Homework 3 — Sample Solutions *14 March 2019*

Return solution by 20:00, 20 March 2019 — electronically in MyCourses

3. (a) Dyadic: Assuming the unit dyadic $\overline{\overline{I}}$ and

$$\overline{\overline{B}} = \overline{\overline{I}} + 2\left(\mathbf{u}_x\mathbf{u}_y + \mathbf{u}_y\mathbf{u}_x\right) + \mathbf{k}\times\overline{\overline{I}}$$

where $\mathbf{k} = \mathbf{u}_z$, compute the following:

i. $\mathrm{tr}\overline{\overline{B}}$,

ii. $\mathrm{spm}\overline{\overline{B}}$,

iii. $\det\overline{\overline{B}}$, and

iv. its inverse $\left(= \overline{\overline{B}}^{-1}\right)$.

v. Check and show that $\overline{\overline{B}}\cdot\overline{\overline{B}}^{-1} = \overline{\overline{I}}$ and $\overline{\overline{B}}^{-1}\cdot\overline{\overline{B}} = \overline{\overline{I}}$.

(b) The electric flux density (displacement) for different electric field excitations is:

$$\mathbf{E} = E_x\mathbf{u}_x \;\rightarrow\; \mathbf{D} = \varepsilon_0\left(3\mathbf{u}_x + 3\mathbf{u}_y - \mathbf{u}_z\right)E_x$$
$$\mathbf{E} = E_y\mathbf{u}_y \;\rightarrow\; \mathbf{D} = \varepsilon_0\left(2\mathbf{u}_x + 4\mathbf{u}_y + 2\mathbf{u}_z\right)E_y$$
$$\mathbf{E} = E_z\mathbf{u}_z \;\rightarrow\; \mathbf{D} = \varepsilon_0\left(\mathbf{u}_x + \mathbf{u}_y + 5\mathbf{u}_z\right)E_z$$

i. Find the relative permittivity dyadic $\overline{\overline{\varepsilon}}$ $\left(\mathbf{D} = \varepsilon_0\overline{\overline{\varepsilon}}\cdot\mathbf{E}\right)$

ii. Calculate $\mathrm{tr}\,\overline{\overline{\varepsilon}}$, $\mathrm{spm}\,\overline{\overline{\varepsilon}}$, and $\det\overline{\overline{\varepsilon}}$

(c) Obviously for the electric field excitations along $\mathbf{u}_x$, $\mathbf{u}_y$, and $\mathbf{u}_z$, the flux density $\mathbf{D}$ is not parallel to $\mathbf{E}$. Find those field directions for which the flux is parallel: $\mathbf{D} \parallel \mathbf{E}$

Sample solutions

3. It is helpful to make use of Appendix D for dyadic identities. For example from the one that expands the double-cross product $\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}}$ one can derive many important special cases, like

$$\overline{\overline{I}}{}^{\times}_{\times}\overline{\overline{I}} = 2\overline{\overline{I}}, \quad \overline{\overline{I}}{}^{\times}_{\times}\overline{\overline{A}} = \overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{I}} = (\mathrm{tr}\overline{\overline{A}})\,\overline{\overline{I}} - \overline{\overline{A}}^T, \quad \overline{\overline{I}}{}^{\times}_{\times}(\mathbf{a}\times\overline{\overline{I}}) = \mathbf{a}\times\overline{\overline{I}}, \quad (\mathbf{a}\times\overline{\overline{I}}){}^{\times}_{\times}(\mathbf{a}\times\overline{\overline{I}}) = 2\mathbf{aa}$$

and furthermore results with dot and double-cross products

$$\overline{\overline{I}}:\overline{\overline{I}} = 3, \quad (\mathbf{a}\times\overline{\overline{I}}):\overline{\overline{I}} = 0, \quad (\mathbf{a}\times\overline{\overline{I}})\cdot(\mathbf{a}\times\overline{\overline{I}}) = \mathbf{aa} - \mathbf{a}\cdot\mathbf{a}\overline{\overline{I}}, \quad (\mathbf{a}\times\overline{\overline{I}}):(\mathbf{a}\times\overline{\overline{I}}) = 2\mathbf{a}\cdot\mathbf{a}$$

Now $\overline{\overline{B}} = \overline{\overline{I}} + 2\left(\mathbf{u}_x\mathbf{u}_y + \mathbf{u}_y\mathbf{u}_x\right) + \mathbf{u}_z \times \overline{\overline{I}} = \overline{\overline{I}} + 3\mathbf{u}_y\mathbf{u}_x + \mathbf{u}_x\mathbf{u}_y$

(a) $\mathrm{tr}\overline{\overline{B}} = 3$

(b) $\overline{\overline{B}}{}^{\times}_{\times}\overline{\overline{B}} = 2\left(\overline{\overline{I}} - 3\mathbf{u}_z\mathbf{u}_z - 3\mathbf{u}_x\mathbf{u}_y - \mathbf{u}_y\mathbf{u}_x\right)$ and $\mathrm{spm}\overline{\overline{B}} = 0$

(c) $\det\overline{\overline{B}} = -2$

(d)

$$\overline{\overline{B}}^{-1} = \frac{\frac{1}{2}(\overline{\overline{B}}{}^{\times}_{\times}\overline{\overline{B}})^T}{\det\overline{\overline{B}}} = \frac{1}{2}\left(-\overline{\overline{I}} + 3\mathbf{u}_z\mathbf{u}_z + \mathbf{u}_x\mathbf{u}_y + 3\mathbf{u}_y\mathbf{u}_x\right)$$

(e)

$$\left(\overline{\overline{I}} + 3\mathbf{u}_y\mathbf{u}_x + \mathbf{u}_x\mathbf{u}_y\right)\cdot\frac{1}{2}\left(-\overline{\overline{I}} + 3\mathbf{u}_z\mathbf{u}_z + \mathbf{u}_x\mathbf{u}_y + 3\mathbf{u}_y\mathbf{u}_x\right) = \frac{1}{2}\left(-\overline{\overline{I}} + 3\mathbf{u}_z\mathbf{u}_z + \mathbf{u}_x\mathbf{u}_y + 3\mathbf{u}_y\mathbf{u}_x\right)\cdot\left(\overline{\overline{I}} + 3\mathbf{u}_y\mathbf{u}_x + \mathbf{u}_x\mathbf{u}_y\right) = \overline{\overline{I}}$$

4. (a) The components of the $\overline{\overline{\varepsilon}}$-dyadic can be picked from the three responses given for the cartesian excitations:

$$\overline{\overline{\varepsilon}} = 3\mathbf{u}_x\mathbf{u}_x + 2\mathbf{u}_x\mathbf{u}_y + \mathbf{u}_x\mathbf{u}_z + 3\mathbf{u}_y\mathbf{u}_x + 4\mathbf{u}_y\mathbf{u}_y + \mathbf{u}_y\mathbf{u}_z - \mathbf{u}_z\mathbf{u}_x + 2\mathbf{u}_z\mathbf{u}_y + 5\mathbf{u}_z\mathbf{u}_z$$

and in the cartesian (xyz) matrix form (note rows and columns)

$$\begin{pmatrix} 3 & 2 & 1 \\ 3 & 4 & 1 \\ -1 & 2 & 5 \end{pmatrix}$$

(b) $\mathrm{tr}\overline{\overline{\varepsilon}} = 12, \quad \mathrm{spm}\overline{\overline{\varepsilon}} = 40, \quad \det\overline{\overline{\varepsilon}} = 32$

5. This means finding the eigenvectors of the dyadic $\overline{\overline{\varepsilon}}$, such directions that $\overline{\overline{\varepsilon}}\cdot\mathbf{u} = \lambda\mathbf{u}$. These are parallel to

$$\mathbf{u}_1 = \frac{\mathbf{u}_x + \sqrt{2}\,\mathbf{u}_y + \mathbf{u}_z}{2}, \quad \mathbf{u}_2 = \frac{-\mathbf{u}_x - 2\mathbf{u}_y + 3\mathbf{u}_z}{\sqrt{14}}, \quad \mathbf{u}_3 = \frac{\mathbf{u}_x - \sqrt{2}\,\mathbf{u}_y + \mathbf{u}_z}{2}$$