# Modeling Securities and Investment in Continuous Time:

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**Preliminary notes (updated during the course)**

# Agenda and References

**Pre-requisite: Derivatives course based on Hull, Options Futures and Other Derivatives, 7th Edition.**

p. 1-22 Wiener Processes and Ito's Lemma (Hull Ch. 12)

p. 23-38 Modeling Stock Price and Black & Scoles Model (Hull Ch. 13)

p. 39-46 Martingales and Risk Neutral Pricing (Hull Ch. 27)

p. 47-56 Term Structure of Interest Rates and Black & Scholes revisited (Hull Ch 30, notes)

P. 57-72 Wealth Evolution and Continuous Time Financial Markets (notes, Additional readings: Merton, Continuous Time Finance)

# 1. Wiener Processes and Ito's Lemma

# Markov Processes

## DEFINITION

- In a Markov process future movements in a variable depend only on where we are, not the history of how we got where we are
- We assume that stock prices follow Markov processes

# Properties of Markov processes

- In Markov processes changes in successive periods of time are independent
- Variances of the process are additive

# A Wiener Process

**DEFINITION**

- We consider a variable $z$ whose value changes continuously
- The change in a small interval of time $dt$ is $dz$
- The variable follows a Wiener process if

1. $dz = \varepsilon\sqrt{dt}$ where $\varepsilon$ is a random drawing from $\phi(0,1)$
2. The values of *dz* for any 2 different (non-overlapping) periods of time are independent

# Properties of a Wiener Process

- Mean of $[z(T) - z(0)]$ is 0
- Variance of $[z(T) - z(0)]$ is $T$
- Standard deviation of $[z(T) - z(0)]$ is $\sqrt{T}$

# Generalized Wiener Processes

- A Wiener process has a drift rate (i.e. average change per unit time) of 0 and a variance rate of 1
- In a generalized Wiener process the drift rate and the variance rate can be set equal to any chosen constants

# Generalized Wiener Processes
## (continued)

The variable $x$ follows a generalized Wiener process with a drift rate of $a$ and a variance rate of $b^2$ if

$$dx = a\,dt + b\,dz$$

# Generalized Wiener Processes
## (continued)

$$dx = a\ dt + b\ \varepsilon\sqrt{dt}$$

- Mean change in $x$ in time $T$ is $aT$
- Variance of change in $x$ in time $T$ is $b^2T$
- Standard deviation of change in $x$ in time $T$ is $b\sqrt{T}$

# Example

- A stock price starts at 40 and has a probability distribution of $\phi(40,10)$ at the end of the year
- If we assume the stochastic process is Markov with no drift then the process is

$$dS = 10dz$$

- If the stock price were expected to grow by $8 on average during the year, so that the year-end distribution is $\phi(48,10)$, the process is

$$dS = 8dt + 10dz$$

# Ito Process

**DEFINITION**

- In an Ito process the drift rate and the variance rate are functions of time

$$dx = a(x,t)dt + b(x,t)dz$$

- The discrete time equivalent

$$\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$$

is only true in the limit as $\Delta t$ tends to zero

# Why a Generalized Wiener Process is not Appropriate for Stocks

- For a stock price we can conjecture that its expected percentage change in a short period of time remains constant, not its expected absolute change in a short period of time
- We can also conjecture that our uncertainty as to the size of future stock price movements is proportional to the level of the stock price

# An Ito Process for Stock Prices

$$dS = \mu S dt + \sigma S dz$$

where $\mu$ is the expected return $\sigma$ is the volatility.

The discrete time equivalent is

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

# Ito's Lemma

- If we know the stochastic process followed by $x$, Ito's lemma tells us the stochastic process followed by some function $G(x, t)$
- Since a derivative security is a function of the price of the underlying and time, Ito's lemma plays an important part in the analysis of derivative securities

# Ignoring Terms of Higher Order Than $\Delta t$

In ordinary calculus we have

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t$$

In stochastic calculus this becomes

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ½\frac{\partial^2 G}{\partial x^2}\Delta x^2$$

because $\Delta x$ has a component which is of order $\sqrt{\Delta t}$

# Substituting for $\Delta x$

Suppose

$$dx = a(x,t)dt + b(x,t)dz$$

so that

$$\Delta x = a\ \Delta t + b\ \varepsilon\sqrt{\Delta t}$$

Then ignoring terms of higher order than $\Delta t$

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ½\frac{\partial^2 G}{\partial x^2}b^2\varepsilon^2\Delta t$$

# The $\varepsilon^2\Delta t$ Term

Since $\varepsilon \approx \phi(0,1)$ $\quad E(\varepsilon) = 0$

$$E(\varepsilon^2) - [E(\varepsilon)]^2 = 1$$

$$E(\varepsilon^2) = 1$$

It follows that $E(\varepsilon^2 \Delta t) = \Delta t$

Hence:

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2 \Delta t$$

# Taking Limits

Taking limits $$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + ½ \frac{\partial^2 G}{\partial x^2} b^2 dt$$

Substituting $$dx = a\,dt + b\,dz$$

We obtain $$dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b\,dz$$

This is Ito's Lemma

# Application of Ito's Lemma to a Stock Price Process

The stock price process is

$$dS = \mu S\, dt + \sigma S\, dz$$

For a function $G$ of $S$ and $t$

$$dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S\, dz$$

For a function $G$ of $S$ and $t$

$$dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S \, dz$$

# Example 1

1. The forward price of a stock for a contract maturing at time T

$$G = S\, e^{r(T-t)}$$

$$\frac{\partial G}{\partial t} =$$

$$\frac{\partial G}{\partial S} =$$

$$\frac{\partial^2 G}{\partial S^2} =$$

=>

$$dG =$$

For a function $G$ of $S$ and $t$

$$dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S \, dz$$

# Example 2

$$2. \; G = \ln S$$

$$\frac{\partial G}{\partial S} =$$

$$\frac{\partial^2 G}{\partial S^2} =$$

$$\frac{\partial G}{\partial t} =$$

$$dG =$$

# Learnings

- Defitions of
  - Markov process,
  - Wiener process
  - Ito process
- Ito’s lemma
  - How to apply Ito’s lemma

# 2. A Model of Stock Price

# The Stock Price Assumption

- Consider a stock whose price is $S$
- In a short period of time of length $\Delta t$, the change in the stock price is

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

- $\mu$ is expected return and $\sigma$ is volatility

# Properties of Lognormality

- It follows from this assumption (recall example 2) that

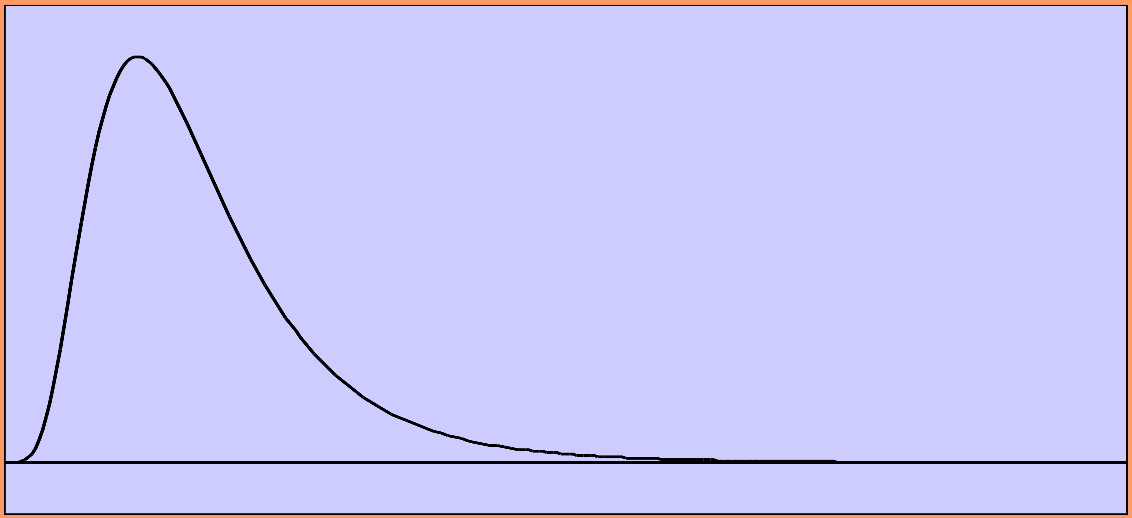
$$\ln S_T - \ln S_0 \approx \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right] \qquad (1)$$

or

$$\ln S_T \approx \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right]$$

- Since the logarithm of $S_T$ is normal, $S_T$ is lognormally distributed

# The Lognormal Distribution



$$E(S_T) = S_0 \, e^{\mu T}$$

$$\text{var}(S_T) = S_0{}^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

# Continuously Compounded Return: η

$$S_T = S_0 \, e^{\eta T}$$

$$\Rightarrow \quad \eta = \frac{1}{T} \ln \frac{S_T}{S_0}$$

$$\Rightarrow \quad \eta \approx \phi\left( \mu - \frac{\sigma^2}{2}, \frac{\sigma}{\sqrt{T}} \right)$$ using (1)

# The Expected Return

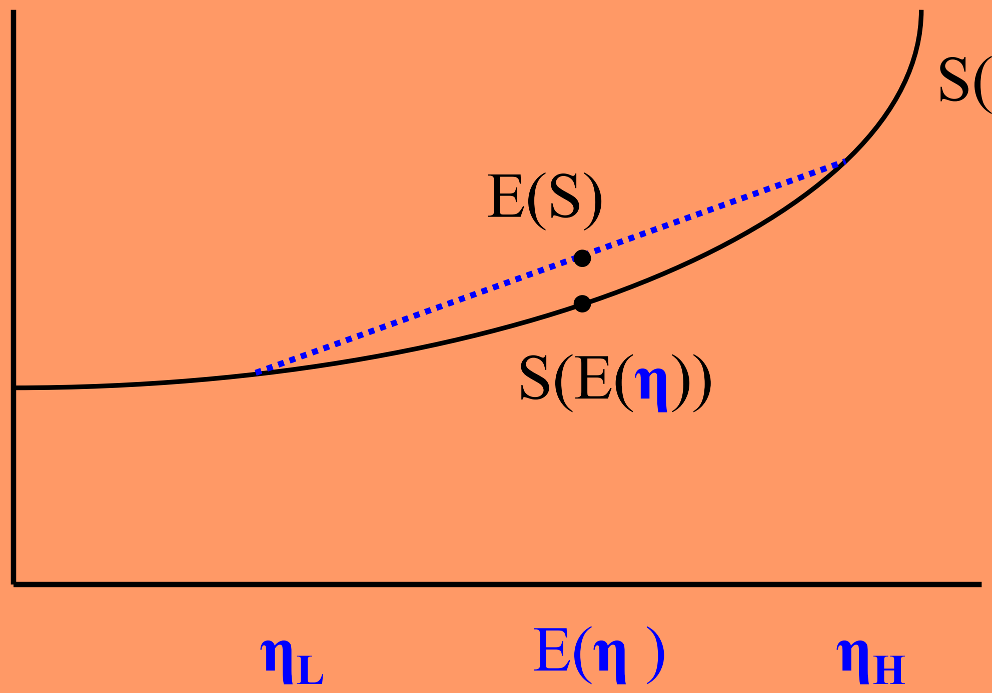
- The expected value of the stock price is

$$E\left[S_0 e^{\eta T}\right] = S_0 e^{E[\eta T] + \frac{1}{2}\sigma^2 T} = S_0 e^{\mu T}$$

- The expected annual return on the stock is $\mu - \sigma^2/2$

$$E\left[\ln(S_T / S_0)\right] = \mu - \sigma^2 / 2$$
$$\ln\left[E(S_T / S_0)\right] = \mu$$

# Intuition why $E(S) \neq S(E(\eta))$




- When x is normally distributed with volatility σ

$$E\left[e^{x}\right] = e^{E[x]+\frac{1}{2}\sigma^{2}}$$

# 3. The Black-Scholes Model

# The Concepts Underlying Black-Scholes

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes differential equation for the value of option: function f (an example of all possible functions G)

# Black-Scholes Differential Equation

$$\Delta S = \mu S \; \Delta t + \sigma S \; \Delta z$$

$$\Delta f = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \tfrac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \; \Delta z$$

We set up a portfolio consisting of

$-1$: derivative

$+\dfrac{\partial f}{\partial S}$: shares

# The Derivation of the Black-Scholes Differential Equation continued

The value of the portfolio $\Pi$ is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time $\Delta t$ is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

# The Derivation of the Black-Scholes Differential Equation

The return on the portfolio must be the risk-free rate. Hence

$$\Delta\Pi = r\,\Pi\Delta t$$

Note: The value of the portfolio π does not depend on z, hence no risk

We substitute for $\Delta f$ and $\Delta S$ in these equations to get the Black-Scholes differential equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r f$$

# The Differential Equation

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation
- In an option the boundary condition is

$$f = \max(S - K;0), \text{ when } t = T$$

# The solution is: Black-Scholes Formula

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where $$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

# The Differential Equation for Forward Contract

- In a forward contract the boundary condition is $f = S - K$ when $t = T$

- The solution to the equation is

$$f = S - K e^{-r(T - t)}$$

Exercise: Check!