Modeling Securities and Investment in Continuous Time II

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Preliminary notes (updated during the course)

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3. The Black-Scholes Model

The Concepts Underlying Black-Scholes

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes differential equation for the value of option: function f (an example of all possible functions G)

Black-Scholes Differential Equation

$$\Delta S = \mu S \ \Delta t + \sigma S \ \Delta z$$

$$\Delta f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + ½ \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \ \Delta z$$

We set up a portfolio consisting of

-1: derivative

$+\frac{\partial f}{\partial S}$: shares

The Derivation of the Black-Scholes Differential Equation continued

The value of the portfolio Π is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

The Derivation of the Black-Scholes Differential Equation

The return on the portfolio must be the risk-free rate. Hence

$$\Delta\Pi = r\,\Pi\Delta t$$

Note: The value of the portfolio π does not depend on z, hence no risk

We substitute for Δf and ΔS in these equations to get the Black-Scholes differential equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

The Differential Equation

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation
- In an option the boundary condition is

$$f = \max(S - K; 0), \text{ when } t = T$$

The solution is: Black-Scholes Formula

$$c = S_0 \, N(d_1) - K \, e^{-rT} N(d_2)$$

$$p = K \, e^{-rT} \, N(-d_2) - S_0 \, N(-d_1)$$

where $$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The Differential Equation for Forward Contract

- In a forward contract the boundary condition is $f = S - K$ when $t = T$

- The solution to the equation is

$$f = S - K e^{-r(T-t)}$$

Exercise: Check!

3. Martingales and Risk Neutral Pricing in Continuous Time

Martingales and Risk Neutral Pricing

- Definition of a Martingale:
  - $E_t \theta_{t+1} = \theta_t$
  - Here: $d\theta = dz$
- Here E_t refers to expectation at time t
- Still assume single source of uncertainty z
- Consider two assets f_1 and f_2
  - $df_1 = \mu_1 f_1 dt + \sigma_1 f_1 dz$
  - $df_2 = \mu_2 f_2 dt + \sigma_2 f_2 dz$

Martingales and Risk Neutral Pricing

- Let $\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2$
- dz terms cancel, Π is riskless → hence drift = $r\Pi dt$
- We obtain:
- $(\sigma_2 f_2) f_1 \mu_1 dt - (\sigma_1 f_1) f_2 \mu_2 dt = r\Pi dt$

$= r[(\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2] dt$

<u>This implies that we must have that (otherwise??)</u>

- $(\mu_1 - r)/\ \sigma_1 = (\mu_2 - r)/\ \sigma_2$

Martingales and Risk Neutral Pricing

- No arbitrage implies that there exists λ such that

  $\lambda = (\mu_i - r)/ \sigma_i$ for all assets i

- Define $\lambda = (\mu_i - r)/ \sigma_i$ as market price of risk
- σ_i is then the level of risk

- Now each security where only source of uncertainty is z must evolve according to:

  – $df / f = (r + \lambda\sigma_f)dt + \sigma_f dz$ (1)

Martingales and Risk Neutral Pricing

- Equivalent to ”risk neutral probabilities” in the binomial model in the derivatives course

- In the derivatives course we defined risk neutral probablity as a ”probability” p such that if E* is the expectation operator that corresponds with p,

$$S_t = E^*_t(S_{t+1}) / (1+r)$$

- Note that p was not a real probability of stock price going up

- Here also we change the ”probability measure” so that drift is equal to zero (or rdt in case of a discounted martingale).

Martingales and Risk Neutral Pricing

- Risk neutral probability measure in continuous time:
- Consider again two securities f, g:

  - $df = \mu_f f dt + \sigma_f f dz$ (2)
  - $dg = \mu_g g dt + \sigma_g g dz$ (3)

  - Now look at the relative price $\Phi = f/g$
  - That is the price of f in units of g (g being the numeraire instead of $)

Equivalent martingale measure

- **Equivalent martingale measure result:** when there are no arbitrage opportunities Φ is a martingale for some choice of market price for risk

- In particular, if market price for risk $\lambda = \sigma_g$ then Φ= f/g is a martingale for all f (p. 595 in Hull).**

- Assuming a market price of risk $\lambda = \sigma_g$ is equivalent to assuming p = (1+r-d)/(u-d) in the binomial framework. If we pretend this is the market price of risk, then pricing of securities is easy as (letting E_g denote expectation under this assumption) it follows that

$$f_0/g_0 = E_{g,0}(f_T/g_T)$$

** HW: Prove this result showing all the steps (including all partial derivatives).

Proof that f/g a martingale under the previous assumption

$$df = \left(r + \sigma_g \sigma_f\right) f\,dt + \sigma_f \,\mathrm{fdz}$$
$$dg = \left(r + \sigma_g^2\right) g\,dt + \sigma_g \,\mathrm{gdz}$$

$$\Rightarrow \quad d\ln f = \left(r + \sigma_g \sigma_f - \sigma_f^2/2\right) dt + \sigma_f \,\mathrm{dz}$$

$$d\ln g = \left(r + \sigma_g^2/2\right) dt + \sigma_g \,\mathrm{dz}$$

$$\Rightarrow \quad d(\ln f - \ln g) = d\ln\left(\frac{f}{g}\right) = \left(\sigma_g \sigma_f - \frac{\sigma_f^2}{2} - \frac{\sigma_g^2}{2}\right) dt + (\sigma_f - \sigma_g)\,\mathrm{dz}$$
$$= -\frac{1}{2}\left(\sigma_f - \sigma_g\right)^2 dt + (\sigma_f - \sigma_g)\,\mathrm{dz}$$

$$\Rightarrow \quad d\left(\frac{f}{g}\right) = (\sigma_f - \sigma_g)\frac{f}{g}\,\mathrm{dz}$$

Example

- A dollar money market account is a security that is worth \$1 at time zero and earns instantaneous risk free rate r at any given time. The variable r may be stochastic. If we set g equal to the money market account, then $\lambda = 0$. Then
  - $dg = rg\,dt$
- Drift is stochastic but volatility of g is zero.
- As $\lambda=0$, E_g is equivalent to taking expectation E^* in the "traditional risk neutral world"
- Hence $\frac{f_0}{g_0} = E^*\left(\frac{f_T}{g_T}\right)$
- In this case: $g_0 = 1$, $g_T = e^{\int_0^T r\,dt}$ and

$$f_0 = E^*\left(e^{\int_0^T -r\,dt} f_T\right) = E^*\left(e^{-\bar{r}T} f_T\right)$$

, where $\bar{r}$ is the average interest rate

Questions

- What probability structure we should assume ”in the traditional risk neutral world”?
- That is, what is the evolution of the interest rate under the risk neutral probability measure?

4. MODELLING TERM STRUCTURE OF INTEREST RATES

Interest rate derivatives and models of short rate

Value of an interest rate derivative

$f_0 = E^* (e^{-\int_0^T r dt} f_T) = E^* (e^{-\bar{r} T} f_T)$, $\bar{r}$ is average interest rate

Define P(t,T) as the period t price of a zero coupon bond that pays off 1 at T.

$P_t = E^* (e^{-\bar{r} (T-t)})$

Interest rate derivatives and models of short rate

If R(t,T) is continuously compounded interest rate from t to T we have

$$R(t,T) = -\ln(P(t,T))/(T-t)$$
$$= -\ln\left(E^*\left(e^{-\bar{r}(T-t)}\right)\right)/(T-t)$$

Models of short rate (to calculate the average rate in risk neutral world)

Vasicek's model

$$dr = a(b-r)dt + \sigma\, dz$$

Cox-Ingersoll Ross model

$$dr = a(b-r)dt + \sigma\sqrt{r}\, dz$$

Hull-White Model:

$$dr = a(\theta(t)/a - r)dt + \sigma\, dz$$

where $\theta(t)$ is calculated from the initial yield curve. These models give closed form solutions to zero coupon bonds of all maturities as functions of the state variables (in the first two models only r).

Models of short rate solution to Vasicek's model

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

$$A(t,T) = \exp\left[\frac{\left(B(t,T) - T + t\right)\left(a^2 b - \sigma^2/2\right)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a}\right]$$

Solutions to the other models can be found in Ch 28 in Hull

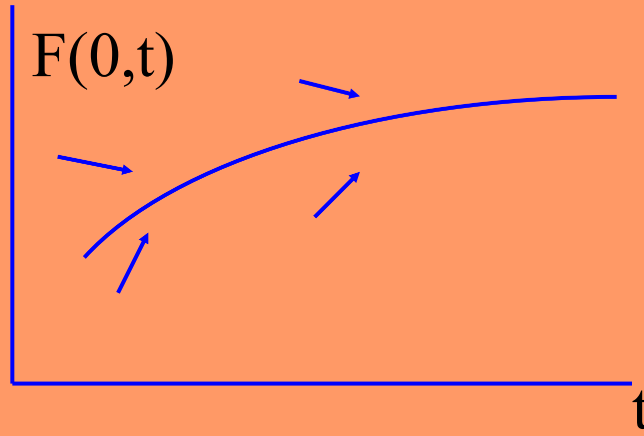
Hull-White Model

$\theta(t) = F_t(0,t) + a\,F(0,t) + \sigma^2/(2a)^*\,(1-\exp(-2at))$

- $\theta(t)$ is selected to fit the initial term structure of interest rates

- the drift in r is towards the initial term structure defined by F(0,t) as if we ignore the last term (which is typically small) the drift is:

$$F_t(0,t) + a[F(0,t) - r]$$



Black& Scoles can also be derived in the risk neutral world

$$C_0 = E^* e^{-rT} \max(S_T - K, 0) = \int_{-\infty}^{\infty} e^{-rT} \max(S_T - K, 0) f(S_T) dS_T$$

$$= \int_{-\infty}^{\infty} e^{-rT} \max\left(S_0 e^{\left(\sigma z + \left(r - \frac{1}{2}\sigma^2\right)T\right)} - K, 0 \right) \frac{e^{-\frac{z^2}{2T}}}{\sqrt{2\pi T}} dz$$

$$= \int_{a}^{\infty} e^{-rT} max\left(S_0 e^{\left(\sigma z + \left(r - \frac{1}{2}\sigma^2\right)T\right)} - K \right) \frac{e^{-\frac{z^2}{2T}}}{\sqrt{2\pi T}} dz$$

where

$$a = \frac{1}{\sigma}\left[\ln K - \ln S_0 - \left(r - \frac{1}{2}\sigma^2 \right) T \right]$$

BS:

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where $d_1 = \dfrac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$

$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Integration gives now the Black&Scholes result

Multiple sources of uncertainty

- Suppose z a vector

$$df = \mu dt + \sum_{i=0}^{n} \sigma_{f,i} f dz_i$$

- Similar results but now market prices λ_i for all sources of uncertainty i

$$\mu - r = \sum_{i=0}^{n} \lambda_i \sigma_i$$

Application: B&S with stochastic interest rate

B&S with stochastic interest rate

- Let $P(t,T)$ be the numeraire security:
  - $g_T = P(T,T) = 1$
  - $g_0 = P(0,T)$

  => $f_0 = P(0,T)\,E^*(f_T) = e^{-RT}E^*(f_T)$

- Let $f_T = \max(S_T - K;0)$

- Whereas in p.53 where we derived B&S with non-stochastic interest rate we had

$$C_0 = E^* e^{-rT} \max(S_T - K,0) = e^{-rT} E^* \max(S_T - K,0)$$

  we now have

$$C_0 = E^* e^{-rT} \max(S_T - K,0) = e^{-RT} E^* \max(S_T - K,0)$$

  implying that the result is similar except R replacing r in the B&S formula.

5. CONTINUOUS TIME FINANCIAL MARKETS AND THE WEALTH EVOLUTION

Stochastic Integrals

- Ito's lemma

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b \, dz$$

implies that

$$G(x_T) - G(x_0) = \int_0^T \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \int_0^T \frac{\partial G}{\partial x} b \, dz$$

Continuous Time Financial Markets: Wealth Evolution

- Suppose only one source of uncertainty: Brownian motion z(t)
- Let Ω_t represent the information generated by the Brownian motion up to time t: $\{z(s): 0 \le s \le t\}$
- Financial markets consist of two assets

- Riskless security 0: $dP_0 = rP_0 dt$, $P_0(0) = 1$

- Risky security 1 (stock): $dP_1 = \mu P_1 dt + \sigma P_1 dz(t)$, $P_1(0) > 0$

HW: Use Ito's Lemma to represent $P_1(t)$ explicitly as a function of z(t) and t

Solution to HW

$$d\ln P = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dz$$

$$\ln P(t) - \ln P(0) = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right)ds + \int_0^t \sigma dz(s) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z(t)$$

$$P(t) = P(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z(t)}$$

Wealth and Utility Maximization

<u>Trading Strategies</u>

Given initial wealth W(0), a trading strategy is a process $\Pi(t)_{t=0}^{T}$ that is non-anticipating and satisfies $\int_0^T \left|\Pi(t)_{t=0}^{T}\right| < \infty.$

Π represents the investment in stock therefore W(t) – Π(t) is invested in riskless asset. Hence this trading strategy is strictly self-financing.

$$dW(t) = [W(t)-\Pi(t)]rdt + \Pi(t)[\mu dt + \sigma dz(t)]$$

This is a controlled diffusion where by changing Π(t) we change the variance of the process

Wealth Evolution

- Using Ito's lemma we can verify that discounted wealth

$$f(W(t),t) = e^{-rt}W(t) =$$

$$W(0) + \int_0^t (\mu - r)e^{-rs}\Pi(s)ds + \int_0^t \sigma e^{-rs}\Pi(s)dz(s)$$

is the solution to the evolution equation (defined below)

Wealth Evolution

$$d\left(e^{-rt}W(t)\right) = \left(e^{-rt}\left[rW(t) + (\mu - r)\Pi(t)\right] - re^{-rt}W(t)\right)dt$$

$$+ e^{-rt}\sigma\Pi(t)dz(t) =$$

$$e^{-rt}(\mu - r)\Pi(t)dt + e^{-rt}\sigma\Pi(t)dz(t)$$

- We soon define a new “probability measure” where drift of this discounted wealth is zero, hence discounted wealth is a martingale.
- Definition: Absence of arbitrage. There exists no non-anticipating trading strategy that starts from W(0)=0 and leads to W(T) where P(W(T)>0)>0 and P(W(T) ≥ 0)=1.

Equivalent Martingale Measure

- The absence of arbitrage is equivalent to existence of a probability measure p* under which discounted security prices are martingales.
- So under this probability measure all trading strategies Π are discounted martingales:

$$E^*\left[e^{-rt}W(t)\right]\left|\Omega_s\right. =$$

$$W(0) + \int_0^s \sigma e^{-rs}\Pi(s)dz(s) + E^*\int_s^t \sigma e^{-ru}\Pi(u)dz(u)\ \left|\Omega_s\right. = e^{-rs}W(s)$$

Martingale representation theorem / Dynamically complete markets

- Martingale Representation Theorem: If $\{M(t)\}$ is a martingale with respect to Ω_t then M(t) can be represented by a stochastic integral

$$M(t) = M(0) + \int_0^t Y(s)dz(s)$$

and this representation is unique.

- Pricing of contingent claims: If $\sigma > 0$, a contingent claim f_T is attainable in the sense that there exists a trading strategy that yields f_T with probability 1. Let W(t) be the value of this trading strategy at time t. Then f(t) = W(t).
  - Why?

APPENDIX: What is the probability measure that makes discounted wealth a martingale?

Theorem(Girsanov Transformation) Suppose z(t) is a standard Brownian motion and X a process such that

$$H_t(X) = e^{\int_0^t X_s dz(s) - \frac{1}{2}\int_0^t X_s{}^2 ds}$$

is a martingale. Then under the measure $P^*(A) = E1_A H_T$, the process

$$z^*(t) = z(t) - \int_0^t X(s) dz(s)$$

is a Brownian motion. We apply this theorem for $X_t=-\theta=-(\mu-r)/\sigma$.

APPENDIX: H(t) a martingale

- We can check that H(t) is a martingale

$$E\left[H(t)\;|\Omega_s\right] = E\left[H(t)\;|z(s)\right] =$$

$$E\left[e^{-\theta z(t)-\frac{1}{2}\theta^2 t}\;|z(s)\right] =$$

$$E\left[e^{-\theta\left(z(t)-z(s)\right)-\frac{1}{2}\theta^2(t-s)}e^{-\theta z(s)-\frac{1}{2}\theta^2 s}\;|z(s)\right] =$$

$$e^{\frac{1}{2}\theta^2(t-s)-\frac{1}{2}\theta^2(t-s)}H(s) = H(s)$$

- So z*(t) = z(t)+θt = $z(t)+\left(\frac{\mu-r}{\sigma}\right)t$ is a Brownian motion under measure P*

Appendix: What is the probability measure that makes discounted wealth a martingale?

$$H_T = \frac{\frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2T}\left(z(T)-\left(\frac{\mu-r}{\sigma}\right)T\right)^2}}{\frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(z(T))^2}}$$

Now H_T is the change of probability measure from the true probability P to the risk neutral probability P*, that is

$H_T = dP^*/dP$
and $H(t) = E[H_T \mid \Omega_t]$

NOTE: f(t) = E*(f(T)) = E(H(T)f(T)) and f(t) thus depends only on H(t) and t. (You can now also verify our earlier statement that $e^{-rt}W(t)$ is a martingale under P*).

Optimal Investment

The optimal wealth for each state and the optimal trading strategy can now be found by looking at the following maximization problem:

- Max E U(W(T))
  s.t. $E^* e^{-rT} W(T) \leq W(0)$

- Max E U(W(T))
  s.t. $e^{-rT} E [H(T)W(T)] \leq W(0)$

- $\max_{W(T)} E\, U(W(T)) + \ell \{W(0) - e^{-rT} H(T)W(T)\}$ $\quad \forall \omega \in \Omega$

$\Rightarrow \hat{W}(T) = Q(e^{-rT} \ell H(T))$, where $Q = U'^{-1}$ and ℓ solves
$E^* e^{-rT} Q(e^{-rT} \ell H(T)) = W(0)$,

Optimal Investment

The optimal trading strategy can be found as follows:

Recall that H(t) = E (H(T))

Now as wealth $\hat{W}(t)$ that replicates $\hat{W}(T)$ depends only on H(t) and t

$$d\hat{W}(t) = \frac{\partial \hat{W}(t)}{\partial H} dH(t) + drift =$$

$$d\hat{W}(t) = \frac{-\partial \hat{W}(t)}{\partial H} H(t)\left(\frac{\mu - r}{\sigma}\right) dz^* + drift$$

On the other hand $d\hat{W}(t) = r\hat{W}(t)dt + (\mu - r)\hat{\Pi}(t)dt + \hat{\Pi}\sigma dz^*$

Equating dz* terms gives:

$$\hat{\Pi}(t) = \frac{-\partial \hat{W}(t)}{\partial H} H(t)\left(\frac{\mu - r}{\sigma^2}\right)$$

Optimal Investment CARA

The optimal trading strategy can be found as follows:

$$U = \frac{-1}{a} e^{-ax} \Rightarrow U'(x) = e^{-ax}$$

$$\hat{W}(T) = \frac{-1}{a} ln(\ell e^{-rT} H(T))$$

$$=> \hat{W}(t) = E_t^* e^{-r(T-t)} * \frac{-1}{a} ln(\ell e^{-rT} H(T))$$

$$= \frac{-e^{-r(T-t)}}{a} ln\left(\ell e^{-rT}\right) - \frac{-e^{-r(T-t)}}{a} E_t^* ln\left(H(T)\right)$$

$$= K(t) - \frac{e^{-r(T-t)}}{a}\left[ln\left(H(t)\right) - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 (T-t)\right]$$

$$=> \frac{\partial \hat{W}(t)}{\partial H(t)} = \frac{-e^{-r(T-t)}}{aH(t)}$$

$$=> \hat{\Pi}(t) = \frac{-e^{-r(T-t)}}{a}\left(\frac{\mu - r}{\sigma^2}\right)$$

Other common problems

- Max $U(W_T)$
  - s.t. $E^* e^{-rT} W(T) = W(0)$
  - s.t. $W(T) \geq A$

HW Many bonus points if you manage to characterize this as HW (see e.g., Grossman and Vila, Journal of Business).

Trick: Note terminal wealth just like an option max(W(T)-A;0)

KEY ISSUES FROM THIS PART

- Wiener process / Brownian motion
- Ito processes
  - Modeling stocks as an Ito-process
  - Applications:
    - Forward price
    - lnS
- $E(e^x)$
- Derivation of the Black&Scholes differential equation
  - Showing using the boundary condition and Black&Scholes differential equation that for the forward price

$$f = S - K e^{-r(T-t)}$$

KEY ISSUES FROM THIS PART

- Concept of a martingale
- Martingales and arbitrage, risk neutral pricing
- Equivalent martingale measure result
  - if market price for risk $\lambda = \sigma_g$ then $\Phi = f/g$ is a martingale for all f
  - Application:
    - Bonds
- Models for the risk free rate and the term structure of interest rates

KEY ISSUES FROM THIS PART

- Wealth processes
- Risk neutral probabilities more generally
- Dynamically complete markets

- Portfolio optimization in dynamically complete markets
  - Trading rule assuming CARA utilities