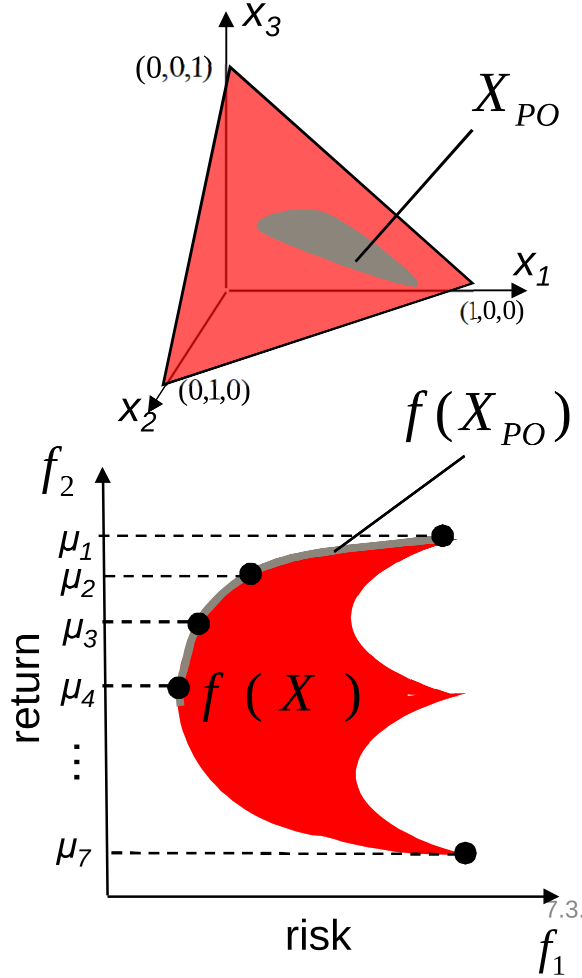


Pareto-optimality in Markowitz model

- ❑ Portfolio *x* is Pareto-optimal, if no other portfolio yields greater or equal expected return with less risk
- ❑ One possibility for computation:
  - Choose d = max number of solutions computed
  - Solve $\mu_1 = \max f_2$, $\mu_d = \min f_2$
  - For all k=2,...,d-1 set μ_k s.t. $\mu_{k-1} > \mu_k > \mu_d$ and solve (1-dimensional) quadratic programming problem

$$\min_{x \in X} \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma_{ij} x_i x_j \text{ such that } \sum_{i=1}^{n} \bar{r}_i x_i = \mu_k$$

  - Discard solutions which are not PO
  - Not attractive when n>2



Algorithms for solving Pareto-optimal solutions (1/2)

- Exact algorithms
  - Guaranteed to find all PO-solutions X_{PO}
  - Only for certain problem types, e.g., Multi-Objective Mixed Integer Linear Programming (MOMILP)

- Use of single-objective optimization algorithms
  - Sequentially solve ordinary (i.e. 1-dimensional) optimization problems to obtain a subset of all PO-solutions, X_{POS}
  - Performance guarantee: $X_{POS} \subseteq X_{PO}$
    - Solutions may not be “evenly” distributed in the sense that majority of the obtained solutions can be very “close” to each other
  - Methods:
    - Weighted sum approach, weighted max-norm approach, ε-constraint approach

Algorithms for solving Pareto-optimal solutions (2/2)

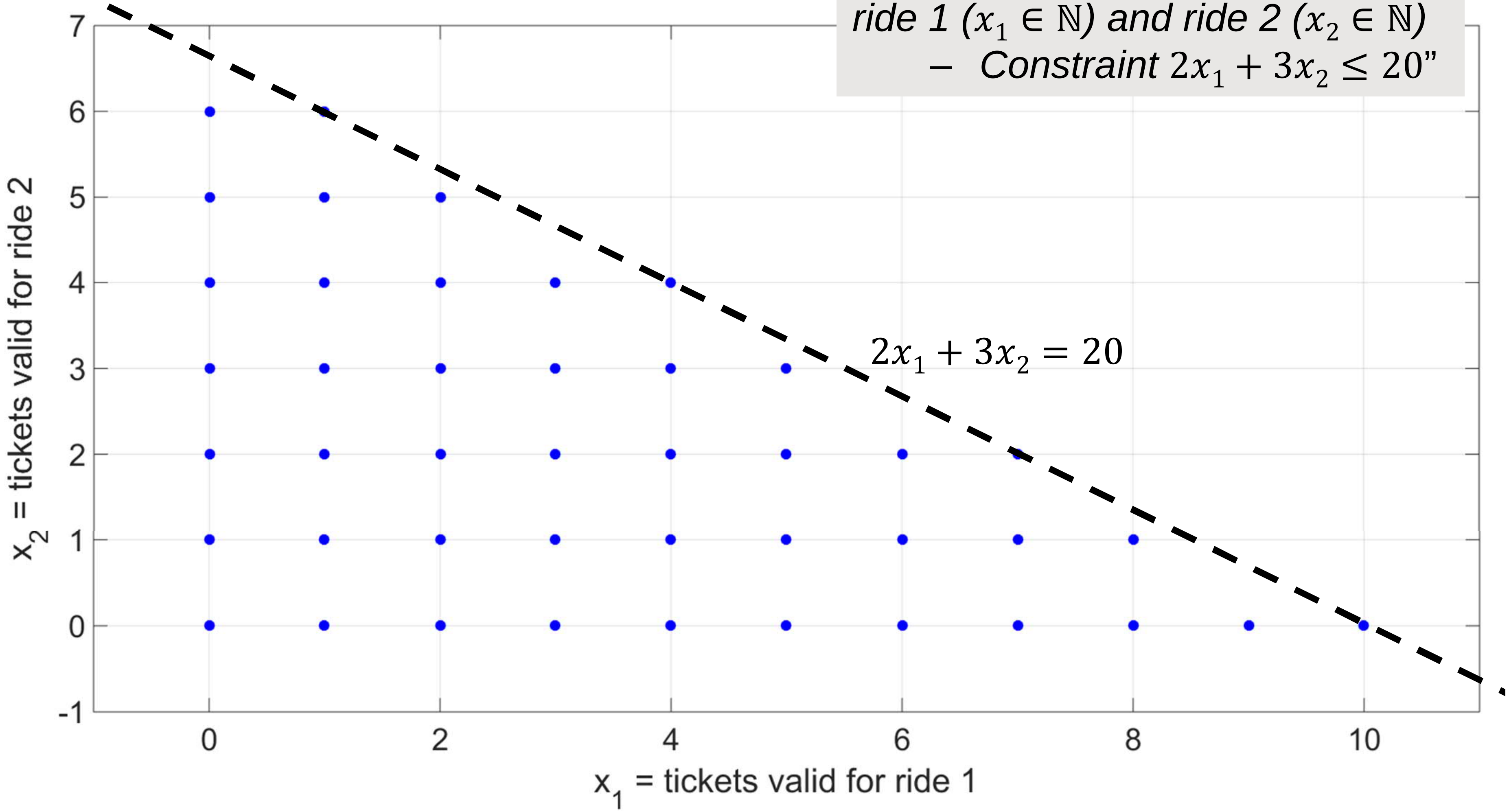
- Approximation algorithms
  - Obtain an approximation X_{POA} of X_{PO} in polynomial time
  - Performance guarantee: For every $x \in X_{PO}$ exists $y \in X_{POA}$ such that $||f(x)-f(y)|| < \varepsilon$
  - Only for very few problem types, e.g., MO knapsack problems
- Metaheuristics
  - No performance guarantees
  - Can handle problems with
    - A large number of variables and constraints
    - Non-linear or non-continuous objective functions/constraints
  - Evolutionary algorithms (e.g., SPEA, NSGA)
  - Stochastic search algorithms (simulated annealing)

Example: Multiobjective integer linear programming (MOILP)

- Ben is at an amusement park that offers 2 different rides:
  - Tickets to ride 1 cost 2 €. Each ticket lets you take the ride twice
  - Tickets to ride 2 are for one ride and cost 3 €
- Ben has 20 euros to spend on tickets to ride 1 ($x_1 \in \mathbb{N}$) and ride 2 ($x_2 \in \mathbb{N}$) → constraint $2x_1 + 3x_2 \leq 20$
- Each time Ben takes ride 2, his grandfather cheers for him
- Ben maximizes the number of (i) rides taken and (ii) cheers → objective functions $f = (f_1, f_2) = (2x_1 + x_2, x_2)$

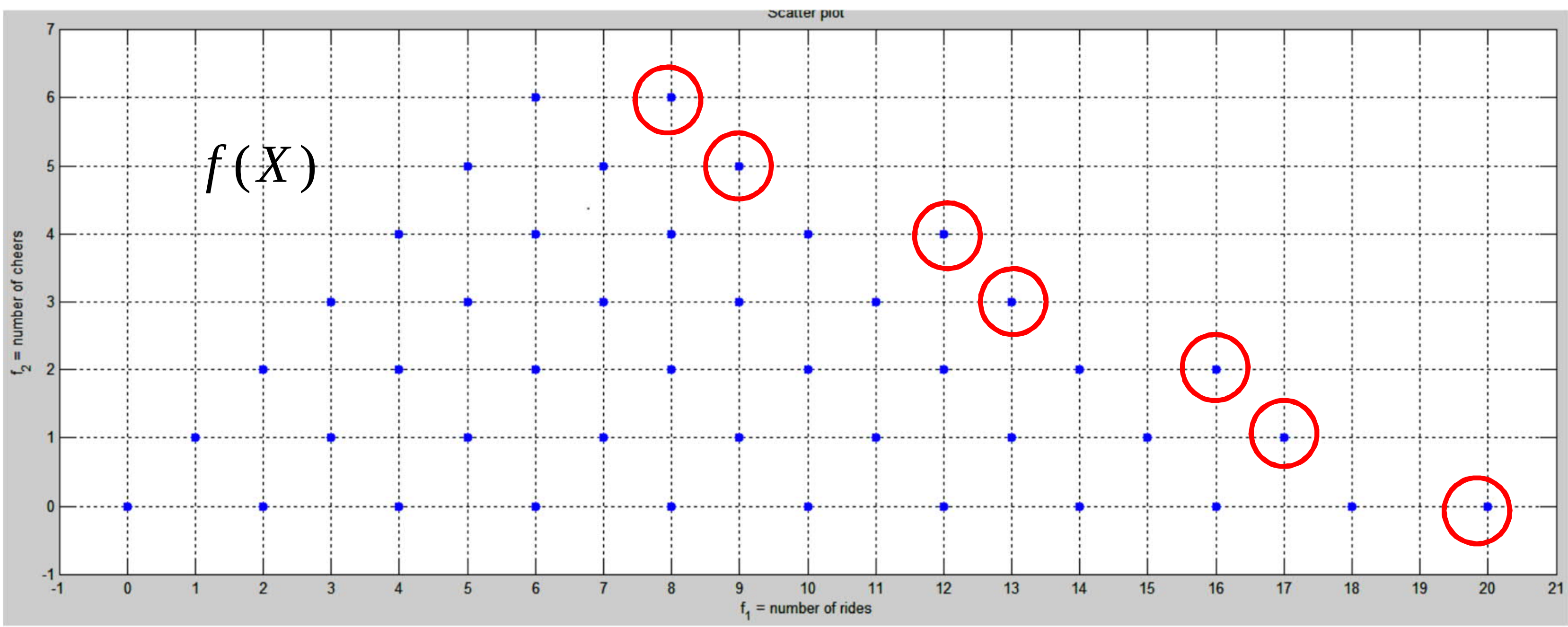
Feasible solutions X

"Ben has 20 euros. He is choosing the number of tickets to ride 1 ($x_1 \in \mathbb{N}$) and ride 2 ($x_2 \in \mathbb{N}$)
- *Constraint $2x_1 + 3x_2 \leq 20$"*

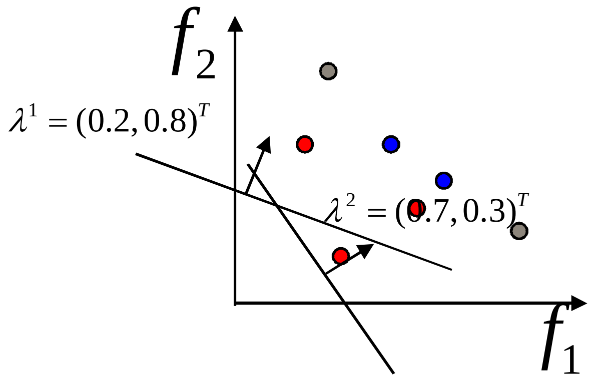


Example: MOILP (cont'd)

- Blue points are feasible solutions; the 7 PO solutions are circled



Weighted sum approach



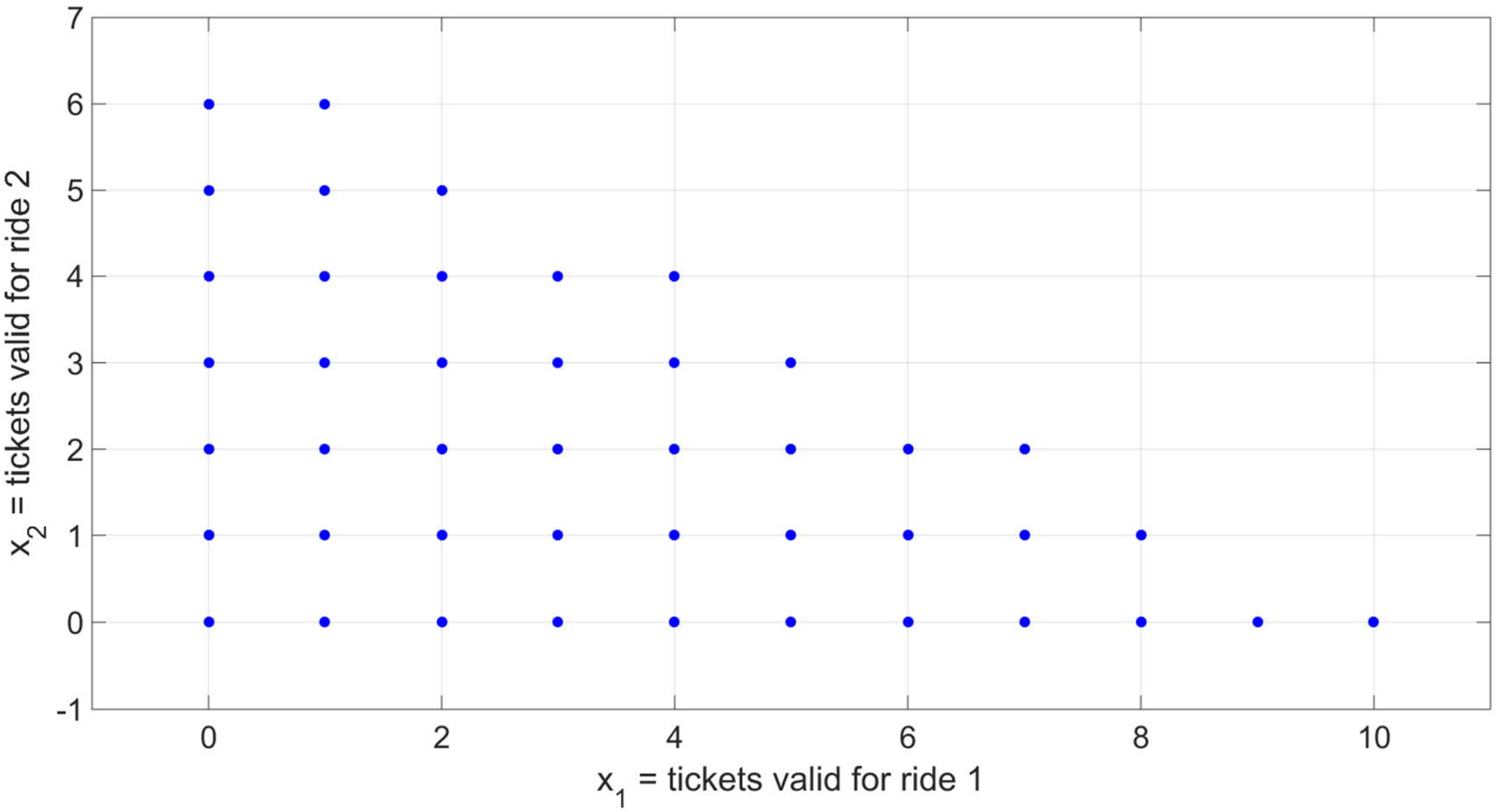
- **Algorithm**
  1. Generate $\lambda \sim UNI(\{\lambda \in [0,1]^n | \sum_{i=1}^{n} \lambda_i = 1\})$
  2. Solve $\max_{x \in X} \sum_{i=1}^{n} \lambda_i f_i(x)$
  3. Solution is Pareto-optimal

  Repeat 1-3 until enough PO-solutions have been found

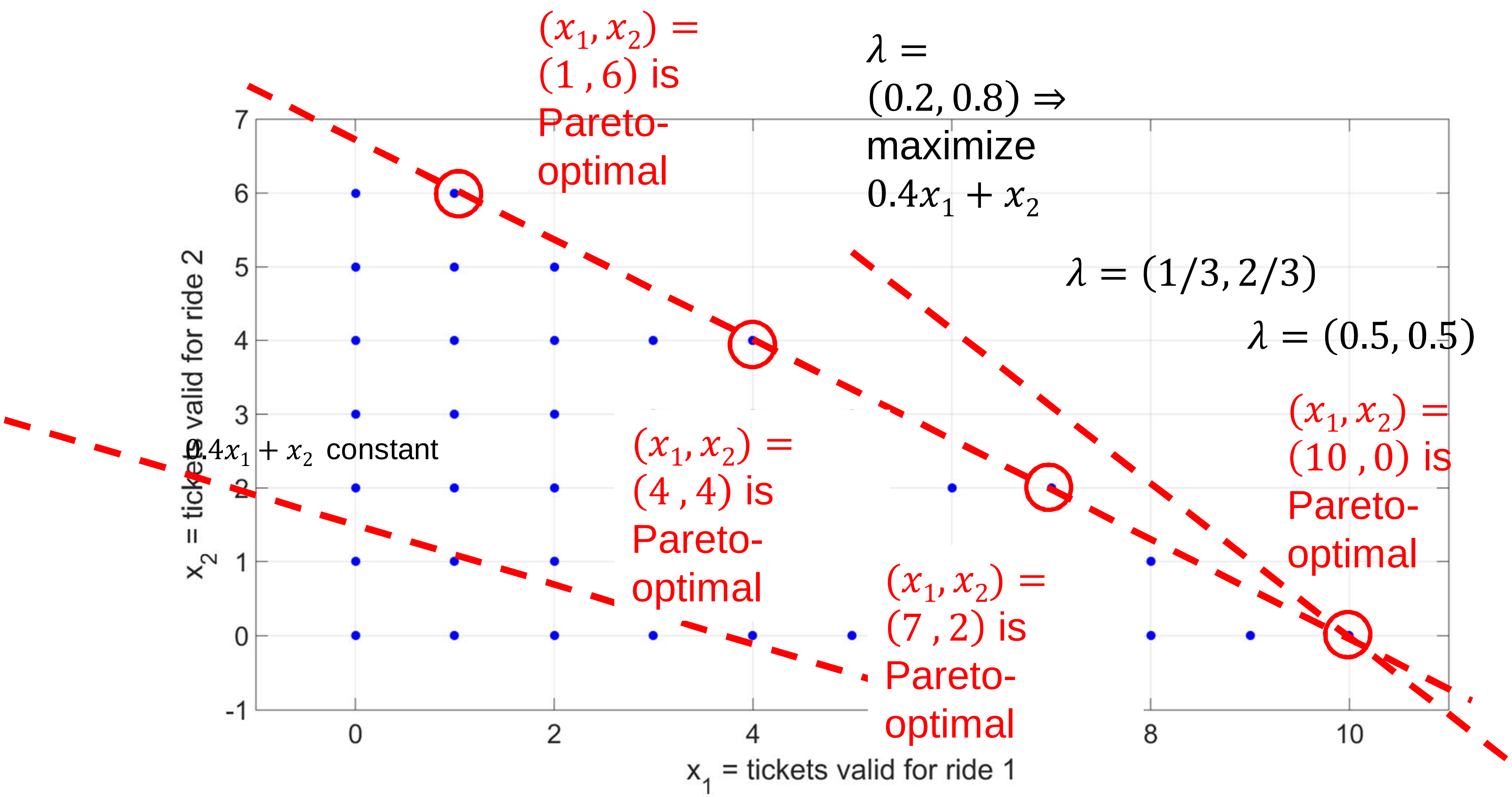
+ Easy to implement

– Cannot find all PO solutions if the problem is non-convex (if PO solutions are not in the border of the convex hull of f(X))

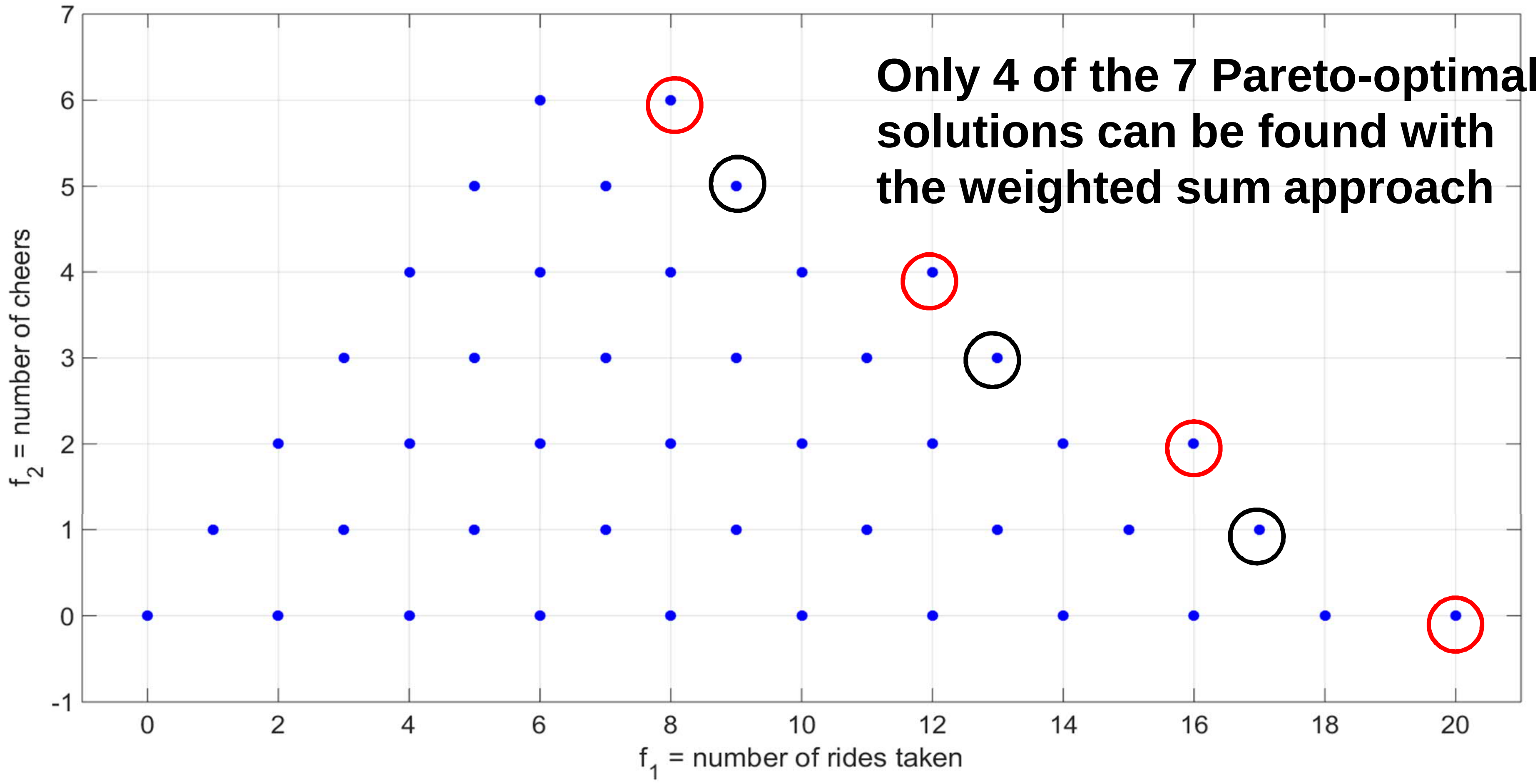
$$\max_{\substack{x_1, x_2 \in \mathbb{N} \\ 2x_1 + 3x_2 \leq 20}} \; [2\lambda_1 x_1 + (\lambda_1 + \lambda_2) x_2]$$



$$\max_{\substack{x_1,x_2\in\mathbb{N}\\ 2x_1+3x_2\le 20}} \left[2\lambda_1 x_1 + (\lambda_1+\lambda_2)x_2\right]$$

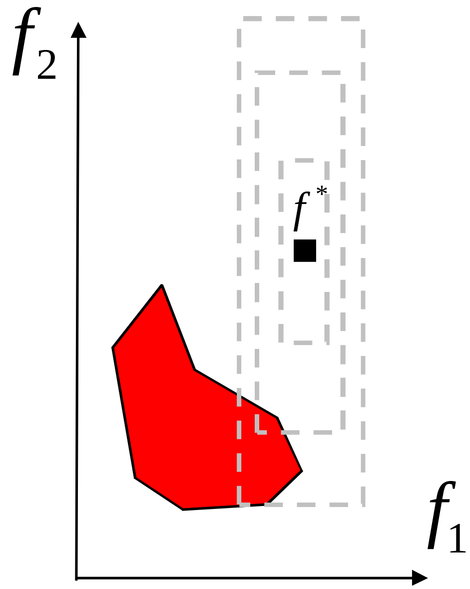


$f(X)$ and Pareto-optimal solutions



Weighted max-norm approach

- Idea: define a utopian vector of objective function values and find a solution for which the distance from this utopian vector is minimized
- Utopian vector: $f^* = [f_1^*, \dots, f_n^*], f_i^* > f_i(x)\ \forall x \in X, i = 1, \dots, n$
- Distance is measured with weighted max-norm $\max_{i=1,\dots,n} \lambda_i d_i$, where d_i is the between f_i^* and $f_i(x)$, and $\lambda_i > 0$ is the weight of objective *i* such that $\sum_{i=1}^n \lambda_i = 1$.
- The solutions that minimize the distance of $f(x)$ from f^* are found by solving:



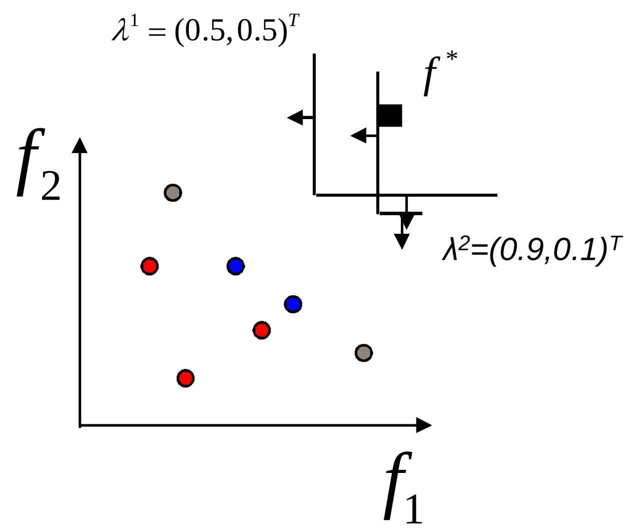
Contours of $\left\| f^* - f(x) \right\|_{\max}^{\lambda}$ when $\lambda = (0.9, 0.1)^T$

$$\min_{x\in X} \|f^* - f(x)\|_{max}^{\lambda} = \min_{x\in X} \max_{i=1,\dots,n} \lambda_i \left(f_i^* - f_i(x)\right)$$
$$= \min_{x\in X, \Delta\in\mathbb{R}} \Delta \quad s.t.\ \lambda_i\left(f_i^* - f_i(x)\right) \le \Delta\ \forall i = 1, \dots, n$$

Weighted max-norm approach (2/2)

- Algorithm
  1. Generate $\lambda \sim UNI(\{\lambda \in [0,1]^n | \sum_{i=1}^n \lambda_i = 1\})$
  2. Solve $\min_{x \in X} \| f^* - f(x) \|_{max}^{\lambda}$
  3. At least one of the solutions of Step 2 is PO

  Repeat 1-3 until enough PO solutions have been found



+ Easy to implement
+ Can find all PO-solutions
– n additional constraints, one additional variable

Example: MOILP (cont'd)

- Find a utopian vector *f**
  - max $f_1 = 2x_1 + x_2$ s.t. $2x_1 + 3x_2 \leq 20$, $x_1, x_2 \geq 0$
    - $x = (10,0)$; $f_1 = 20$
  - max $f_2 = x_2$ s.t. $2x_1 + 3x_2 \leq 20$, $x_1, x_2 \geq 0$
    - $x = (0, 20/3)$; $f_2 = 20/3$
  - Let $f^* = (21,7)$
- Minimize the distance from the utopian vector:

$$\min_{\Delta \in \mathbb{R}} \Delta \text{ s.t.}$$
$$\lambda_1\big(21 - (2x_1 + x_2)\big) \leq \Delta$$
$$\lambda_2(7 - x_2) \leq \Delta$$
$$2x_1 + 3x_2 \leq 20, x_1, x_2 \in \mathbb{N}$$

$\lambda_1 = 0.1$, $\lambda_2 = 0.9$:

$$\min_{\Delta \in \mathbb{R}} \Delta \text{ s.t.}$$
$$2.1 - 0.2x_1 - 0.1x_2 \leq \Delta$$
$$6.3 - 0.9x_2 \leq \Delta$$
$$2x_1 + 3x_2 \leq 20$$
$$x_1, x_2 \in \mathbb{N}$$

Solution: $\Delta = 1.3$, $x = (1,6) \Rightarrow$
$x = (1,6)$, $f = (8,6)$ is PO

Example: MOILP revisited

1. λ_1=0.1; solution: {Δ=1.3, *x*=(1,6)} ⇒ *x*=(1,6), *f*=(8,6) is PO

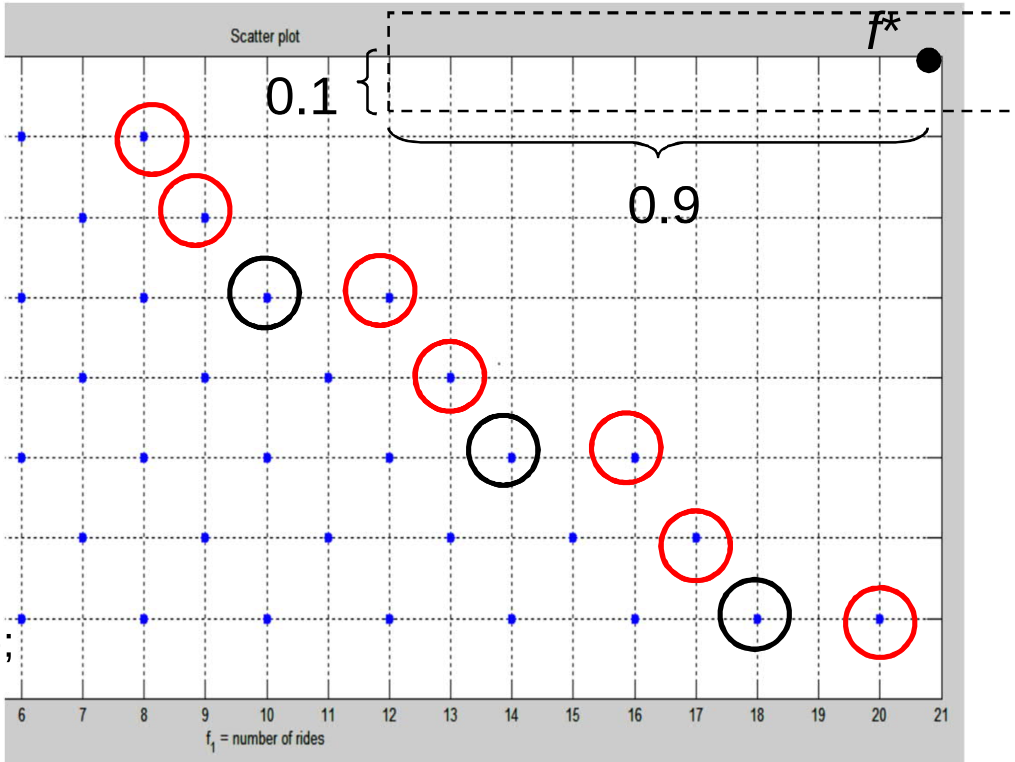
2. λ_1=0.2; 3 solutions *x*=(2,5), *x*=(3,4), *x*=(4,4). Only *x*=(2,5), *f*=(9,5) and *x*=(4,4), *f*=(12,4) are PO

3. λ_1=0.35; *x*=(5,3); *f*=(13,3) is PO

4. λ_1=0.4; 2 solutions *x*=(6,2) and *x*=(7,2); *x*=(7,2), *f*=(16,2) is PO

5. λ_1=0.55; *x*=(8,1); *f*=(17,1) is PO

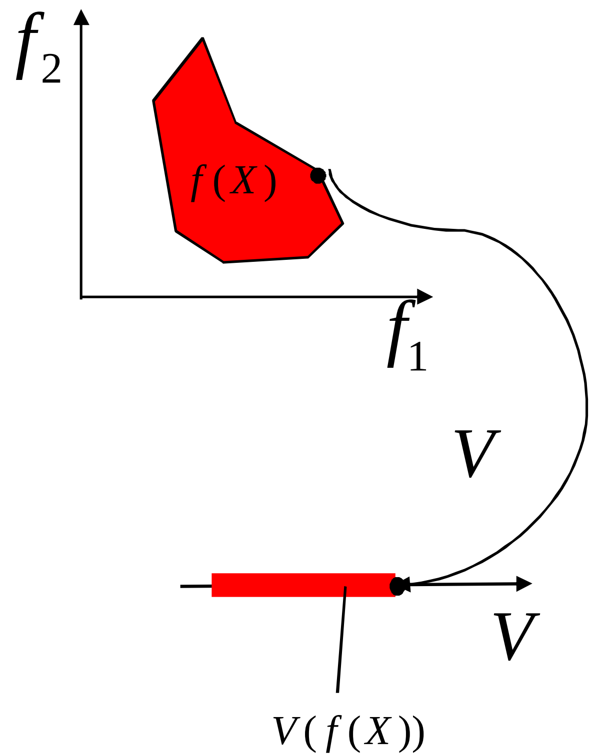
6. λ_1=0.70; 2 solutions *x*=(9,0) and *x*=(10,0); *x*=(10,0), *f*=(20,0) is PO



Value function methods (1/2)

- Use value function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ to transform the MOO problem into a single-objective problem
  - E.g., the additive value function $V\big(f(x)\big) = \sum_{i=1}^{n} w_i v_i(f_i(x))$

- **Theorem:** Feasible solution *x** with the highest value *V(x*)* is Pareto-optimal



Value function methods (2/2)

- Consider the additive value function $V\big(f(x)\big) = \sum_{i=1}^{n} w_i v_i(f_i(x))$ with incomplete weight information $w \in S \subseteq S^0$

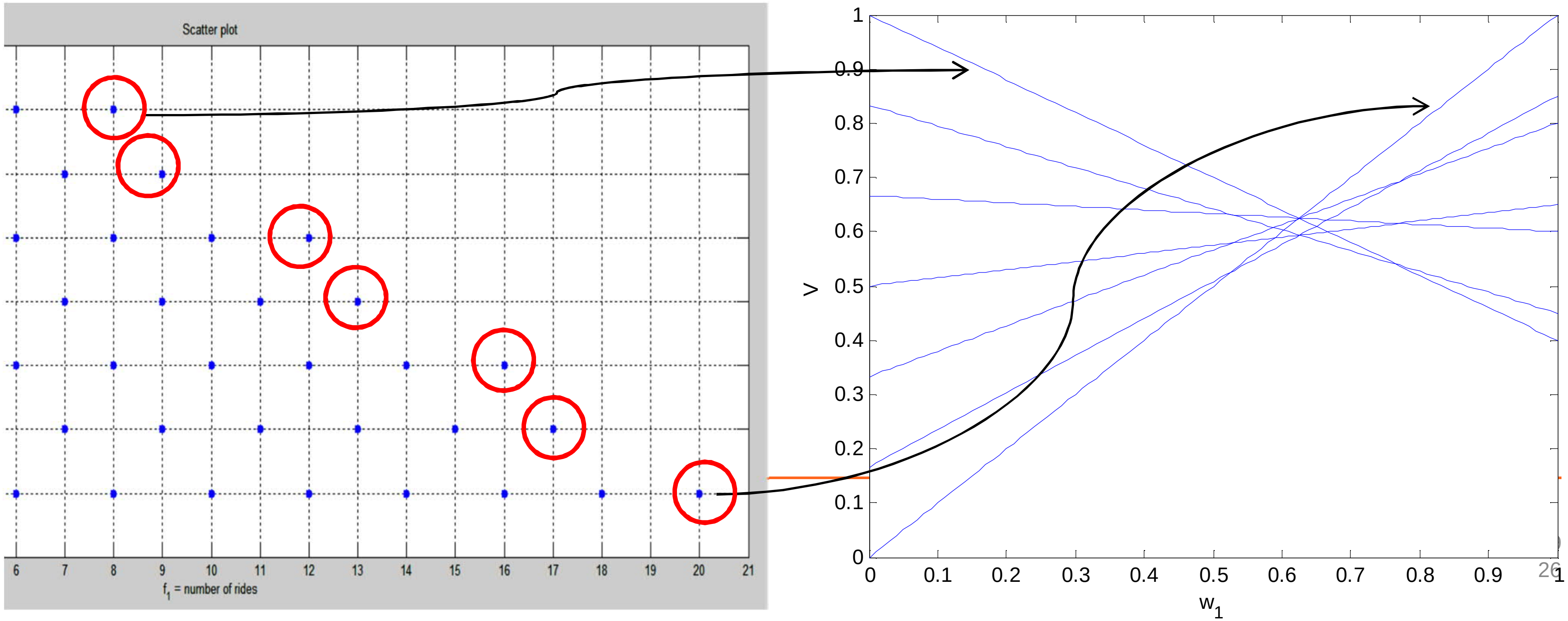
- Set of Pareto-optimal solutions X_{PO}= set of non-dominated solutions with no weight information $X_{ND}(S^0)$

- Preference statements on weights decrease the set of feasible weights to $S \subseteq S^0 \rightarrow$ focus on preferred PO-solutions $X_{ND}(S) \subseteq X_{ND}(S^0) = X_{PO}$

Example: MOILP revisited

- Choose $v_i(f_i(x))=f_i(x)/C_i^*$, normalization constants $C_1^*=20$, $C_2^*=6$

$$V(f(x),w)=\sum_{i=1}^{n} w_i v_i(f(x)) = w_1 v_1\big(f_1(x)\big)+(1-w_1)v_2\big(f_2(x)\big)=\frac{w_1(2x_1+x_2)}{20}+(1-w_1)(x_2/6)$$



Example: Bridge repair program (1/7)

- Total of 313 bridges calling for repair

- Which bridges should be included in the repair program under the next three years?

- Budget of 9,000,000€

- Program can contain *maximum* of 90 bridges
  - Proxy for limited availability of equipment and personnel etc.

- Program must repair the total sum of damages by *at least* 15,000 units

P. Mild, J. Liesiö and A. Salo (2015): Selecting Infrastructure Maintenance Projects with Robust Portfolio Modeling, *Decision Support Systems*

Example: Bridge repair program (2/7)

- Set of feasible solutions *X* defined by linear constraints and binary decision variables:

$$X = \{x \in \{0,1\}^{313} | g(x) \leq 0\}, \quad g(x) = \begin{bmatrix} \sum_{j=1}^{313} c_j x_j - 9000000 \\ \sum_{j=1}^{313} x_j - 90 \\ 15000 - \sum_{j=1}^{313} d_j x_j \end{bmatrix}$$

- x_j = a decision variable: x_j =1 repair bridge j
- $x=[x_1,\ldots,x_{313}]$ is a repair program
- c_j = repair cost of bridge j
- d_j = sum of damages of bridge j

Example: Bridge repair program (3/7)

- Six objective indexes measuring urgency for repair
  1. Sum of Damages ("SumDam")
  2. Repair Index ("RepInd")
  3. Functional Deficiencies ("FunDef")
  4. Average Daily Traffic ("ADTraf")
  5. Road Salt usage ("RSalt")
  6. Outward Appearance ("OutwApp")
- All objectives additive over bridges: $f_i(x) = \sum_{j=1}^{313} v_i^j x_j$,

  where v_i^j is the score of bridge *j* with regard to objective *i:*

Example: Bridge repair program (4/7)

- A multi-objective zero-one linear programming (MOZOLP) problem

$$\text{v}-\max_{x \in X}\left(\sum_{j=1}^{313} v_1^j x_j\,, \dots, \sum_{j=1}^{313} v_6^j x_j\right)$$

- Pareto-optimal repair programs X_{PO} generated using the weighted max-norm approach

$$\min_{x \in X, \Delta \in \mathbb{R}} \Delta$$

$$\Delta \geq \lambda_i\left(f_i^* - \sum_{j=1}^{313} x_j v_i^j\right) \forall i = 1, \dots, 6$$

Example: Bridge repair program (5/7)

- Additive value function applied for modeling preferences between the objectives: $V(x,w) = \sum_{i=1}^{6} w_i f_i(x) = \sum_{i=1}^{6} w_i \sum_{j=1}^{313} v_i^j x_j$
- Incomplete ordinal information about objective weights: {SumDam,RepInd} ≥{FunDef, ADTraf} ≥ {RSalt,OutwApp}

$$S = \left\{ w \in S^0 \,\middle|\, w_i \geq w_j \geq w_k, \forall i = 1{,}2; j = 3{,}4; k = 5{,}6 \right\}$$

- Non-dominated repair programs

$$X_{ND}(S) = \left\{ x \in X \middle| \nexists x' \in X \text{ s.t. } \begin{Bmatrix} V(x',w) \geq V(x,w) \text{ for all } w \in S \\ V(x',w) > V(x,w) \text{ for some } w \in S \end{Bmatrix} \right\}$$

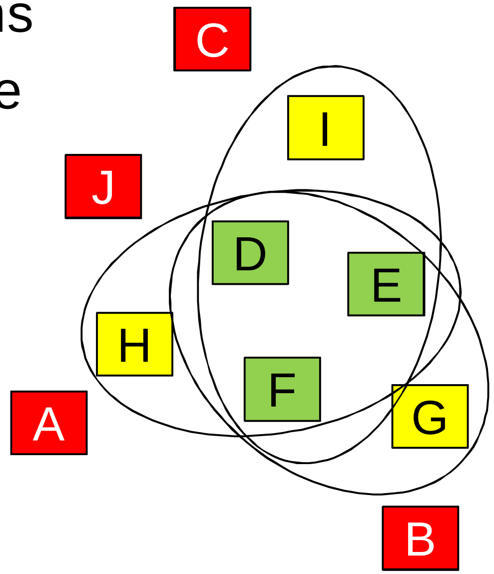
$$X_{PO} = X_{ND}(S^0) \supseteq X_{ND}(S)$$

Example: Bridge repair program (6/7)

- Ca. 10,000 non-dominated bridge repair programs
- Bridge-specific decision recommendations can be obtained through a concept of *core index*:

$$CI_j = \frac{|\{x \in X_{ND}(S) | x_j = 1\}|}{|X_{ND}(S)|}$$

- Of the 313 bridges:
  - 39 were included in all non-dominated repair programs (CI=1)
  - 112 were included in some but not all non-dominated programs (0<CI<1)
  - 162 were included in none of the non-dominated programs (CI=0)



Example: Bridge repair program (7/7)

- Bridges listed in decreasing order of core indices
  - Tentative but not binding priority list
  - Costs and other characteristics displayed

- The list was found useful by the program managers

Bridge number and name	Core Index	BRIDEGES' SCORES						Cost
		DamSum	RepInd	FunDef	ADTraf	Rsalt	OutwApp	
2109 Lavusjoen silta	**1.00**	5.00	1.65	4	2.6	1	2.6	50000
2218 Joroisvirran silta	**1.00**	5.00	5.00	2	5	5	2.6	180000
2217 Rautatieylikulkusilta	**1.00**	3.49	5.00	1.5	5	5	1.8	130000
763 Hurukselantien risteyssilta	**1.00**	2.27	2.33	1	3.4	5	1	280000
80 Suolammenojan silta	**1.00**	1.36	1.53	2	4.2	5	1.8	10000
257 Villikkalan silta	**0.81**	1.97	1.96	5	1	1	1.8	20000
1743 Huuman silta II	**0.76**	1.64	1.53	1	5	5	1.8	140000
730 Mälkiän itäinen risteyssilta	**0.63**	1.33	1.58	1.5	5	5	1	120000
2804 Raikuun kanavan silta	**0.60**	3.93	1.12	2.5	1	1	1	20000
856 Ojaraitin alikulkukäytävä I	**0.54**	1.46	1.46	1	5	5	1	20000
2703 Grahnin alikulkukäytävä	**0.43**	1.70	1.23	1	5	5	1	60000
817 Petäjäsuon risteyssilta	**0.39**	1.52	1.37	1	5	5	1	50000
725 Mustolan silta	**0.29**	1.98	1.93	2	1.8	1	4.2	190000
2189 Reitunjoen silta	**0.24**	1.90	1.63	3	1.8	1	1.8	10000
2606 Haukivuoren pohjoinen ylikulkusilta	**0.15**	1.84	2.09	1.5	2.6	1	1	70000
125 Telataipaleen silta	**0.14**	1.38	1.12	1	5	5	1.8	40000
608 Jalkosalmen silta	**0.03**	1.54	1.50	3	1.8	1	2.6	10000
556 Luotolan silta	**0.00**	1.74	1.26	3	1	1	1.8	10000
661 Raikan silta	**0.00**	1.95	1.58	2	1	1	1.8	10000
2613 Pitkänpohjanlahden silta	**0.00**	1.27	1.16	1	4.2	5	2.6	20000
738 Hyypiälän ylikulkusilta	**0.00**	1.72	1.79	1	3.4	1	1.8	90000
2549 Uitonsalmen silta	**0.00**	1.71	1.37	3	1	1	1	30000
703 Tokkolan silta	**0.00**	1.82	1.70	2	1.8	1	1	10000
870 Tiviän alikulkukäytävä	**0.00**	1.10	1.07	1	5	5	1	20000
377 Sudensalmen silta	**0.00**	1.88	1.66	1	2.6	1	1.8	20000
953 Sydänkylän silta	**0.00**	1.23	1.33	3.5	1	1	1.8	10000
700 Kirjavalan ylikulkusilta	**0.00**	1.42	1.98	1.5	1	1	1	60000
2142 Latikkojoen silta	**0.00**	1.43	1.58	2.5	2.6	1	1.8	20000
464 Jokisilta	**0.00**	1.19	1.25	3.5	1.8	1	1	20000
1025 Hartunsalmen silta	**0.00**	1.18	1.09	3.5	1.8	1	2.6	20000
95 Touksuon silta	**0.00**	1.83	1.18	2	2.6	1	2.6	20000
418 Laukassalmen silta	**0.00**	1.54	1.35	1.5	2.6	1	1.8	10000
420 Sillanmäenojan silta	**0.00**	1.20	1.07	1.5	2.6	1	1.8	10000

Summary

- MOO differs from MAVT in that
  - Alternatives are not explicit but defined implicitly through constraints
  - MOO problems are computationally much harder
- MOO problems are solved by
  - Computing the set of all Pareto-optimal solutions – or at least a subset or an approximation
  - Introducing preference information about trade-offs between objectives to support the selection of one of the PO-solutions

Aalto University
School of Science

Decision making and problem solving – Lecture 9

- Analytic Hierarchy Process
- Outranking methods

Liesiö, Punkka, Salo, Vilkkumaa

Motivation

- When alternatives are evaluated w.r.t. multiple attributes / criteria, decision-making can be supported by methods of
  - Multiattribute value theory (certain attribute-specific performances)
  - Multiattribute utility theory (uncertain attribute-specific performances)

- MAVT and MAUT have a strong axiomatic basis

- Yet, other popular multicriteria methods exist

Analytic Hierarchy Process (AHP)

- Thomas L. Saaty (1977, 1980)

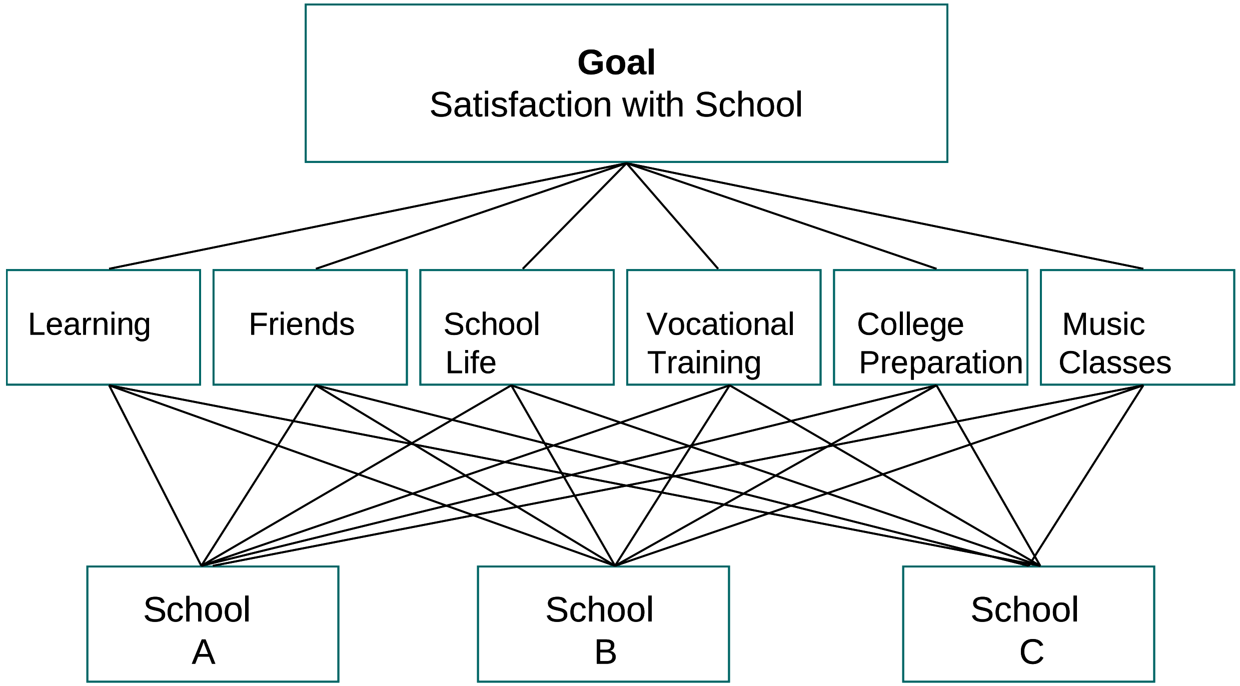
- Enormously popular
  - Thousands of reported applications
  - Dedicated conferences and scientific journals

- Several decision support tools
  - Expert Choice, WebHipre etc.

- Not based on the axiomatization of preferences – therefore remains controversial

Problem structuring in AHP

- Objectives, sub-objectives / criteria, and alternatives are represented as a <u>hierarchy</u> of elements (cf. value tree)



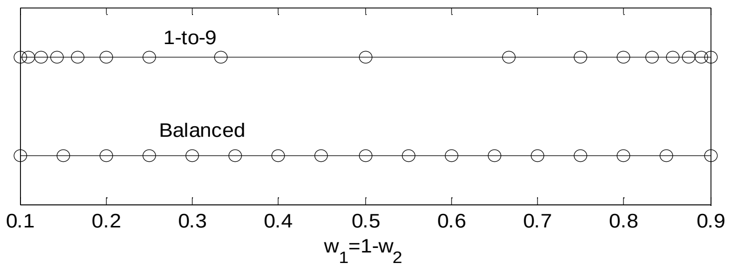
Local priorities

- For each objective / sub-objective, a local priority vector is determined to reflect the relative importance of those elements placed immediately below the objective / sub-objective

- Pairwise comparisons:
  - For (sub-)objectives: ”Which sub-objective / criterion is more important for the attainment of the objective? How much more important is it?”
  - For alternatives: ”Which alternative contributes more to the attainment of the criterion? How much more does it contribute?”

- Responses on a verbal scale correspond to weight ratios

Verbal statement	Scale 1-to-9	Scale Balanced
Equally important	1	1.00
-	2	1.22
Slightly more important	3	1.50
-	4	1.86
Strongly more important	5	2.33
-	6	3.00
Very strongly more important	7	4.00
-	8	5.67
Extremely more important	9	9.00



Pairwise comparison matrix

- Weight ratios $r_{ij} = \frac{w_i}{w_j}$ form a pairwise comparison matrix *A*:

$$A = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} = 1/r_{1n} & \cdots & r_{nn} \end{bmatrix}$$

	L	F	SL	VT	CP	MC
Learning	1	4	3	1	3	4
Friends	1/4	1	7	3	1/5	1
School life	1/3	1/7	1	1/5	1/5	1/6
Voc. training	1	1/3	5	1	1	1/3
College prep.	1/3	5	5	1	1	3
Music classes	1/4	1	$\boxed{6}$	3	1/3	1

Music classes are strongly – very strongly more important than school life

	Learning		
	A	B	C
A	1	1/3	½
B	3	1	3
C	2	1/3	1

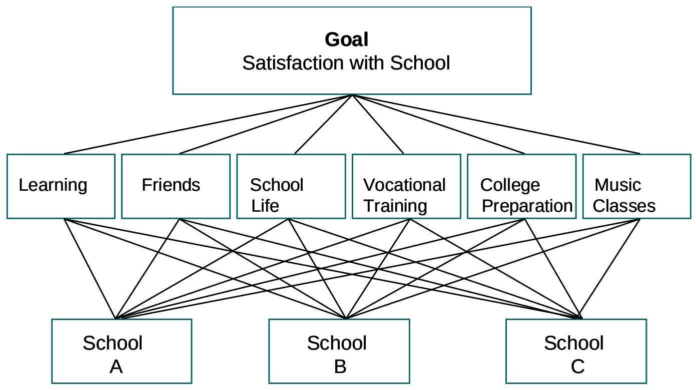
	Friends		
	A	B	C
A	1	1	1
B	1	1	1
C	1	1	1

	School life		
	A	B	C
A	1	5	1
B	1/5	1	1/5
C	1	5	1

	Voc. training		
	A	B	C
A	1	9	7
B	1/9	1	5
C	1/7	1/5	1

	College prep.		
	A	B	C
A	1	1/2	1
B	2	1	2
C	1	1/2	1

	Music classes		
	A	B	C
A	1	6	4
B	1/6	1	1/3
C	1/4	3	1



Incosistency in pairwise comparison matrices

- ❑ **Problem:** pairwise comparisons are not necessarily consistent

- ❑ E.g., if learning is slightly more importannt (3) than college preparation, which is strongly more important (5) than school life, then learning should be 3 × 5 times more important than school life … but this is impossible with the applied scale

→ Weights need to be estimated

Local priority vector

- The local priority vector *w* (=estimated weights) is obtained by normalizing the eigenvector corresponding to the largest eigenvalue of matrix *A:*

$$Aw = \lambda_{max} w,$$
$$w := \frac{1}{\sum_{i=1}^{n} w_i} w.$$

- Matlab:
  - [v,lambda]=eig(A) returns the eigenvectors and eigenvalues of A

```
>> real(v(:,1))/sum(real(v(:,1)))

ans =

    0.1571
    0.5936
    0.2493
```

	Learning			W
	A	B	C	
A	1	1/3	1/2	0.16
B	3	1	3	0.59
C	2	1/3	1	0.25

Only one eigenvector with all real elements: (0.237, 0.896, 0.376) → normalized eigenvector *w*=(0.16, 0.59, 0.25).

```
>> A=[1 1/3 .5; 3 1 3; 2 1/3 1]

A =

    1.0000    0.3333    0.5000
    3.0000    1.0000    3.0000
    2.0000    0.3333    1.0000

>> [v,l]=eig(A)

v =

   0.2370 + 0.0000i   0.1185 + 0.2052i   0.1185 - 0.2052i
   0.8957 + 0.0000i  -0.8957 + 0.0000i  -0.8957 + 0.0000i
   0.3762 + 0.0000i   0.1881 - 0.3258i   0.1881 + 0.3258i

l =

   3.0536 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.0268 + 0.4038i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0268 - 0.4038i
```

Local priority vectors = ”weights”

	Learning			W
	A	B	C	
A	1	1/3	1/2	0.16
B	3	1	3	0.59
C	2	1/3	1	0.25

	Friends			W
	A	B	C	
A	1	1	1	0.33
B	1	1	1	0.33
C	1	1	1	0.33

	School life			W
	A	B	C	
A	1	5	1	0.45
B	1/5	1	1/5	0.09
C	1	5	1	0.46

	Voc. training			W
	A	B	C	
A	1	9	7	0.77
B	1/9	1	5	0.05
C	1/7	1/5	1	0.17

	College prep.			W
	A	B	C	
A	1	1/2	1	0.25
B	2	1	2	0.50
C	1	1/2	1	0.25

	Music classes			W
	A	B	C	
A	1	6	4	0.69
B	1/6	1	1/3	0.09
C	1/4	3	1	0.22

	L	F	SL	VT	CP	MC	W
Learning	1	4	3	1	3	4	0.32
Friends	1/4	1	7	3	1/5	1	0.14
Schoo life	1/3	1/7	1	1/5	1/5	1/6	0.03
Voc. Training	1	1/3	5	1	1	1/3	0.13
College prep.	1/3	5	5	1	1	3	0.24
Music classes	1/4	1	6	3	1/3	1	0.14

Consistency checks

- The consistency of the pairwise comparison matrix *A* is studied by comparing the *consistency index* (*CI*) of *A* to the *average consistency index RI* of a random pairwise comparison matrix:

$$CI = \frac{\lambda_{max} - n}{n-1}, \qquad CR = \frac{CI}{RI}$$

n	3	4	5	6	7	8	9	10
RI	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49

- Rule of thumb: if CR>0.10, comparisons are so inconsistent that they should be revised

Three alternatives, *n=3:*

- Learning: λ_{max} = 3.05, CR = 0.04
- Friends: λ_{max} = 3.00, CR = 0
- School life: λ_{max} = 3.00, CR = 0
- Voc. training λ_{max} = 3.40, CR = 0.34
- College prep: λ_{max} = 3.00, CR = 0
- Music classes: λ_{max} = 3.05, CR = 0.04

Six attributes, *n=6:*

- All attributes: λ_{max} = 7.42, CR = 0.23

```
>> real(max(l))

ans =

    3.0536   -0.0268   -0.0268
```

Total priorities

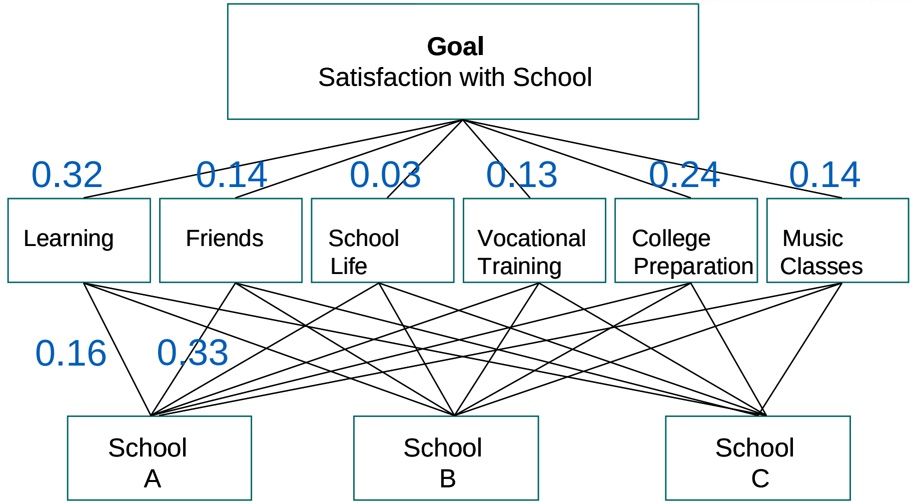
- The total (overall) priorities are obtained recursively:

$$w_k = \sum_{i=1}^{n} w_i \, w_k^i,$$

where

- w_i is the total priority of criterion i,
- w_k^i is the local priority of criterion / alternative k with regard to criterion i,
- The sum is computed over all criteria i below which criterion / alternative k is positioned in the hierarchy

	L	F	SL	VT	CP	MC	W
Learning	1	4	3	1	3	4	0.32
Friends	1/4	1	7	3	1/5	1	0.14
Schoo life	1/3	1/7	1	1/5	1/5	1/6	0.03
Voc. Training	1	1/3	5	1	1	1/3	0.13
College prep.	1/3	5	5	1	1	3	0.24
Music classes	1/4	1	6	3	1/3	1	0.14



	Learning			W
	A	B	C	
A	1	1/3	1/2	$\boxed{0.16}$
B	3	1	3	0.59
C	2	1/3	1	0.25

	Friends			w
	A	B	C	
A	1	1	1	$\boxed{0.33}$
B	1	1	1	0.33
C	1	1	1	0.33

$$w_A = \sum_{i=1}^{6} w_i \, w_k^i = 0.32 \cdot 0.16 + 0.14 \cdot 0.33 + \ldots$$

Total priorities

	Learning			w
	A	B	C	
A	1	1/3	1/2	0.16
B	3	1	3	0.59
C	2	1/3	1	0.25

	Friends			w
	A	B	C	
A	1	1	1	0.33
B	1	1	1	0.33
C	1	1	1	0.33

	School life			w
	A	B	C	
A	1	5	1	0.45
B	1/5	1	1/5	0.09
C	1	5	1	0.46

	Voc. training			w
	A	B	C	
A	1	9	7	0.77
B	1/9	1	5	0.05
C	1/7	1/5	1	0.17

	College prep.			w
	A	B	C	
A	1	1/2	1	0.25
B	2	1	2	0.50
C	1	1/2	1	0.25

	Music classes			w
	A	B	C	
A	1	6	4	0.69
B	1/6	1	1/3	0.09
C	1/4	3	1	0.22

	L	F	SL	VT	CP	MC	w
Learning	1	4	3	1	3	4	0.32
Friends	1/4	1	7	3	1/5	1	0.14
Schoo life	1/3	1/7	1	1/5	1/5	1/6	0.03
Voc. Training	1	1/3	5	1	1	1/3	0.13
College prep.	1/3	5	5	1	1	3	0.24
Music classes	1/4	1	6	3	1/3	1	0.14

	0.32	0.14	0.03	0.13	0.24	0.14	
	L	F	SL	VT	CP	MC	Total w
A	0.16	0.33	0.45	0.77	0.25	0.69	0.37
B	0.59	0.33	0.09	0.05	0.50	0.09	**0.38**
C	0.25	0.33	0.46	0.17	0.25	0.22	0.25

E.g.,
w_B=0.32*0.59+0.14*0.33+0.03*0.09+
0.13*0.05+0.24*0.50+0.14*0.09

Problems with AHP

- **Rank reversals**: the introduction of an additional alternative may change the relative ranking of the other alternatives
- Example:
  - Alternatives A and B are compared w.r.t. two ”equally important” criteria C_1 and C_2 ($w_{C1} = w_{C2} = 0.5$)
  - A is better than B:

$$w_A = \frac{1}{2} \times \frac{1}{5} + \frac{1}{2} \times \frac{5}{6} \approx 0.517, \qquad w_B = \frac{1}{2} \times \frac{4}{5} + \frac{1}{2} \times \frac{1}{6} \approx 0.483$$

  - Add C which is identical to A:

$$w_A = w_C = \frac{1}{2} \times \frac{1}{6} + \frac{1}{2} \times \frac{5}{11} \approx 0.311, \qquad w_B = \frac{1}{2} \times \frac{4}{6} + \frac{1}{2} \times \frac{1}{11} \approx 0.379$$

  - Now B is better than A!

	C_1	C_2
A	1	5
B	4	1
C	**1**	**5**

Methods based on outranking relations

- Basic question: is there enough preference information / evidence to state that an alternative is at least as good as some other alternative?

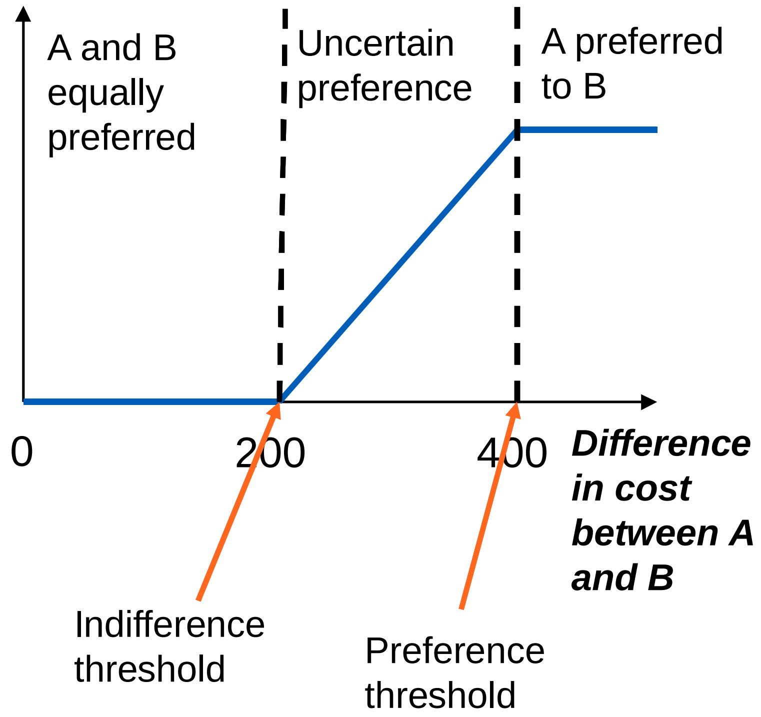
- I.e., does an alternative *outrank* some other alternative?

Indifference and preference thresholds divide the measurement scale into three parts

- If the difference between the criterion-specific performances of A and B is below a pre-defined **indifference threshold**, then A and B are ”equally good” w.r.t. this criterion

- If the difference between the criterion-specific performances of A and B is above a pre-defined **preference threshold**, then A is preferred to B w.r.t this criterion

- Between indifference and preference thresholds, the DM is uncertain about preference



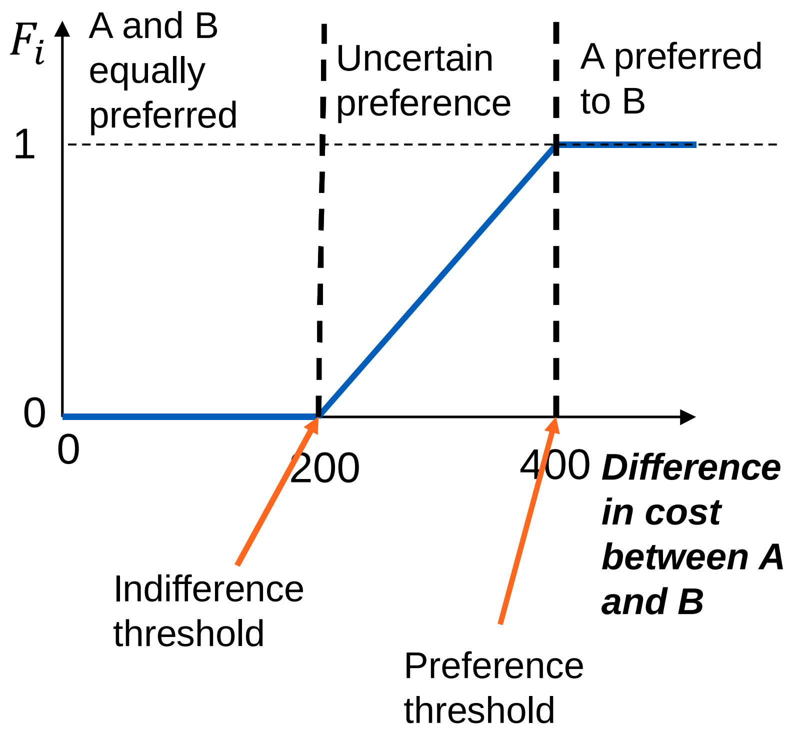
PROMETHEE I & II

- In PROMETHEE methods, the degree to which alternative *k* is preferred to *l* is

$$\sum_{i=1}^{n} w_i F_i(k,l) \geq 0,$$

where

- w_i is the weight of criterion i
- $F_i(k,l)$ =1, if k is preferred to l w.r.t. criterion i,
- $F_i(k,l)$ =0, if the DM is indifferent between k and l w.r.t. criterion i, or l is preferred to k
- $F_i(k,l) \in (0,1)$, if preference between k and l w.r.t. criterion i is uncertain



PROMETHEE I & II

- PROMETHEE I: *k* is preferred to *k'*, if

$$\sum_{l\neq k}\sum_{i=1}^{n} w_i F_i(k,l) > \sum_{l\neq k'}\sum_{i=1}^{n} w_i F_i(k',l)$$
$$\sum_{l\neq k}\sum_{i=1}^{n} w_i F_i(l,k) < \sum_{l\neq k'}\sum_{i=1}^{n} w_i F_i(l,k')$$

There is more evidence in favor of *k* than *k'*

There is less evidence against *k* than *k'*

- PROMETHEE II: *k* is preferred to *k'*, if

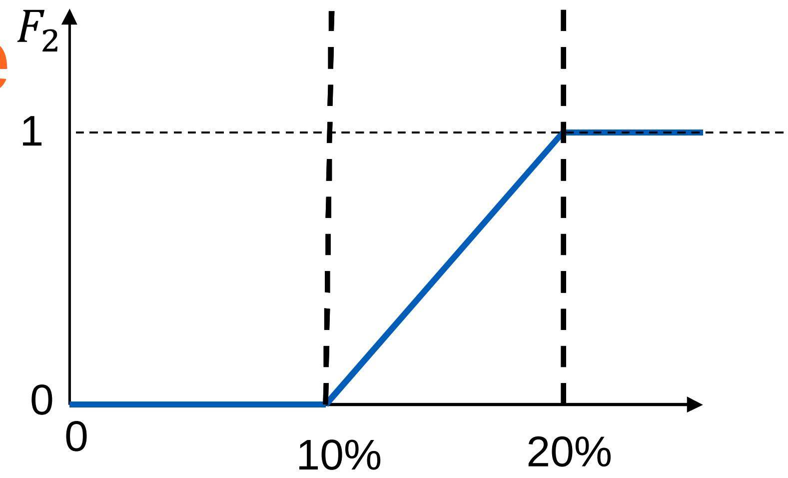
$$F_{net}(k) = \sum_{l\neq k}\sum_{i=1}^{n} w_i[F_i(k,l) - F_i(l,k)] > \sum_{l\neq k'}\sum_{i=1}^{n} w_i[F_i(k',l) - F_i(l,k')] = F_{net}(k')$$

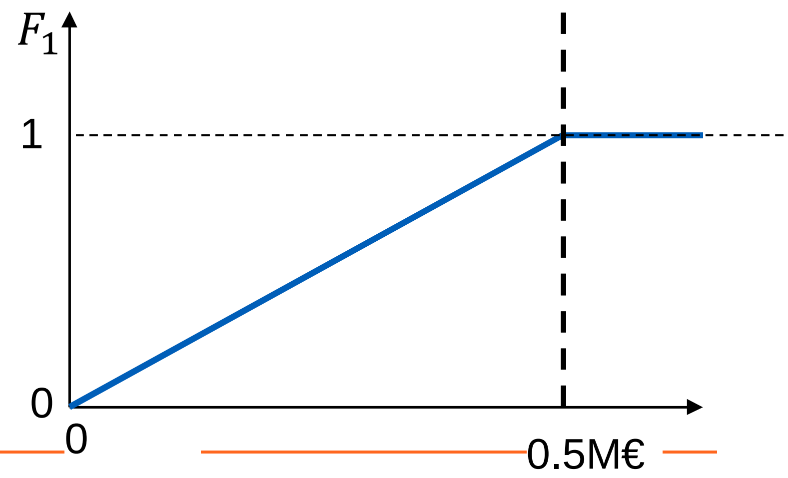
The "net evidence" for *k* is larger than for *k'*

PROMETHEE: Example

	Revenue	Market share
x^1	1M€	10%
x^2	0.5M€	20%
x^3	0	30%
Indiff. threshold	0	10%
Pref. threshold	0.5M€	20%
Weight	1	1

	Revenue F_1	Market share F_2	Weighted $F_w=w_1F_1+w_2F_2$
x^1, x^2	1	0	1
x^2, x^1	0	0	0
x^1, x^3	1	0	1
x^3, x^1	0	1	1
x^2, x^3	1	0	1
x^3, x^2	0	0	0





PROMETHEE I: Example

	F_1	F_2	F_w
x^1, x^2	1	0	1
x^2, x^1	0	0	0
x^1, x^3	1	0	1
x^3, x^1	0	1	1
x^2, x^3	1	0	1
x^3, x^2	0	0	0

- PROMETHEE I:
  - x^1 is preferred to x^2, if

$$\underbrace{\sum_{i=1}^{2}\left(F_i(x^1,x^2)+F_i(x^1,x^3)\right)}_{=1+1=2} > \underbrace{\sum_{i=1}^{2}\left(F_i(x^2,x^1)+F_i(x^2,x^3)\right)}_{=0+1=1}$$

$$\underbrace{\sum_{i=1}^{2}\left(F_i(x^2,x^1)+F_i(x^3,x^1)\right)}_{=0+1=1} < \underbrace{\sum_{i=1}^{2}\left(F_i(x^1,x^2)+F_i(x^3,x^2)\right)}_{=1+0=1}$$

  - x^1 is not preferred to x^2 due to the latter condition
  - x^2 is not preferred to x^1 due to both conditions
  - x^1 is preferred to x^3
  - x^2 is not preferred to x^3 and vice versa
- Note: preferences are not transitive
  - $x^1 \succ x^3 \sim x^2 \nRightarrow x^1 \succ x^2$

PROMETHEE I: Example (Cont'd)

- PROMETHEE I is also prone to rank reversals:
  - Remove x^2
  - Then,

$$\underbrace{\sum_{i=1}^{2}\left(F_i(x^1,x^3)\right)}_{=1} \nsucc \underbrace{\sum_{i=1}^{2}\left(F_i(x^3,x^1)\right)}_{=1}$$

$$\underbrace{\sum_{i=1}^{2}\left(F_i(x^3,x^1)\right)}_{=1} \nprec \underbrace{\sum_{i=1}^{2}\left(F_i(x^1,x^3)\right)}_{=1}$$

$\rightarrow x^1$ is no longer preferred to x^3

	F_1	F_2	F_w
~~x^1, x^2~~	~~1~~	~~0~~	~~1~~
~~x^2, x^1~~	~~0~~	~~0~~	~~0~~
x^1, x^3	1	0	1
x^3, x^1	0	1	1
~~x^2, x^3~~	~~1~~	~~0~~	~~1~~
~~x^3, x^2~~	~~0~~	~~0~~	~~0~~

PROMETHEE II: Example

- The "net flow" of alternative x^j

$$F_{net}(x^j) = \sum_{k \neq j} [F_w(x^j, x^k) - F_w(x^k, x^j)]$$

– $F_{net}(x^1) = (1-0) + (1-1) = 1$
– $F_{net}(x^2) = (0-1) + (1-0) = 0$
– $F_{net}(x^3) = (1-1) + (0-1) = -1$

$\rightarrow x_1 \succ x_2 \succ x_3$

	F_1	F_2	F_w
x^1, x^2	1	0	1
x^2, x^1	0	0	0
x^1, x^3	1	0	1
x^3, x^1	0	1	1
x^2, x^3	1	0	1
x^3, x^2	0	0	0

PROMETHEE II: Example (Cont'd)

- PROMETHEE II is also prone to rank reversals
  - Add two altrenatives that are equal to x^3 in both criteria. Then, x^2 becomes the most preferred:
    $$F_{net}(x^1) = (1-0) + 3 \times (1-1) = 1$$
    $$F_{net}(x^2) = (0-1) + 3 \times (1-0) = 2$$
    $$F_{net}(x^{3:5}) = (1-1) + (0-1) = -1$$
  - Add two alternatives that are equal to x^1 in both criteria. Then, x^2 becomes the least preferred:
    $$F_{net}(x^{1,4,5}) = (1-0) + (1-1) + 2 \times (0-0) = 1$$
    $$F_{net}(x^2) = 3 \times (0-1) + (1-0) = -2$$
    $$F_{net}(x^3) = 3 \times (1-1) + (0-1) = -1$$
  - Remove x^2. Then, x^1 and x^3 are equally preferred.
    $$F_{net}(x^1) = F_{net}(x^3) = (1-1) = 0$$

	F_1	F_2	F_w
x^1, x^2	1	0	1
x^2, x^1	0	0	0
x^1, x^3	1	0	1
x^3, x^1	0	1	1
x^2, x^3	1	0	1
x^3, x^2	0	0	0

Summary

- AHP and outranking methods are commonly used for supporting multiattribute decision-making
- Unlike MAVT (and MAUT), these methods do not build on the axiomatization of preferences →
  - Rank reversals
  - Preferences are not necessarily transitive
- Elicitation of model parameters can be difficult
  - Weights have no clear interpretation
  - In outranking methods, statement ”I prefer 2€ to 1€” and ”I prefer 3€ to 1€” are both modeled with the same number (1); to make a difference, indifference and preference thresholds need to be carefully selected

Decision making and problem solving – Lecture 10

- Group techniques
- Voting
- MAVT for group decisions

Liesiö, Punkka, Salo, Vilkkumaa

Motivation

- Thus far we have assumed that
  - Objectives, attributes/criteria, and decision alternatives are given
  - There is a single decision maker

- This time we'll learn
  - How a group of experts / DMs can be used to generate objectives, attributes, and/or decision alternatives
  - How to aggregate the views and preferences of the group members into a single decision recommendation

Idea generation and evaluation techniques

- Goals:
  - Generate topics / ideas / decision alternatives
  - Evaluate these topics / ideas / alternatives
  - Agree on a prioritization of the topics / ideas / alternatives

- Methods:
  - Brainstorming
  - Nominal group technique
  - Delphi method
  - ...and variants of the above

Brainstorming

- **Goal:** to generate a large number of possible solutions for a problem

- **Participants:** Facilitator, recorder, and max 8-12 panel members
  - Step 1 Prior notification: time for individual idea generation
  - Step 2 Session for idea generation: all ideas are listed, spotaneous ideas are encouraged, <u>no criticism is allowed</u>
  - Step 3 Review and evaluation: a list of ideas is sent to the panel members for further study

- **Principles:**
  - Focus on quantity
  - Withhold criticism
  - Welcome unusual ideas
  - Combine and improve ideas

Brainstorming

+ A large number of ideas can be generated in a short period of time
+ Simple – no expertise or knowledge required from the facilitator

- Blocking: during the process, participants may forget their ideas or not share them because they no longer find them relevant
- Collaborative fixation: Exchanging ideas in a group may decrease the novelty and variety of ideas

Nominal group technique

- **Goal:** to generate a large number of possible solutions for a problem and decide on a solution

- **Participants:** Faciliator, recorder, and max 6-12 panel members
  - Step 1: Silent generation of ideas – group work not allowed
  - Step 2: Round-robin sharing of ideas. Facilitator lists all ideas on a flip chart, no comments at this point.
  - Step 3: Group discussion to facilitate common understanding of the presented ideas. No ideas are eliminated, judgment and criticism are avoided.
  - Step 4: Ranking of the ideas (by, e.g., voting)

Nominal group technique

+ A large number of ideas can be generated in a short period of time
+ Silent generation of ideas decreases blocking
+ Round-robin process ensures equal participation

- Not suitable for settings where consensus is required
- Can be time-consuming

Delphi technique

- **Goal:** To obtain quantitative estimates about some future events (e.g., estimated probabilities, impacts, and time spans of negative trends for Finland)
- **Participants:** Faciliator and a panel of experts
- **Principles:**
  - Anonymous participation
  - Structured gathering of information through questionnaires: numerical estimates and arguments to support these estimates
  - Iterative process: participants comment on each other's estimates and are encouraged to revise their own estimates in light of such comments
  - Role of the facilitator: sends out the questionnaires, organizes the information, identifies common and conflicting viewpoints, works toward synthesis

Example: Decision analysis based real world conflict analysis tools

- Workshop organized by the Finnish Operations Research Society (FORS) Monday 5.10.2015

- Goal: to practice DA-based conflict analysis tools that Crisis Management Initiative (CMI) uses regularly in its operations:
  - Trend identification,
  - Data collection,
  - Visualization,
  - Root-cause analysis.

Example cont'd

- Prior to the workshop, each participant was asked to
  - List 3-5 negative trends for Finland (title and brief description)
  - Provide time-spans for the impacts of these trends (<10 years, 10-20 years, >20 years)

Trend identification exercise: Negative trends for Finland

Negative trend is a, possibly escalating, course of events that would lead to harmful consequences. Please provide three to five negative trends that can have harmful impacts on the development of Finland. There are no limitations regarding the scope of the trend; it can be either broad or specific trend.

These trends can be related to
demographics,
economic situation,
welfare of citizens,
environment,
political situation
or other topics.

Give a short title for each trend and a longer explanation of why this trend can be harmful.

OBJECTIVES OF THE WORKSHOP. READING THIS MAY HELP YOU TO SUGGEST RELEVANT TRENDS:
The objective of this workshop is to evaluate and discuss these trends. This includes evaluation of (i) the probabilities that these trends cause significant harmful impacts, and (ii) the magnitudes of these impacts. The next step would be to use this information to design policy actions that can help to mitigate these trends and to adapt to them.

*Required

Trend 1 (title) *

Trend 1 (explanation) *

Example cont'd

- ❑ Trends listed by the participants were organized by the workshop facilitators
  - – Similar trends combined
  - – Marginal trends eliminated

- ❑ A final list of 21 trends was emailed to the participants prior to the workshop

Trend evaluation exercise

Trends, time-scales and explanations.

1.) Urbanization (10-20 years)
A worsening economic situation can send people to seek employment in urban areas, leaving much of the Finnish rural areas depopulated. As these rural areas already have functioning infrastructure, this causes inefficiency.

2.) Bifurcation of Finns and political radicalization (<10)
Tough economic times combined with other crises can create rifts between Finns. In many political issues, there seems to be an increased tendency to polarize the matter, creating only two sides with little discourse. For example, worker's unions vs. employers, urban vs. rural, pro-immigration vs. anti-immigration.

3.) The "welfare trap" (<10)
The social security system can discourage the unemployed to accept low-paying part-time work. This can lead to the situation where individuals would rather receive constant benefits rather than risk losing or decreasing their income by taking a job.

4.) Passive political system (<10)
In the past years, the government has shown an inability to react with speed and decisiveness to many issues facing Finland today. Delayed preventive actions can cause crisis situations to escalate.

5.) Socially excluded youth (10-20)
During a recession, getting a job and joining society as a productive member can be challenging, especially for young people applying for schooling or work. To maintain social stability and ensure future economic success, the youth should be integrated into society, or else there is a risk they become permanent outsiders.

6.) The Retirement Bomb (20+)
The current pension system might be unable to handle the aging population. The number of employed may be too low to pay for pensions.

7.) Brain drain (10-20)
Talented and educated people who are dissatisfied with the current situation in Finland might emigrate to find more suitable conditions to work in.

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Example cont'd

- At the workshop, each participant was asked to evaluate
  - The probability of each trend being realized (scale 0-5)
  - The impact that the trends would have upon realization (scale 0-5)

Prioritization

You are asked to evaluate each trend with respect to (1) the probability of the trend realizing in significant negative consequences and (2) the impact the trend would have upon realization.

In both dimensions we use a scale of 0 to 5. For the probabilities, 0 means close to impossible and 5 means next to certain. For the impacts, 0 means close to no significant negative impact and 5 means a very significant negative impact.

Probability of each trend realizing in significant negative consequences *

Tick your best guess for each. 0 means close to impossible and 5 means next to certain.

	0	1	2	3	4	5
1.) Urbanization	○	○	○	○	○	○
2.) Bifurcation of Finns and political radicalization	○	○	○	○	○	○
3.) The "welfare trap"	○	○	○	○	○	○

.
.
.

Impacts that the trends would have upon realization *

Tick your best guess for each. 0 means close to no significant negative impact and 5 means a very significant negative impact.

	0	1	2	3	4	5
1.) Urbanization	○	○	○	○	○	○
2.) Bifurcation of Finns and political radicalization	○	○	○	○	○	○
3.) The "welfare trap"	○	○	○	○	○	○
4.) Passive political system	○	○	○	○	○	○

Example cont'd

- The participants were also asked to assess *cross-impacts* among trends
  - Which other trends does this trend enhance?

Cross-impact analysis

Some of the trends enhance another trends. In this exercise you are asked to identify for each trend 0 to 3 other trends that it strongly enhances.

1.) Urbanization

Choose 0-3 trends that this trend impacts strongly

☐ 2.) Bifurcation of Finns and political radicalization
☐ 3.) The "welfare trap"
☐ 4.) Passive political system

.
.
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2.) Bifurcation of Finns and political radicalization

Choose 0-3 trends that this trend impacts strongly

☐ 1.) Urbanization
☐ 3.) The "welfare trap"
☐ 4.) Passive political system

.
.
.

21.) Economic stagnation

Choose 0-3 trends that this trend impacts strongly

☐ 1.) Urbanization
☐ 2.) Bifurcation of Finns and political radicalization
☐ 3.) The "welfare trap"
☐ 4.) Passive political system

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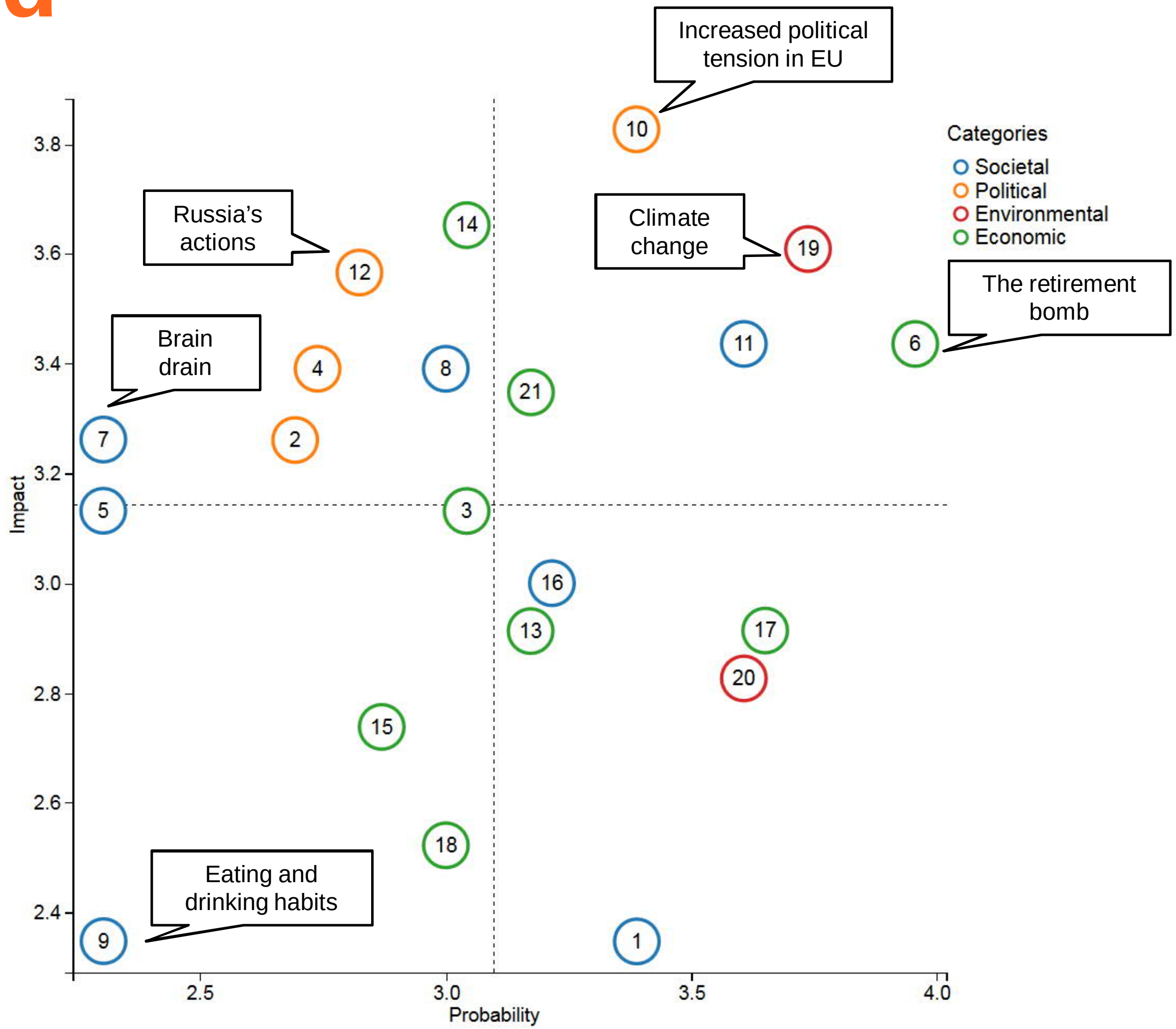
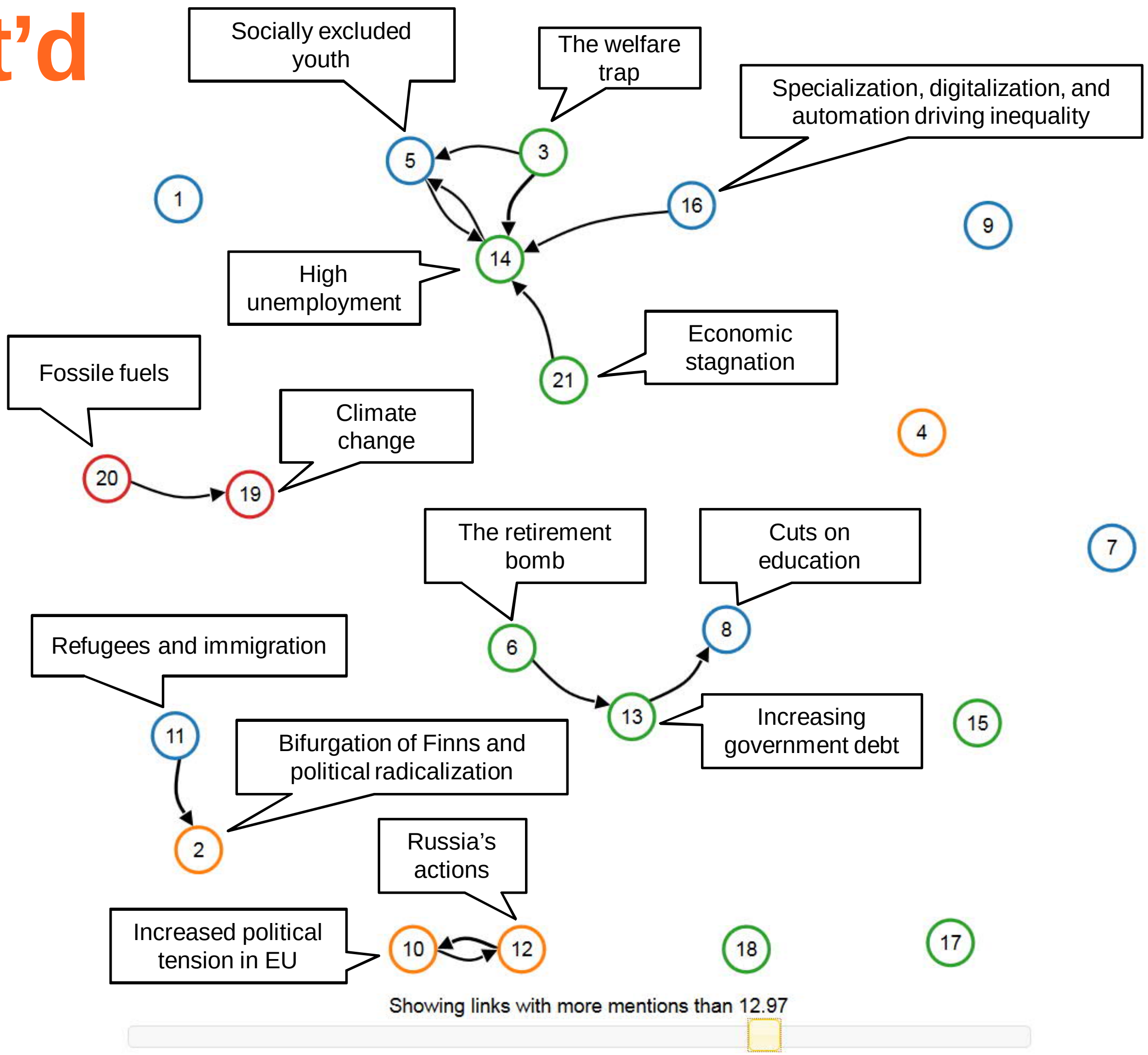
Example cont'd

- Visualizations on the probability and impact assessments were shown to the participants to facilitate discussion



Example cont’d

- Cross-impacts were visualized, too



Example cont'd

- Goal of such analysis:
  - To create a shared understanding of the problem
  - To identify possible points of disagreement

- Next steps:
  - Possible revision of estimates in light of the discussion
  - The determination of policy actions to help mitigate / adapt to the most important negative trends
  - Agreement on which policy actions to pursue
  - The implementation of these policy actions

Aggregation of preferences

- Consider N alternatives $x_1, \ldots, x_N$

- Consider K decision makers $DM_1, \ldots DM_K$ with different preferences about the alternatives

- How to aggregate the DMs' preferences into a group choice?
  - Voting
  - MAVT

Plurality voting

- Each voter casts one vote to his/her most preferred candidate
- The candidate with the most votes wins
- Plurality voting with runoff:
  - The winner must get over 50% of the votes
  - If this condition is not met, alternatives with the least votes are eliminated
  - Voting is continued until the condition is met
  - E.g., Finnish presidential election: in the second round only two candidates remain

Plurality voting

- Suppose, there are three alternatives A, B, C, and 9 voters
  - 4 think that A > B > C
  - 3 think that B > C > A
  - 2 think that C > B > A

<u>Plurality voting</u>

4 votes for A
3 votes for B
2 votes for C

⇨ A is the winner

<u>Run-off</u>

C eliminated

4 votes for A
3+2 = 5 votes for B

⇨ B is the winner

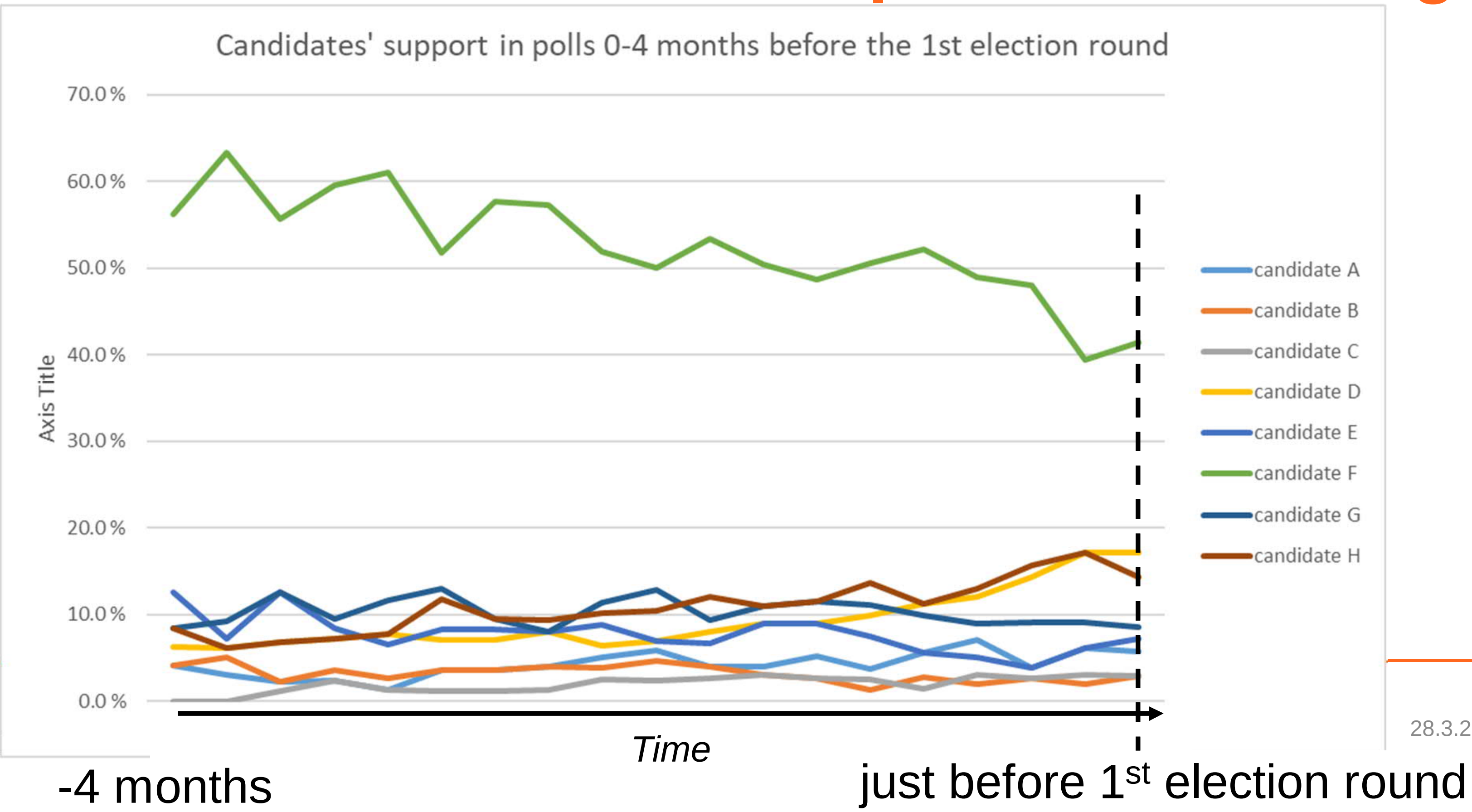
Example: Finnish Presidential elections

- **Organized every 6 six years**

- **Plurality voting with runoff**
  - 2 most voted candidates to the 2nd round, unless some candidate receives over 50 % of votes already on the 1st round

- **7-11 candidates in 1994-2018**
  - Some candidates can have moderate support, but strong opposition
    - I.e., they are ranked 1st by some, but last or close to last by many other voters

Polls just before the 1st election round suggest that candidate F is the strongest, but a 2nd will be needed. The battle for the 2nd position will be tight



A

Time for 1st round vote!

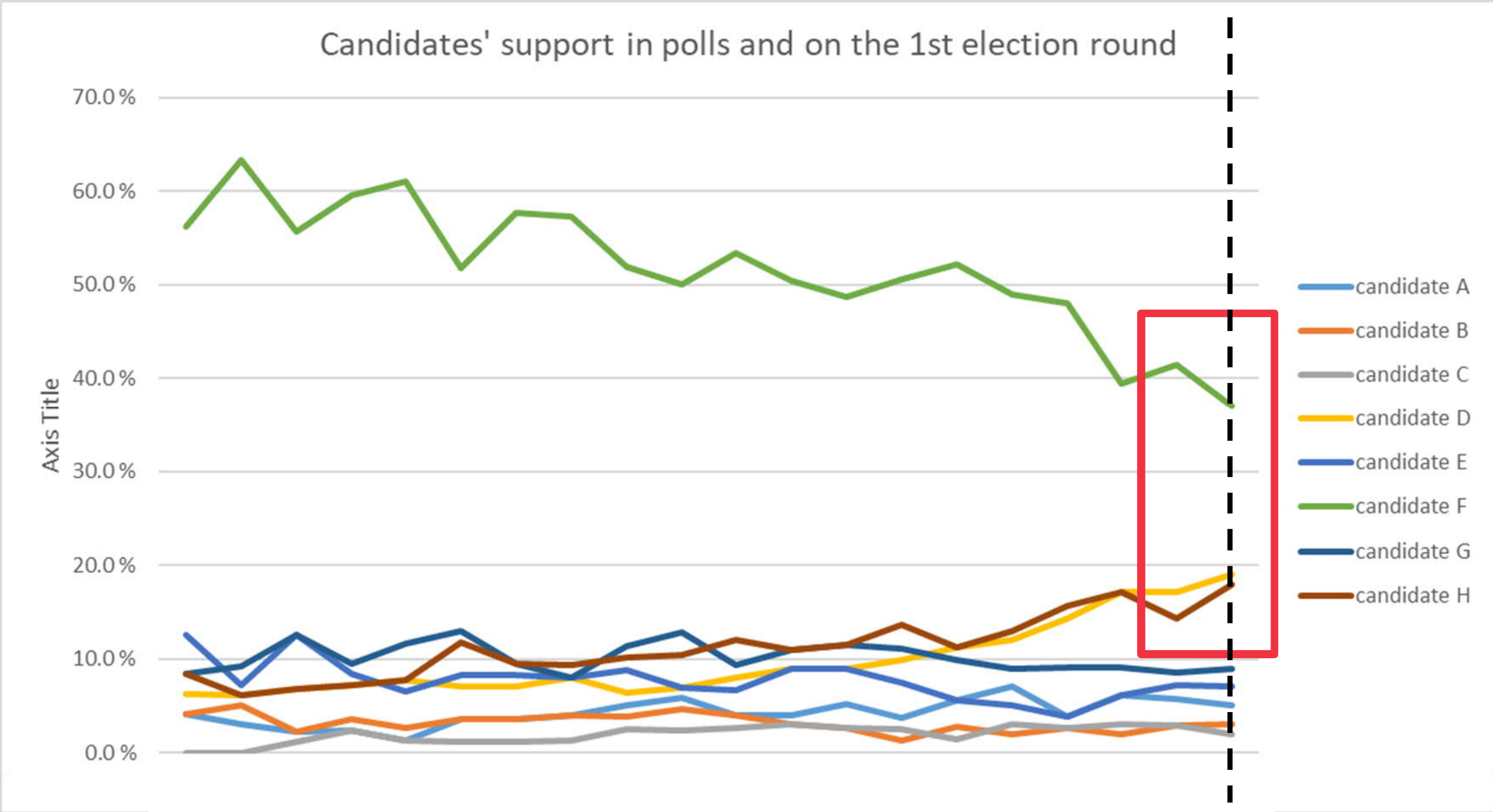
- **Based on polls, D and H are battling for the second position**
- **Supporters of A,B,C,E,F,G: who to vote?**
  - Supporters of F could vote against an unwanted 2nd round competitor (D or H)
  - Supporters A,B,C,E,G could vote against or for D or H
- **Your preferences are given on the piece of paper provided to you**

ranking of D: 2; ranking of F: 1; ranking of g: 3; vote according to preferences

= F > D > G

- **Go to https://presemo.aalto.fi/votingexample/ and vote!**

Do voters actually vote tactically?



1st round results

Condorcet

- All voters rank-order the alternatives
- Each pair of alternatives is compared - the one with more votes is the winner
- If an alternative wins all its one-to-one comparisons, it is the Condorcet winner
- There might not be a Condorcet winner – some other rule must be applied, e.g.,
  - Copeland's method: the winner is the alternative with the most wins in one-to-one comparisons
  - Eliminate the alternative(s) with the least votes and recompute

Condorcet - example

- 33 voters and alternatives A, B, C
  - 17 voters: A>B>C
  - 1 voter: A>C>B
  - 15 voters: B>C>A
  - 0 voters: C>B>A, C>A>B, B>A>C

- A is the Condorcet winner, because it wins both one-on-one comparisons
  - 17+1=18>15 out of 33 favor A over B and 18 favor A over C

Condorcet completion

- There might not be a Condorcet winner
  - Copeland's completion method: the winner is the alternative with the most wins in one-to-one comparisons

5 voters and 5 alternatives A, B, C, D, E

- 1 voter: A>B>C>D>E
- 1 voter: A>D>E>C>B
- 2 voters: D>E>B>C>A
- 1 voter: C>B>A>D>E

D wins more one-on-one comparisons than other alternatives

	A	B	C	D	E	wins
A		2	2	3	3	2
B	2+1= 3		3	2	2	2
C	3	2		2	2	1
D	2	3	3		5	3
E	2	3	3	0		2

Condorcet completion

- Another possibility for Condorcet completion: Eliminate the one with least wins and recompute results

- First C is eliminated
  - B,D,E lose one win

- B and E with one win are elimitated
  - A and D remain

- A wins D by 3 votes to 2

	A	B	~~C~~	D	E	wins
A		2	~~2~~	3	3	2
B	3		~~3~~	2	2	~~2~~ 1
~~C~~	~~3~~	~~2~~		~~2~~	~~2~~	~~1~~
D	2	3	~~3~~		5	~~3~~ 2
E	2	3	~~3~~	0		~~2~~ 1

Borda

- Each voter gives
  - n-1 points to the most preferred alternative,
  - n-2 points to the second most preferred,
  - ...
  - 0 points to the least preferred alternative
- The alternative with the highest total number of points wins

4 state that A > B > C	A : $4 \cdot 2 + 3 \cdot 0 + 2 \cdot 0 =$ 8 points
3 state that B > C > A	B : $4 \cdot 1 + 3 \cdot 2 + 2 \cdot 1 =$ 12 points
2 state that C > B > A	C : $4 \cdot 0 + 3 \cdot 1 + 2 \cdot 2 =$ 7 points

Approval voting

- Each voter casts one vote for each alternative he/she approves

- The alternative with the highest number of votes is the winner

	DM_1	DM_2	DM_3	DM_4	DM_5	DM_6	DM_7	DM_8	DM_9	total	
A	X	-	-	X	-	X	-	X	-	4	
B	**X**	**X**	**X**	**X**	**X**	**X**	**-**	**X**	**-**	7	the winner!
C	-	-	-	-	-	-	X	-	X	2	

- “If you want to vote against some, cast your votes to all others”

Problems with voting: The Condorcet paradox (1/2)

- Consider the following rank-orderings of three alternatives

	DM1	DM2	DM3
A	1	3	2
B	2	1	3
C	3	2	1

- Paired comparisons:
  - A is preferred to B by 2 out of 3 voters
  - B is preferred to C by 2 out of 3 voters
  - C is preferred to A by 2 out of 3 voters

Problems with voting: The Condorcet paradox (2/2)

- Three voting orders:
  1. (A-B) → A wins, (A-C) → C is the winner
  2. (B-C) → B wins, (B-A) → A is the winner
  3. (A-C) → C wins, (C-B) → B is the winner

	DM1	DM2	DM3
A	1	3	2
B	2	1	3
C	3	2	1

The outcome depends on the order in which votes are cast!

- No matter what the outcome is, the majority of voters would prefer some other alternative:
  - If C wins, 2 out of 3 voters would change it to B
  - ...But B would be changed to A by 2 out of 3 voters
  - ...And then A would be changed to C by 2 out of 3 voters...

Problems with voting: tactical voting

- DM_1 knows the preferences of the other voters and the voting order (A-B, winner-C)
- If DM_2 and DM_3 vote according to their true preferences, then the favourite of DM_1 (A) cannot win:
  - 1st round: A gets 2 votes
  - 2nd round: A loses to C
- Could DM_1 avoid the selection of C, her worst outcome?
  - 1st round: vote for B; B wins 2-1
  - 2nd round: vote for B; B wins 2-1

	DM1	DM2	DM3
A	1	3	2
B	2	1	3
C	3	2	1

Social choice function

- Assume that the preferences of DM_i are represented by a complete and transitive *weak preference order* R_i:

$$\text{DM}_i \text{ thinks that x is at least as good as y} \Leftrightarrow x\ R_i\ y$$

- What is the *social choice function f* that determines the collective preference $R=f(R_1,\ldots,R_K)$ of a group of K decision-makers?
  - Voting procedures are examples of social choice functions

Requirements on the social choice function

1. **Universality:** For any set of R_i, the social choice function should yield a unique and complete preference ordering R for the group

2. **Independence of irrelevant alternatives:** The group's preference between two alternatives (x and y) does not change if we remove an alternative from the analysis or add an alternative to the analysis.

3. **Pareto principle**: If all group members prefer x to y, the group should prefer x to y

4. **Non-dictatorship:** There is no DM_i such that $x \, R_i \, y \Rightarrow x \, R \, y$

The big problem with voting: Arrow’s theorem

There is no complete and transitive social choice function *f* such that conditions 1-4 would always be satisfied.

Arrow's theorem – an example

- Borda criterion:

	DM_1	DM_2	DM_3	DM_4	DM_5	**Total**
x_1	3	3	1	2	1	10
x_2	2	2	3	1	3	**11**
x_3	1	1	2	0	0	4
x_4	0	0	0	3	2	5

Alternative x_2 is the winner!

- Suppose that the DMs' preferences do not change. A ballot between alternatives 1 and 2 gives

	DM_1	DM_2	DM_3	DM_4	DM_5	**Total**
x_1	1	1	0	1	0	**3**
x_2	0	0	1	0	1	2

Alternative x_1 is the winner!

- **Independence of irrelevant alternatives is not satisfied!**

Aggregation of values

Theorem (Harsanyi 1955, Keeney 1975):

Let $v_k(\cdot)$ be a cardinal value function describing the preferences of DM_k. There exists a K-dimensional differentiable (ordinal) function $V^G()$ with positive partial derivatives describing group preferences $\succ_g$ in the definition space such that

$$a \succ_g b \Leftrightarrow V^G[v_1(a),\ldots,v_K(a)] \geq V^G[v_1(b),\ldots,v_K(b)]$$

and conditions 1-4 are satisfied.

Note: Voting procedures use only ordinal information (i.e., rank ordering) about the DMs' preferences – strength of preference should be considered, too

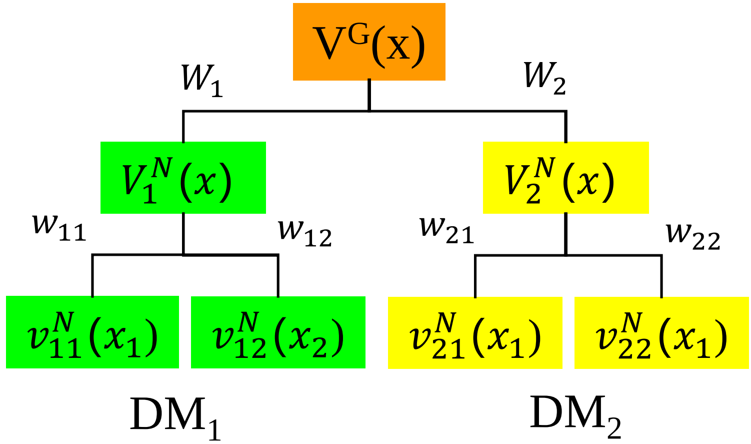
MAVT in group decision support

- From MAVT, we already know how to combine cardinal value functions into an overall value function:

$$V^G(x)=\sum_{k=1}^{K} W_k V_k^N(x),\ W_k \geq 0, \sum_{k=1}^{K} W_k = 1.$$

- This can be done for multiattribute cardinal value functions as well:

$$V^G(x)=\sum_{k=1}^{K} W_k \sum_{i=1}^{n} w_{ki} v_{ki}^N(x_i)$$



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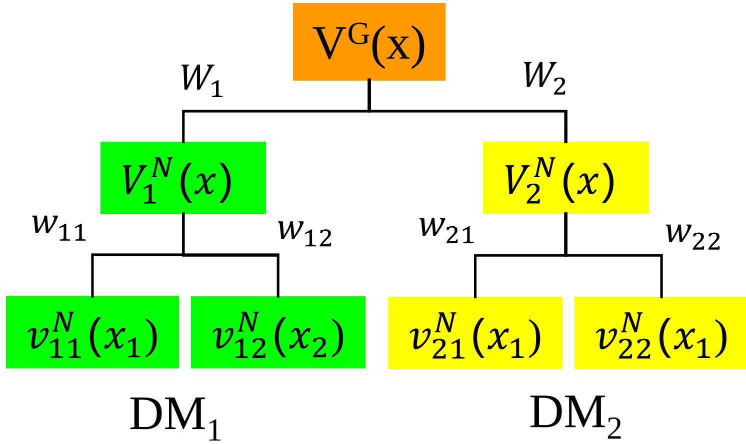
MAVT in group decision support

- Weights W_1, W_2 measure the value difference between the worst and best achievement levels x^0, x^* for DM_1 and DM_2, respectively
- How to compare these value differences – i.e., how to make trade-offs **between people**?
  - *"Compared to my preference for apples over oranges, how strong is yours?"*
- Group weights $W_1 = W_2 = 0.5$ would mean that the value differences are equally valuable, but…
- Who gets to define x^0 and x^*?



MAVT for group decision support

- Example: for both DMs, v_i's are linear, DM_1 has preferences (1,0)~(0,2) and DM_2 (2,0)~(0,1)
- Let $x^0=(0,0)$, $x^*=(2,4)$ for both DMs, and $W_1=W_2=0.5$
  - Then $v_{k1}^N=0.5x_1$, $v_{k2}^N=0.25x_2$ for both k=1,2

DM_1
- $(1,0)\sim(0,2) \Rightarrow V_1^N(1,0)=V_1^N(0,2) \Rightarrow$
  $0.5w_{11}=0.5w_{12} \Rightarrow$
  $w_{11}=w_{12}=0.5$
- $V_1^N(1,0)=0.25$, $V_1^N(0,1)=0.125$

DM_2
- $(2,0)\sim(0,1) \Rightarrow V_2^N(2,0)=V_2^N(0,1) \Rightarrow$
  $w_{21}=0.25w_{22} \Rightarrow$
  $w_{21}=0.2$, $w_{22}=0.8$
- $V_2^N(1,0)=0.1$, $V_2^N(0,1)=0.2$

- **$V^G(1,0)=0.5*0.25+0.5*0.1=0.175 > V^G(0,1)=0.1625$**

MAVT for group decision support

- Interpretation of the result
  - For DM_1 $(1,0) \leftarrow (0,1)$ is an improvement. The "group" values this more than the value of change $(0,1) \leftarrow (1,0)$ for DM_2

- Let $x^0=(0,0)$, $x^*=(4,2)$ for both DMs, and $W_1=W_2=0.5$
  - $V^G(1,0)=0.1625 < V^G(0,1)=0.175$

- Interpretation of the result
  - $(0,1) \leftarrow (1,0)$ - which is an improvement for DM_2 - is now more valuable for the group than change $(1,0) \leftarrow (0,1)$

Summary

- Techniques for involving a group of experts or DMs can be helpful for
  - Problem identification and definition,
  - Generating objectives, attributes, and alternatives,
  - Defining common terminology

- Individual preferences can be easily aggregated into a group preference through voting procedures, but…
  - Arrow’s impossibility theorem states that no ”good” voting procedure exists

- MAVT provides a sound method for aggregating preferences, but…
  - The determination of group weights = interpersonal comparisons can be difficult

  → Aim at a joint model e.g. by exploiting incomplete preference information