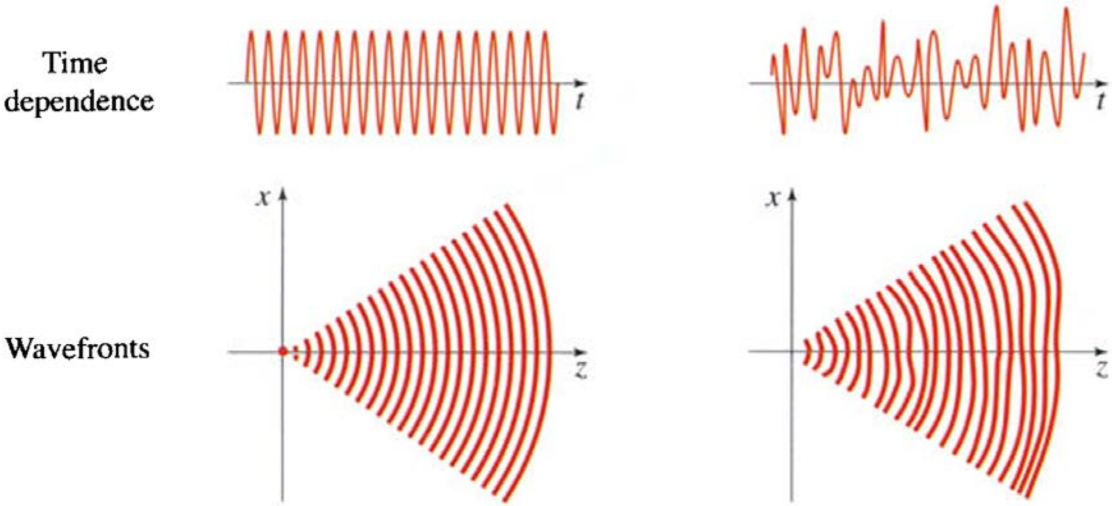
Chapter 11

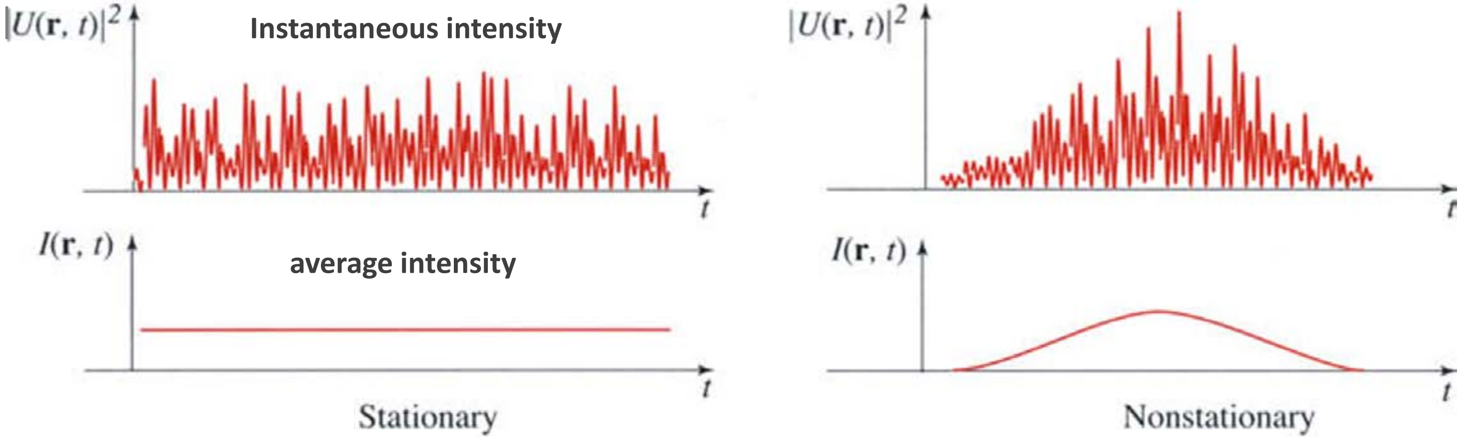
STATISTICAL OPTICS

Random light



coherent

random



Instantaneous intensity

average intensity

Temporal coherence

Temporal coherence function:

$$G(\tau) = \langle U^*(t)U(t+\tau)\rangle$$

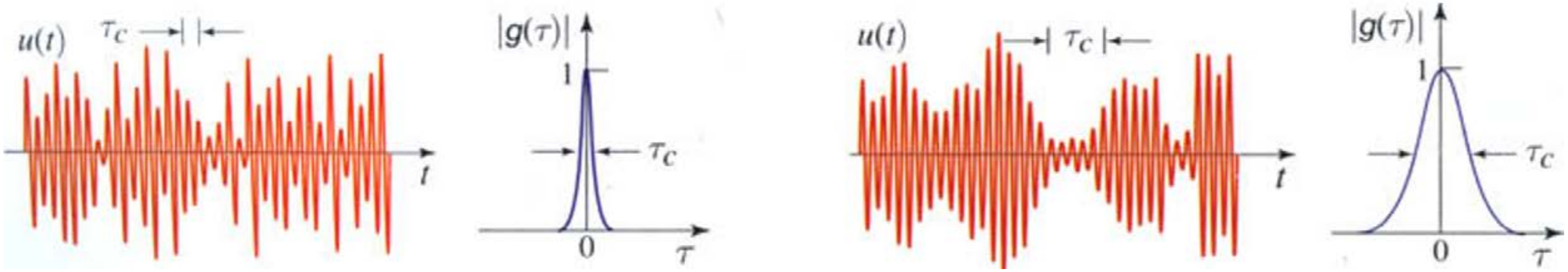
$$\Rightarrow I = G(0).$$

Degree of temporal coherence:

$$\boxed{g(\tau) = \frac{G(\tau)}{G(0)} = \frac{\langle U^*(t)U(t+\tau)\rangle}{\langle U^*(t)U(t)\rangle}}$$

$$\underset{\nearrow \atop \textit{incoherent}}{0} \leq |g(\tau)| \leq \underset{\nwarrow \atop \textit{coherent}}{1}$$

Coherence time and length:



$$\tau_c = \int_{-\infty}^{\infty} |g(\tau)|^2 \, d\tau$$

Coherence length is defined as a distance $l_c = c\tau_c$ along light propagation direction.

Power spectral density $S(\nu)$

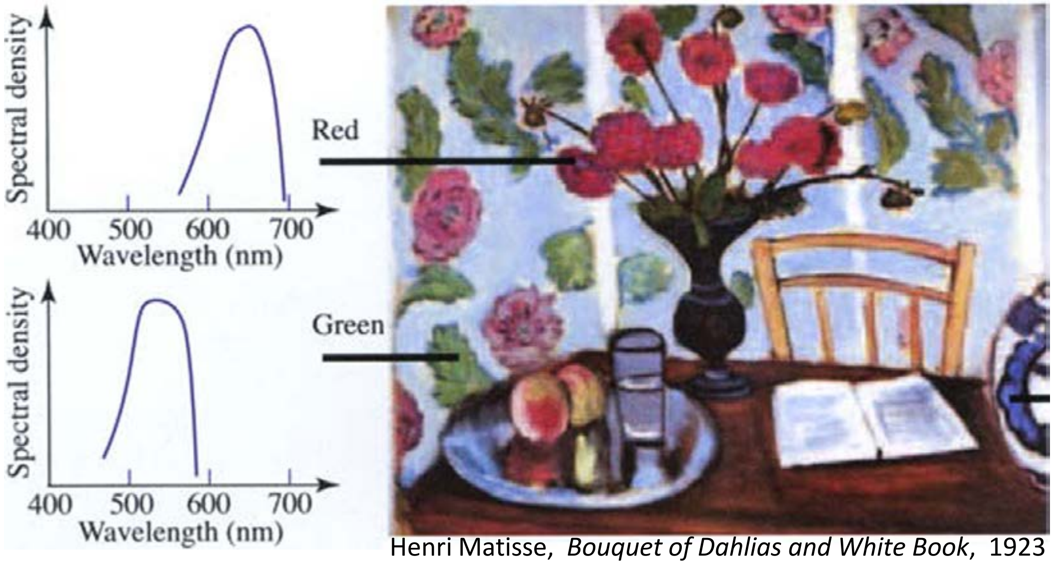
A ν–component of the field in an interval T is $V_T(\nu) = \int_{-T/2}^{T/2} U(t) \exp(-j2\pi\nu t)\, dt$.

$$S(\nu) = \lim_{T\to\infty} \frac{1}{T} \langle |V_T(\nu)|^2 \rangle$$

$$I = \int_0^\infty S(\nu)\, d\nu$$

The *Wiener-Khinchin theorem*:

$$S(\nu) = \int_{-\infty}^{\infty} G(\tau) \exp(-j2\pi\nu\tau)\, d\tau.$$



Henri Matisse, *Bouquet of Dahlias and White Book*, 1923

$$\Delta\nu_c = \frac{\left(\int_0^\infty S(\nu)\, d\nu\right)^2}{\int_0^\infty S^2(\nu)\, d\nu}$$

$$\Rightarrow \quad \Delta\nu_c = \frac{1}{\tau_c}$$

Spatial coherence

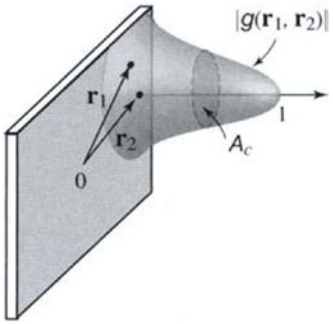
Mutual coherence function:

$$G(\tau) = \langle U^*(\mathbf{r}_1, t)U(\mathbf{r}_2, t+\tau)\rangle$$

Degree of mutual coherence:

$$\boxed{g(\mathbf{r}_1, \mathbf{r}_2, \tau) = \frac{G(\mathbf{r}_1, \mathbf{r}_2, \tau)}{\sqrt{I(\mathbf{r}_1)I(\mathbf{r}_2)}}}$$

$$\underset{\text{incoherent}}{0} \le |g(\mathbf{r}_1, \mathbf{r}_2, \tau)| \le \underset{\text{coherent}}{1}$$



coherence area

Cross-spectral density:

$$S(\mathbf{r}_1, \mathbf{r}_2, \nu) = \int_{-\infty}^{\infty} G(\mathbf{r}_1, \mathbf{r}_2, \tau) \exp(-j2\pi\nu\tau)\, d\tau.$$

It is equal to the power spectral density at $\mathbf{r} = \mathbf{r}_1 = \mathbf{r}_2$.

❖ Interference of two partially coherent waves:

$$I = \langle |U_1 + U_2|^2\rangle = \langle |U_1|^2\rangle + \langle |U_2|^2\rangle + \langle U_1^* U_2\rangle + \langle U_1 U_2^*\rangle$$

$$\Rightarrow \; I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,|g_{12}|\cos\varphi$$

phase difference = $\arg\{g_{12}\}$

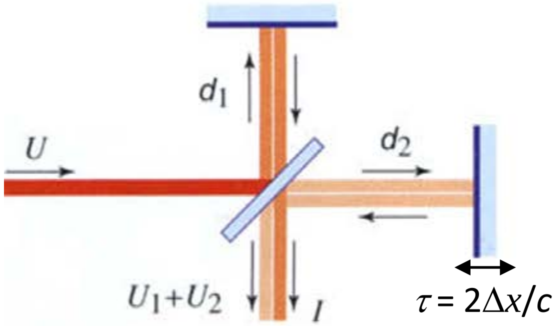
Visibility of interference fringes is given by

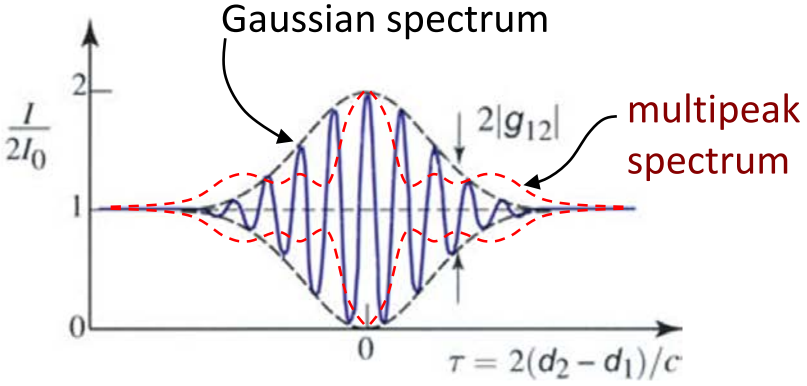
$$\mathcal{V} = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} |g_{12}|$$

If the intensities I_1 and I_2 are equal, we have $\mathcal{V} = |g_{12}|$, and the *interferogram* is

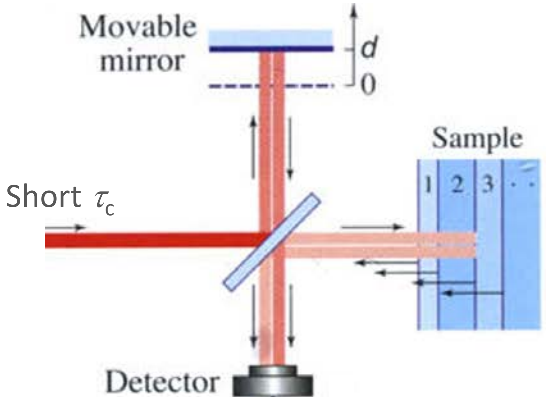
$$I = 2I_0 \left[1 + |g(\tau)| \cos\varphi(\tau)\right] \Rightarrow I = 2\int_0^\infty S(\nu) \left[1 + \cos(2\pi\nu\tau)\right] d\nu \Rightarrow$$ **FT spectroscopy**

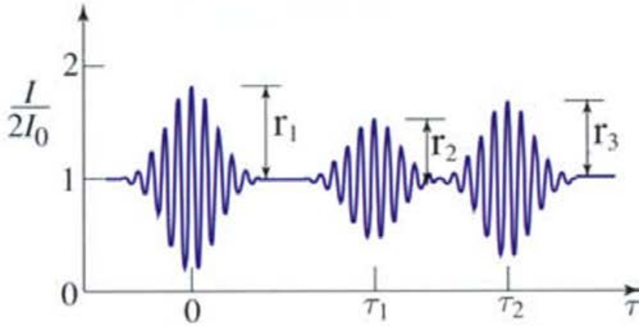
Measurement of longitudinal coherence:



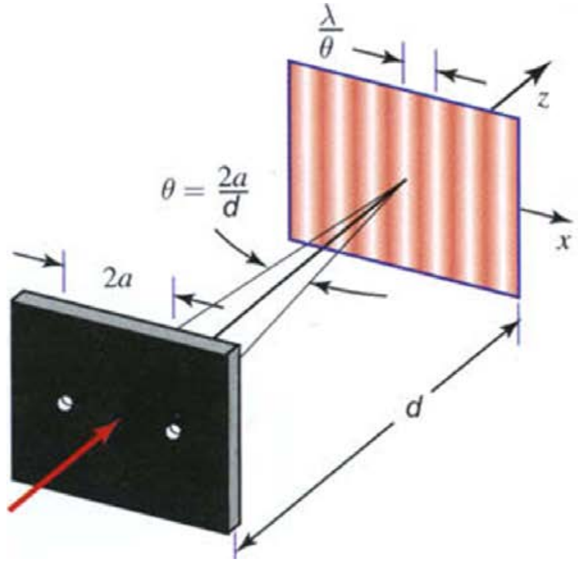


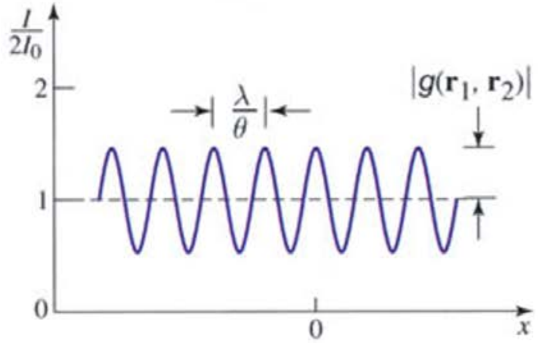
Optical coherence tomography:



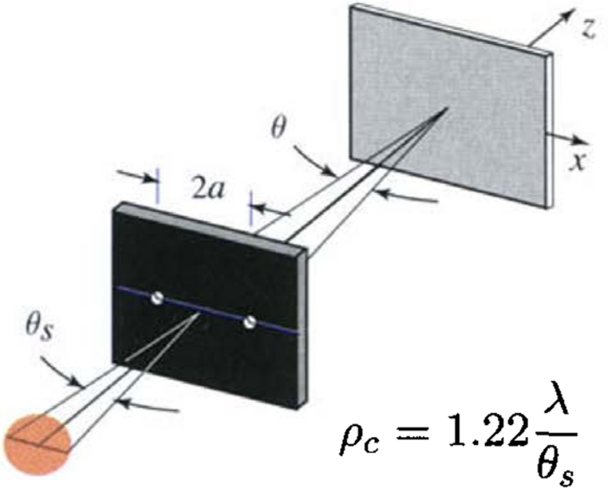


Measurement of transverse coherence with Young's double-pinhole interferometer:

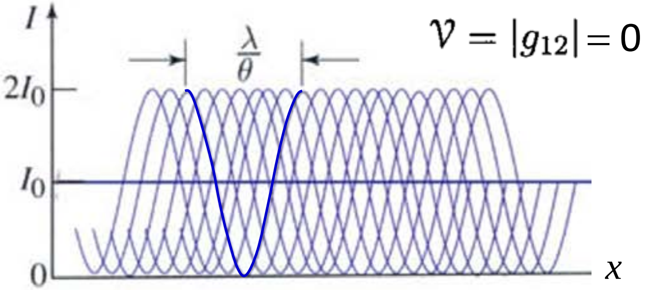




For a field generated by an extended spatially incoherent source:



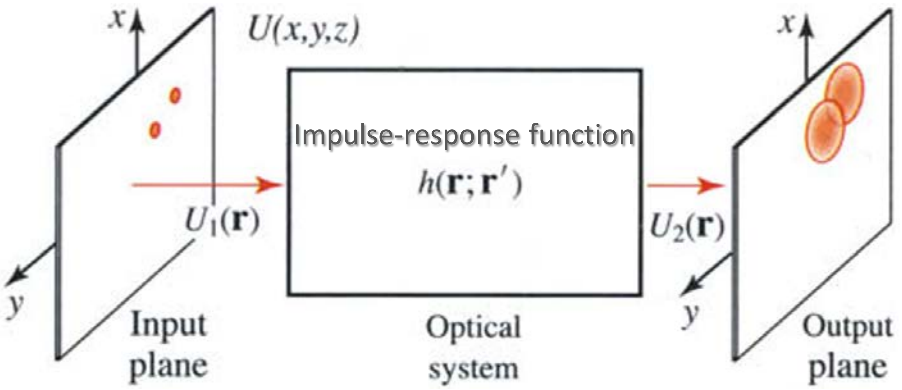
$$\rho_c = 1.22\frac{\lambda}{\theta_s}$$



$$\mathcal{V} = |g_{12}| = 0$$

If $2a > \rho_c$, the fringes disappear

Optical systems with partially coherent light



By definition, the output mutual intensity is

$$G_2(\mathbf{r}_1,\mathbf{r}_2) = \iint h^*(\mathbf{r}_1;\mathbf{r}_1')\,h(\mathbf{r}_2;\mathbf{r}_2')\,G_1(\mathbf{r}_1',\mathbf{r}_2')\,d\mathbf{r}_1'\,d\mathbf{r}_2'$$

The image intensity is

$$I_2(\mathbf{r}) = \iint h^*(\mathbf{r};\mathbf{r}_1')\,h(\mathbf{r};\mathbf{r}_2')\,G_1(\mathbf{r}_1',\mathbf{r}_2')\,d\mathbf{r}_1'\,d\mathbf{r}_2'$$

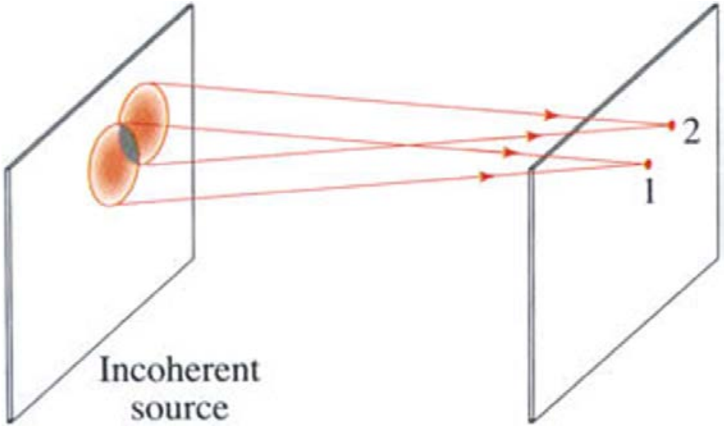
For *spatially incoherent illumination*, we have

$$G_1(\mathbf{r}_1,\mathbf{r}_2) \approx \sigma\sqrt{I_1(\mathbf{r}_1)I_1(\mathbf{r}_2)}\;\delta(\mathbf{r}_1-\mathbf{r}_2)$$

$$\Rightarrow I_2(\mathbf{r}) \approx \int I_1(\mathbf{r}')\,h_i(\mathbf{r};\mathbf{r}')\,d\mathbf{r}',$$

$$\underline{h_i(\mathbf{r};\mathbf{r}') = \sigma|h(\mathbf{r};\mathbf{r}')|^2}$$

❖ Gain of spatial coherence by propagation



For *incoherent illumination*, when $G_1(\mathbf{r}_1, \mathbf{r}_2) \approx \sigma \sqrt{I_1(\mathbf{r}_1) I_1(\mathbf{r}_2)}\; \delta(\mathbf{r}_1 - \mathbf{r}_2)$ we have

$$G_2(\mathbf{r}_1, \mathbf{r}_2) = \sigma \int h^*(\mathbf{r}_1; \mathbf{r})\, h(\mathbf{r}_2; \mathbf{r})\, I_1(\mathbf{r})\, d\mathbf{r}.$$

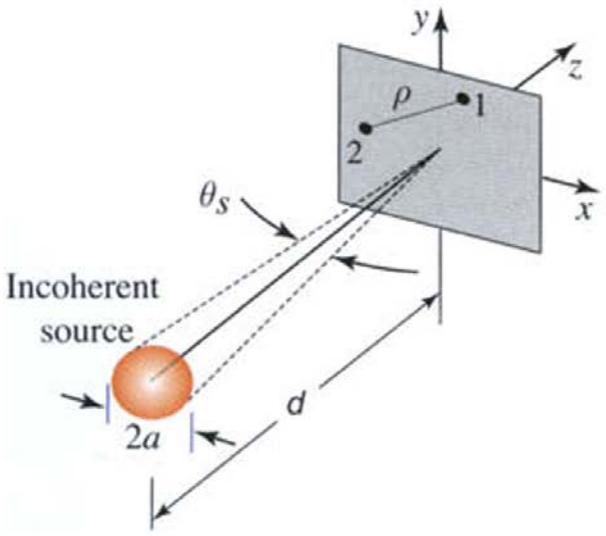
$$\Rightarrow G_2(\mathbf{0}, \mathbf{r}_2) = \sigma \int h^*(\mathbf{0}; \mathbf{r})\, h(\mathbf{r}_2; \mathbf{r})\, I_1(\mathbf{r})\, d\mathbf{r}$$

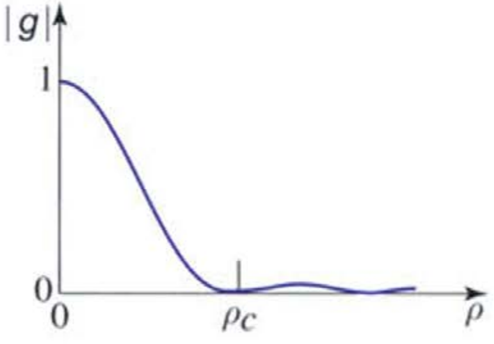
Defining $U_1(\mathbf{r}) = \sigma h^*(\mathbf{0}; \mathbf{r}) I_1(\mathbf{r})$, we obtain

$$G_2(\mathbf{0}, \mathbf{r}_2) = U_2(\mathbf{r}_2) = \int h(\mathbf{r}_2; \mathbf{r})\, U_1(\mathbf{r})\, d\mathbf{r}$$

Van Cittert-Zernike theorem: The spatial coherence of initially incoherent light behaves upon propagation similarly to the amplitude of a certain spatially coherent light traveling through the same system.

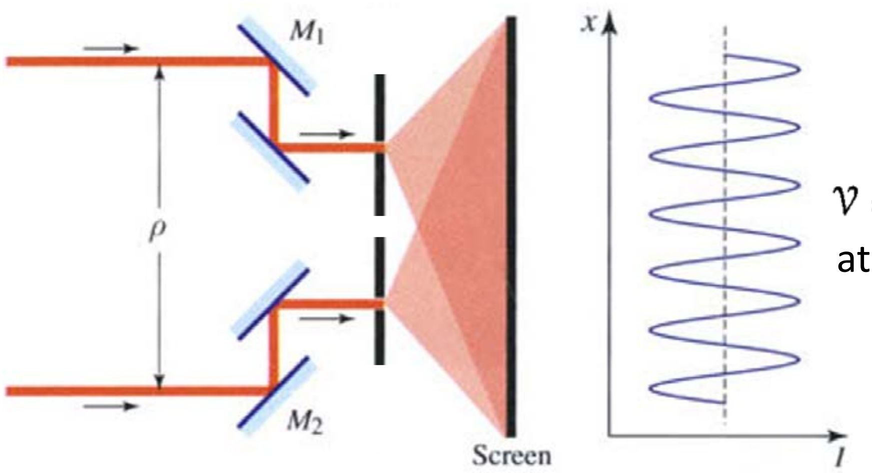
Example





$$\rho_c = 1.22\frac{\lambda}{\theta_s} = 1.22\,\frac{\lambda d}{2a}$$

Michelson stellar interferometer for measuring the angular diameters of stars:



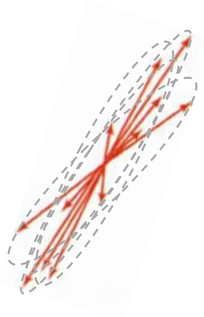
$\mathcal{V} = |g_{12}| = 0$

at $\rho = \rho_c$.

Partial polarization

For a transverse field, the correlation functions of the vector components are

$$G_{xx}(\tau) = \langle U_x^*(t) U_x(t+\tau) \rangle$$
$$G_{yy}(\tau) = \langle U_y^*(t) U_y(t+\tau) \rangle$$
$$G_{xy}(\tau) = \langle U_x^*(t) U_y(t+\tau) \rangle.$$



Coherency matrix:

$$\mathbf{G} = \begin{bmatrix} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{bmatrix}$$

Examples of $\mathbf{G}(\tau = 0)$:

x-polarized

$$\mathbf{G} = \bar{I} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

right-circularly polarized

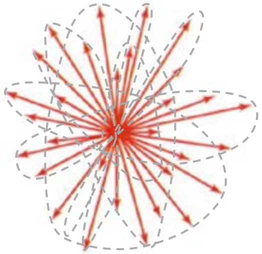
$$\mathbf{G} = \tfrac{1}{2}\bar{I} \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}$$

unpolarized

$$\mathbf{G} = \tfrac{1}{2}\bar{I} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Degree of polarization:

$$\mathbb{P} = \sqrt{1 - \frac{4 \det \mathbf{G}}{(\mathrm{Tr}\,\mathbf{G})^2}} = \sqrt{1 - 4\left[\frac{I_x I_y}{(I_x + I_y)^2}\right](1 - |g_{xy}|^2)} = \frac{\sqrt{\mathsf{S}_1^2 + \mathsf{S}_2^2 + \mathsf{S}_3^2}}{\mathsf{S}_0}$$



$$0 \leq \mathbb{P} \leq 1$$

unpolarized

polarized