Aalto University
School of Science

Department of Computer Science

Combinatorics of Efficient Computations

# Approximation Algorithms

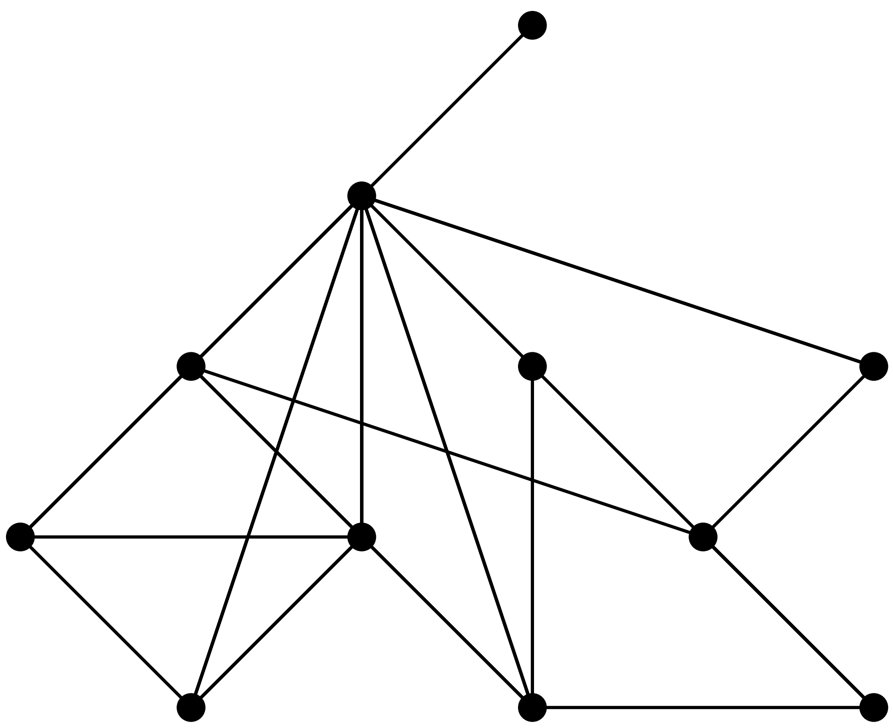
## Lecture 7: Min. Degree Spanning Trees via Local Search

Joachim Spoerhase

2019

# MinimumDegreeSpanningTree

**Given**: A connected graph Graph $G = (V, E)$.
**Find**: A spanning tree $T$ which has the minimum maximum degree $\Delta(T)$ among all spanning trees of G.



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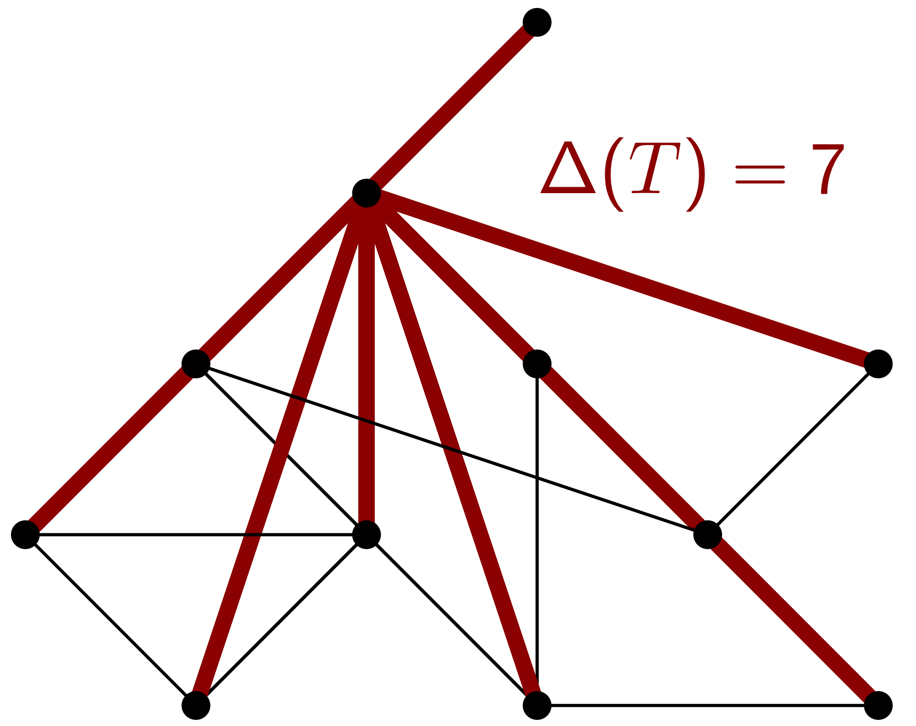

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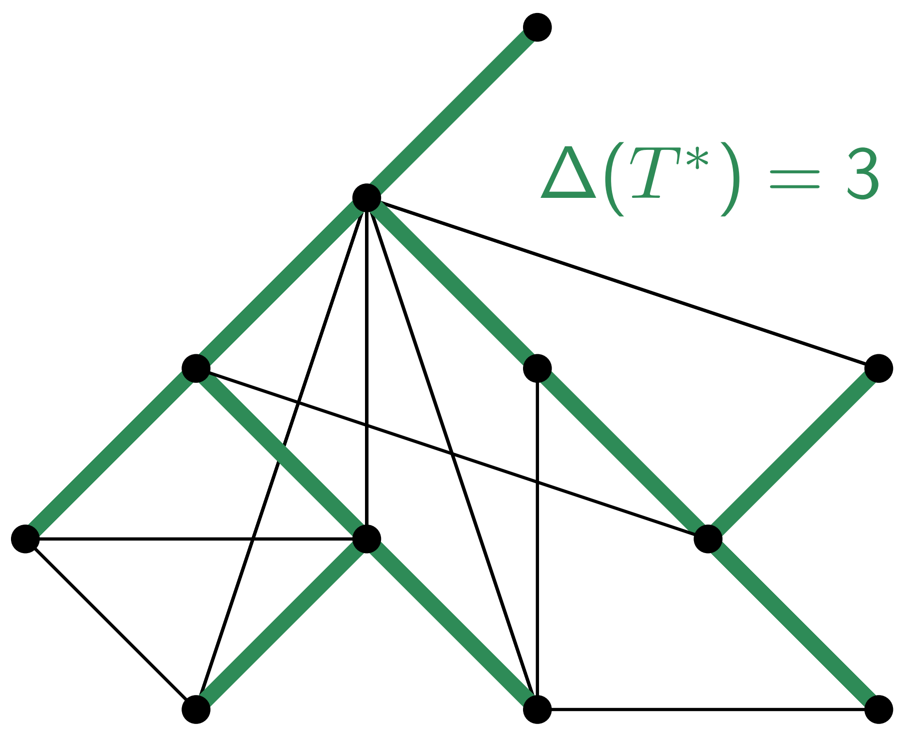

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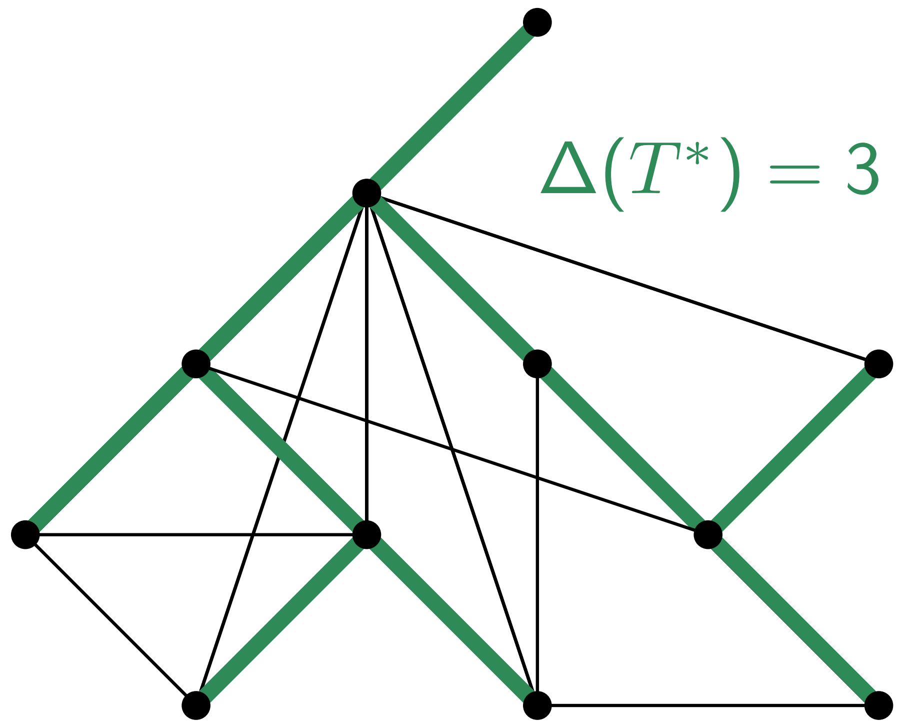

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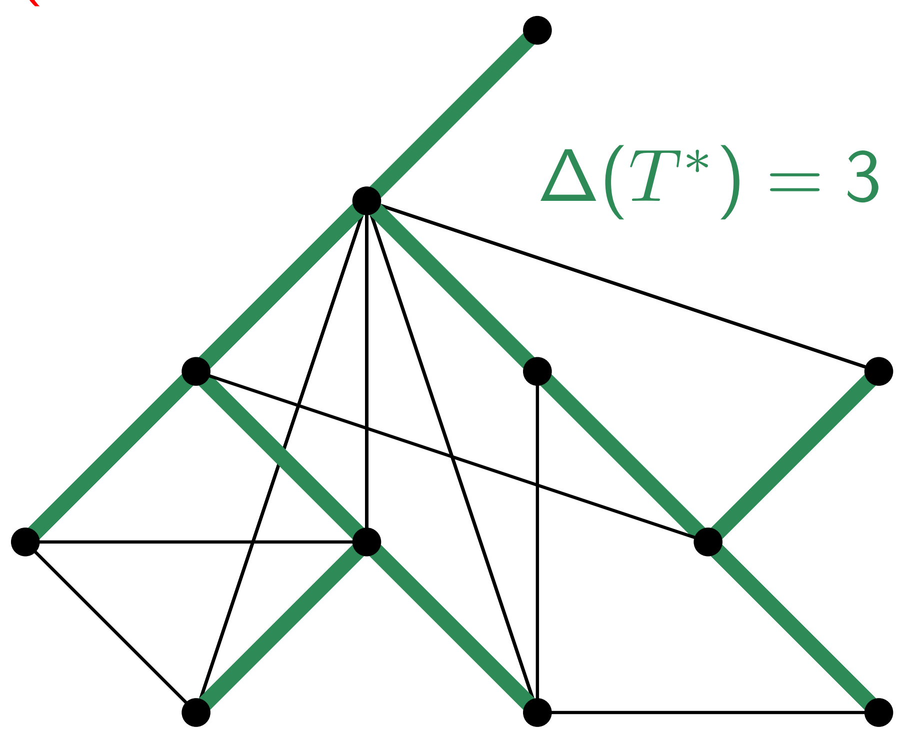

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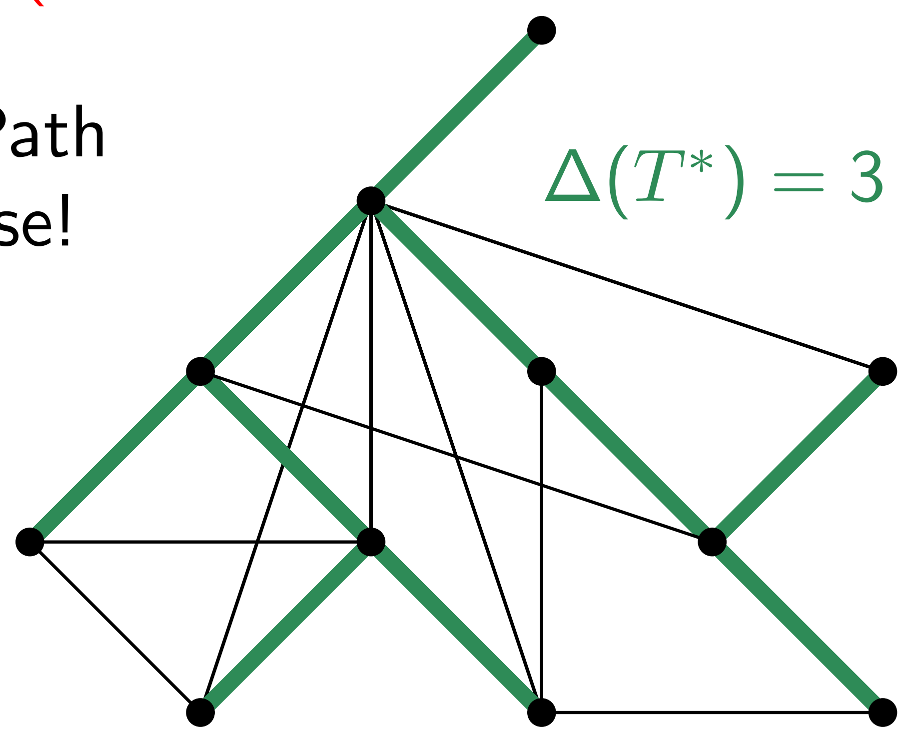
Hamiltonian Path
is a special case!



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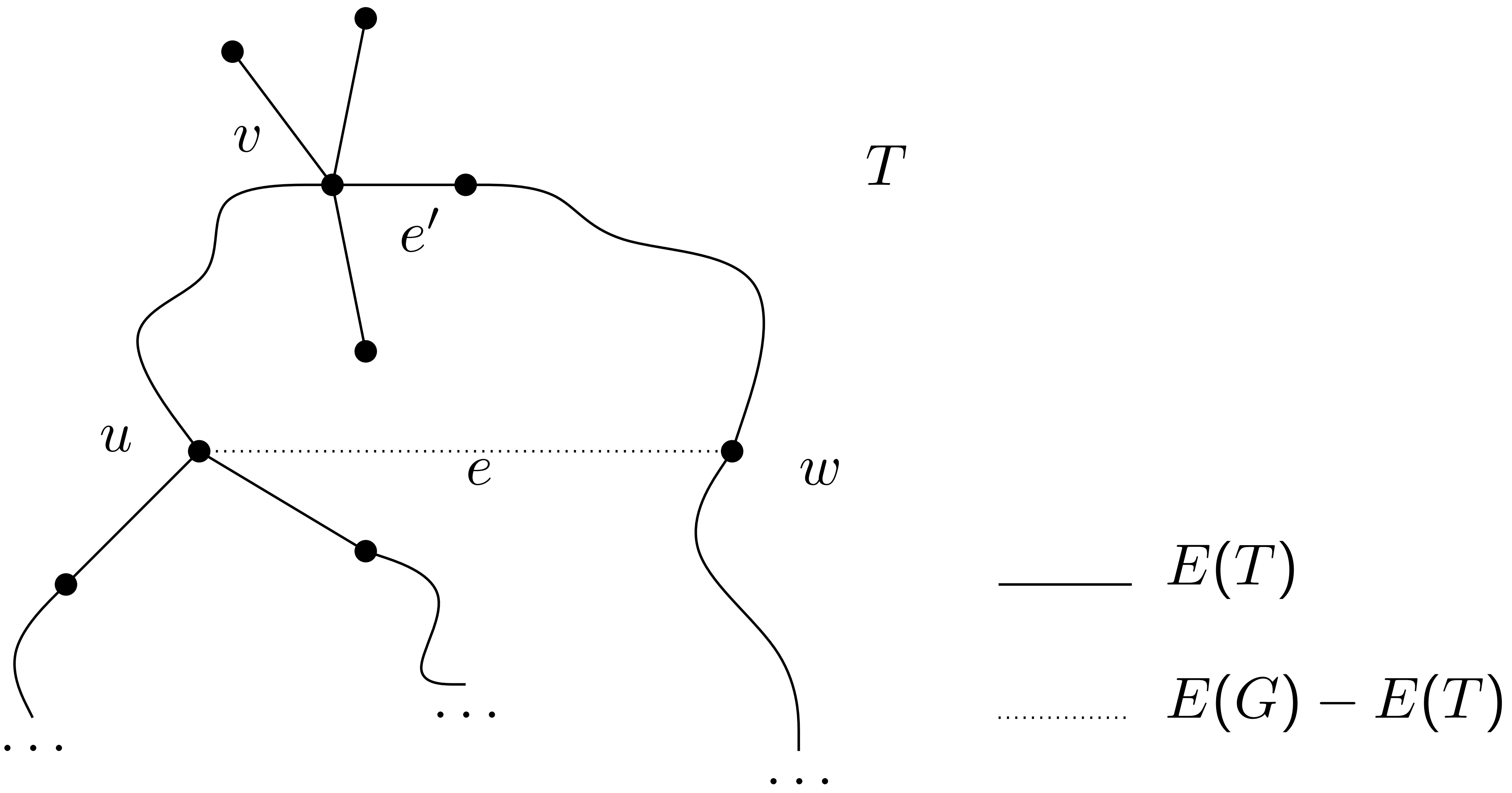

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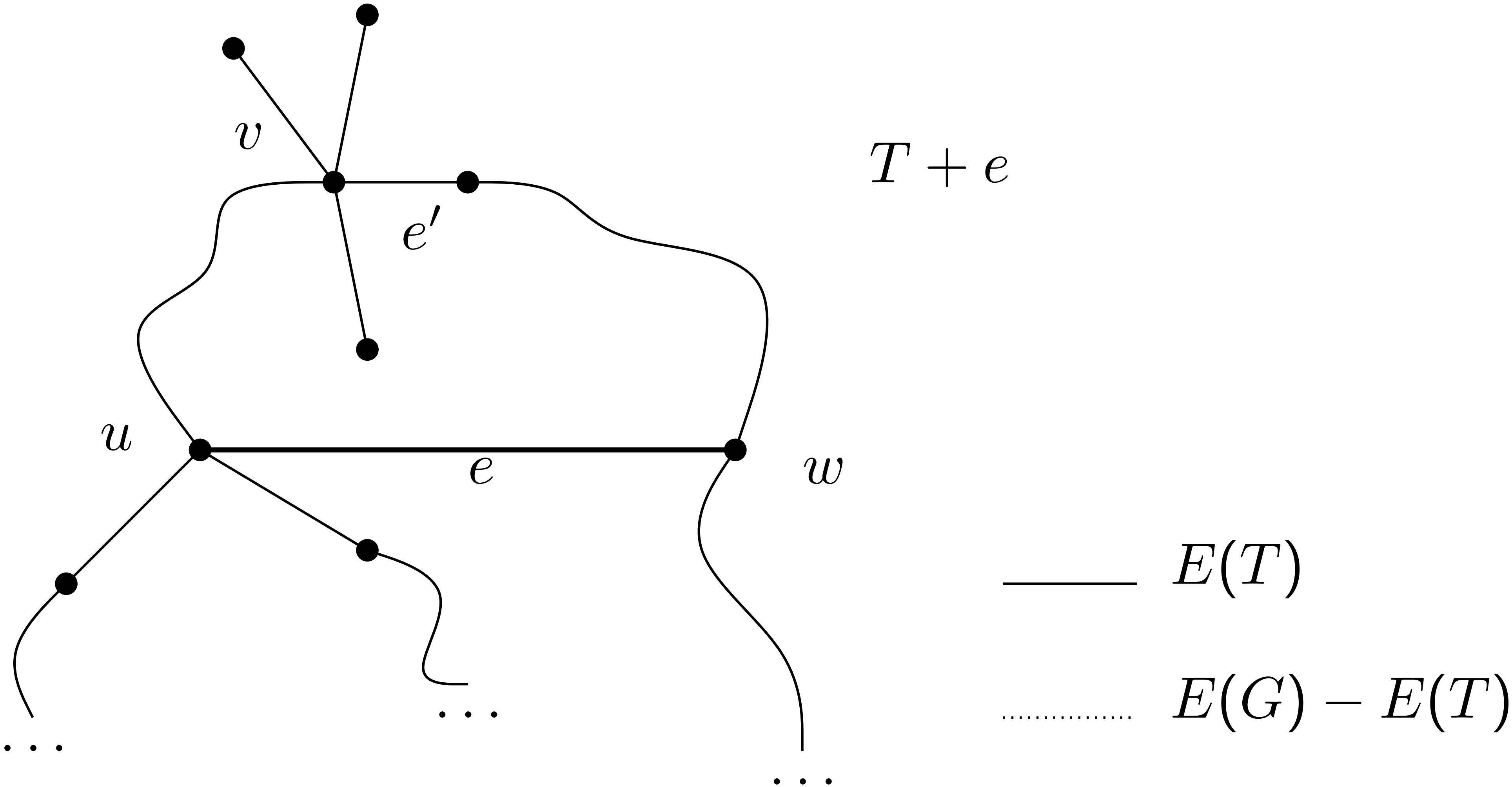

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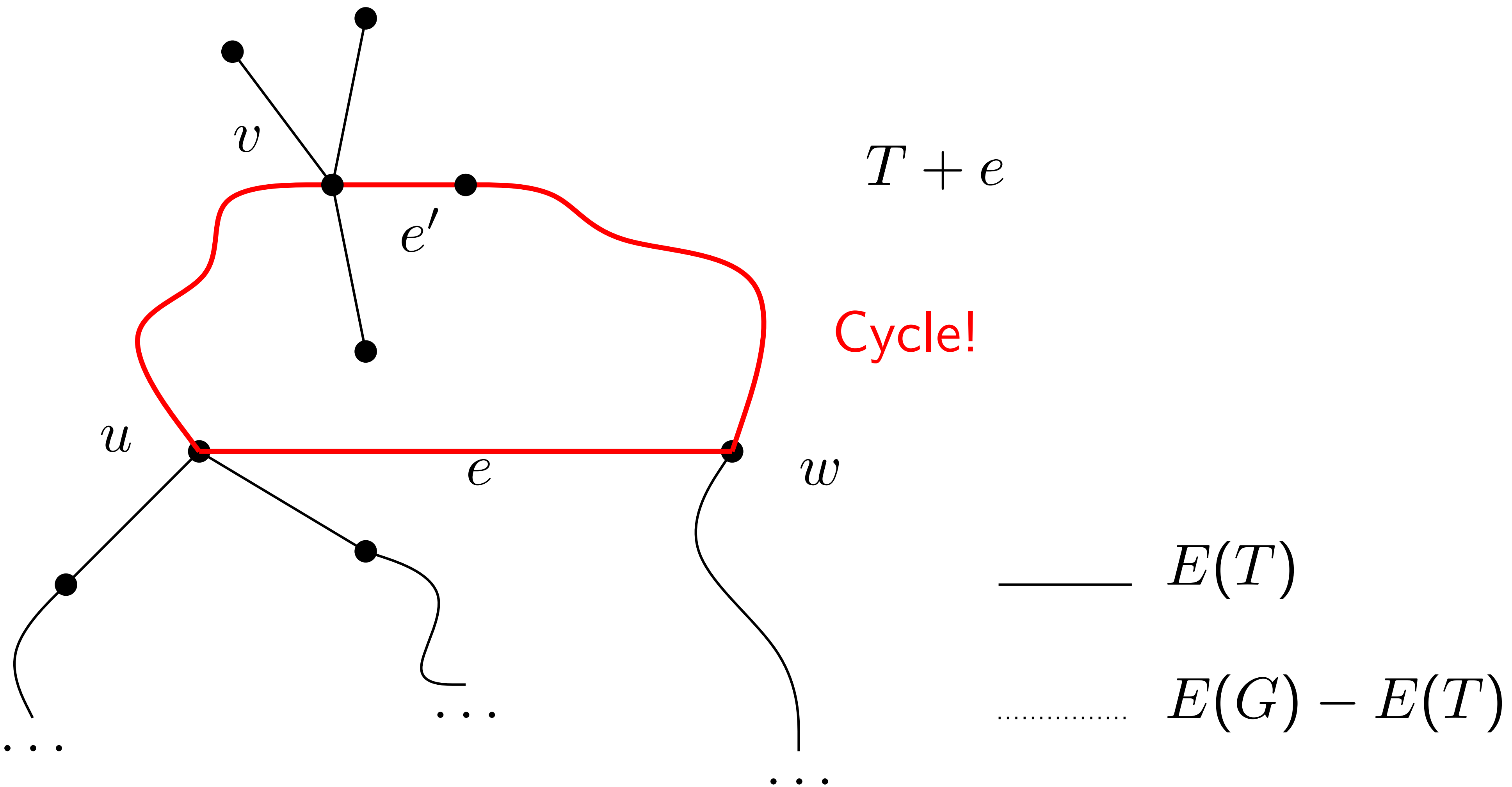

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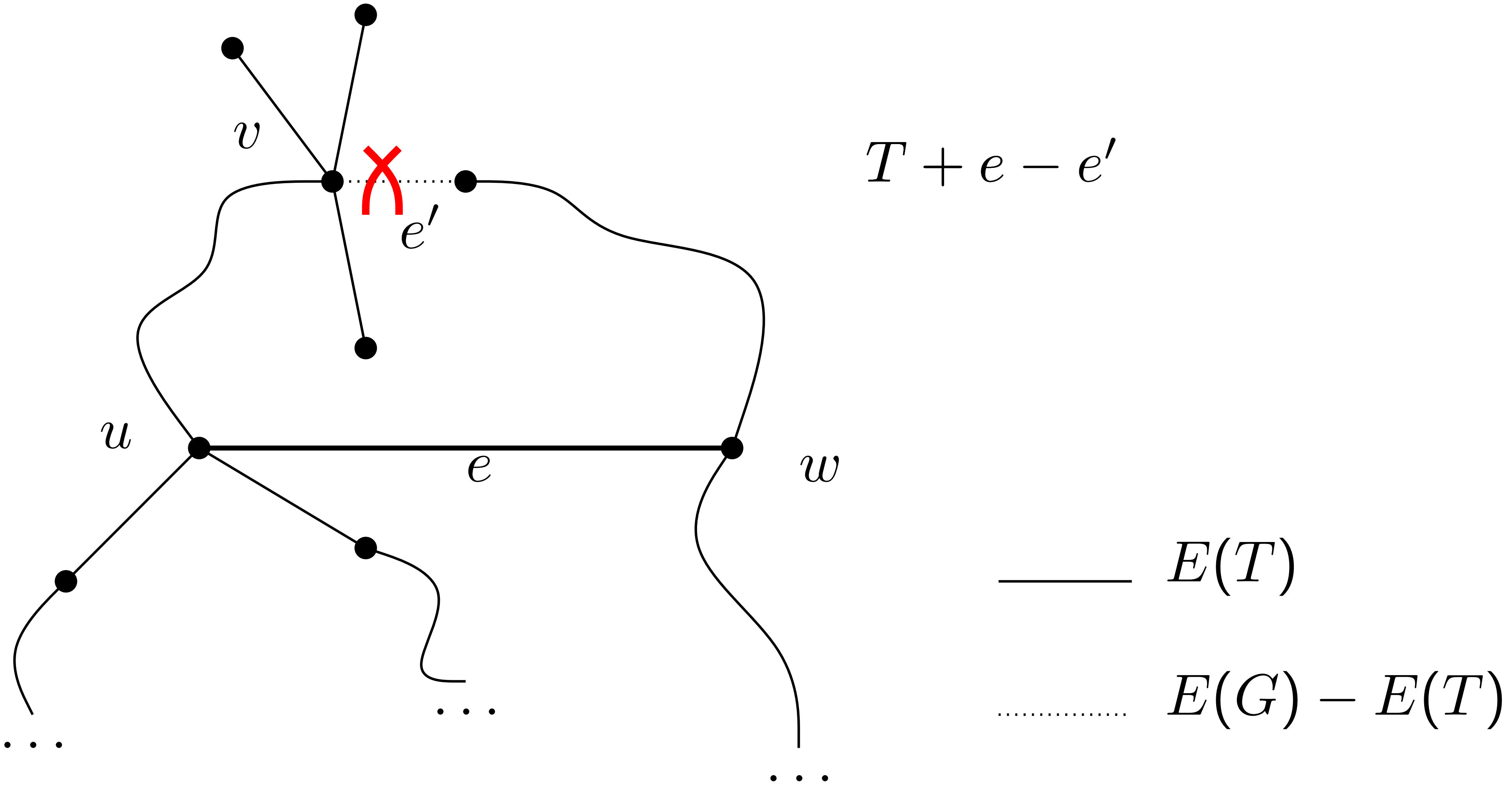

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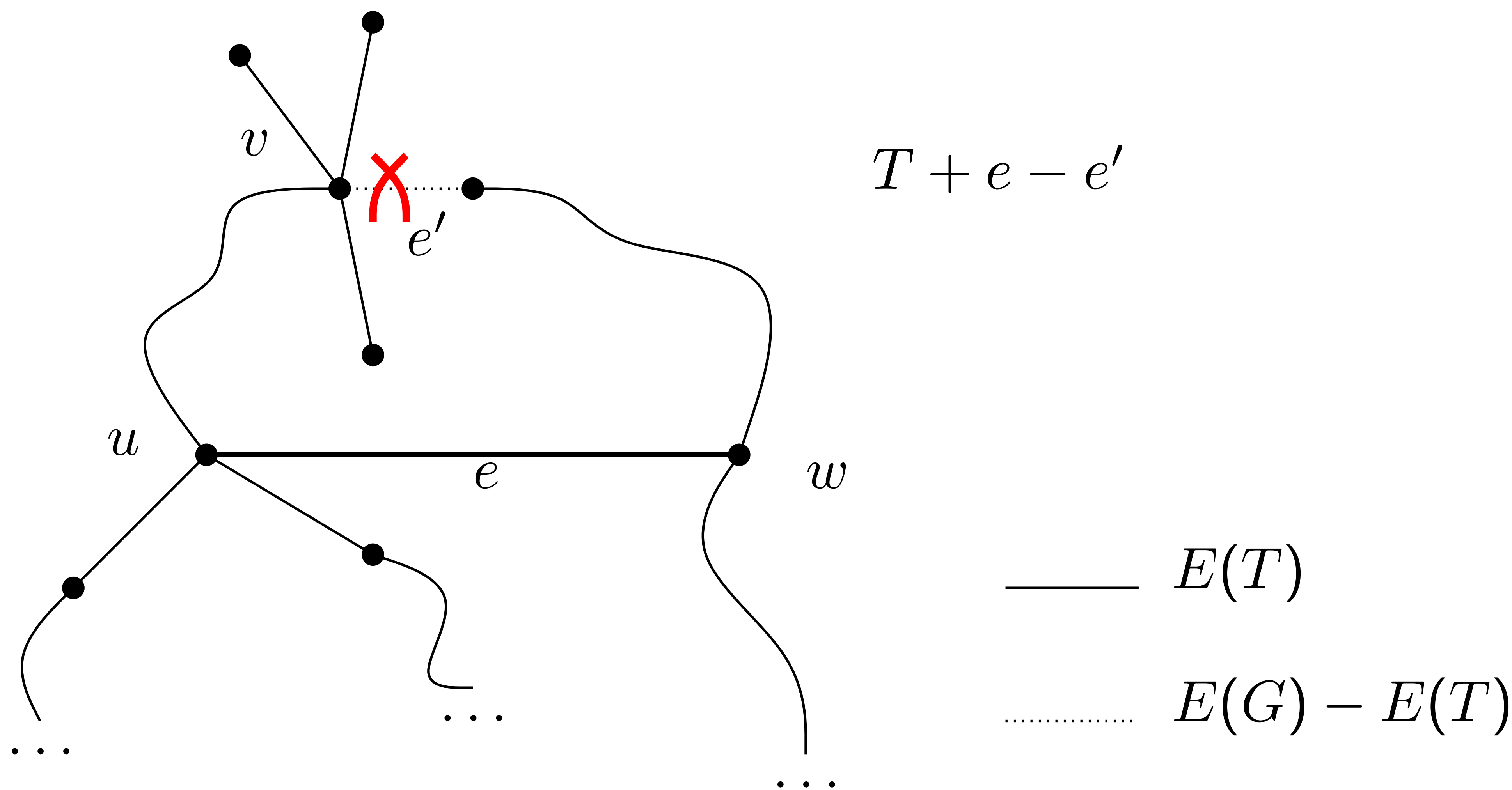
Improvement when $\deg_T(v) - 1 > \max\{\deg_T(u), \deg_T(w)\}$



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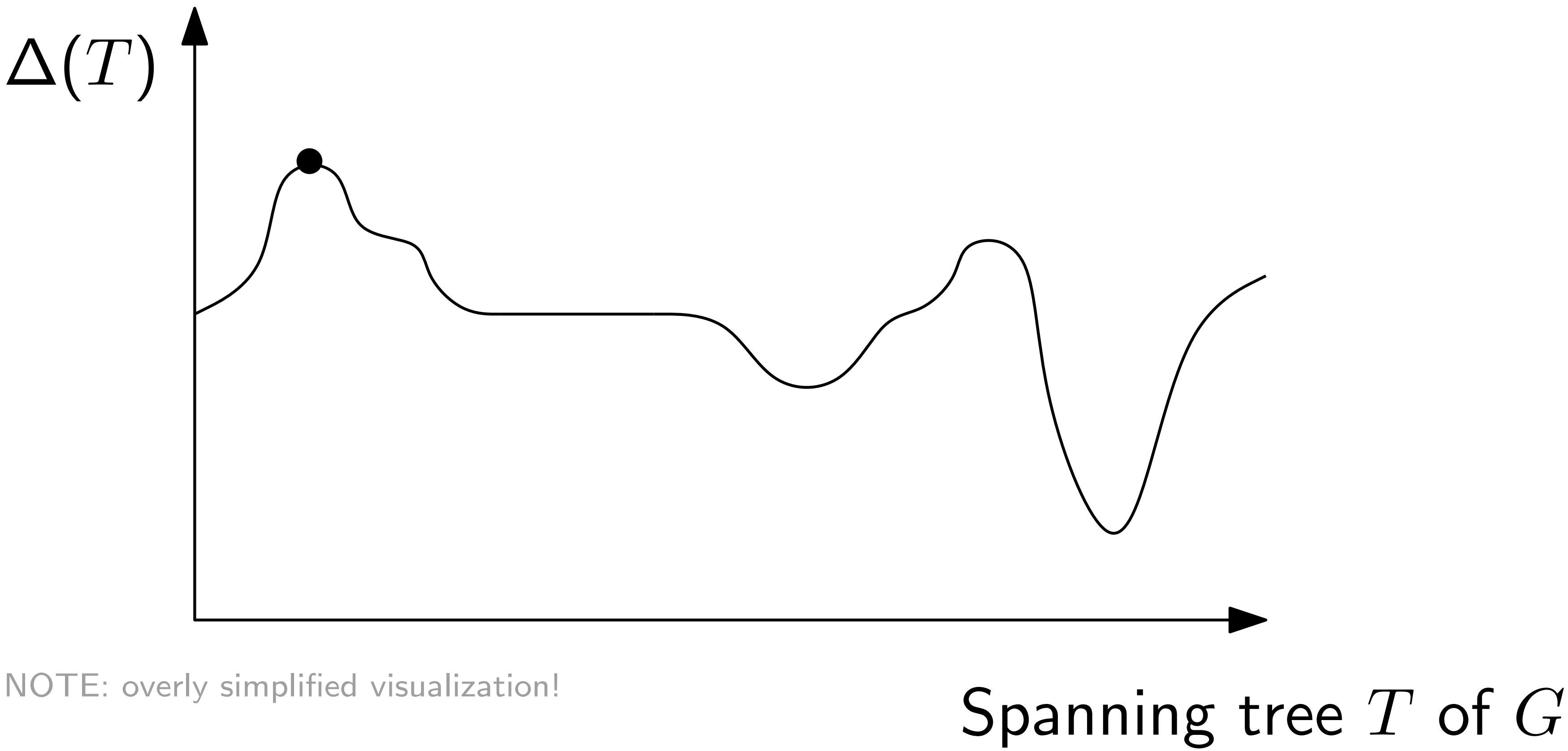


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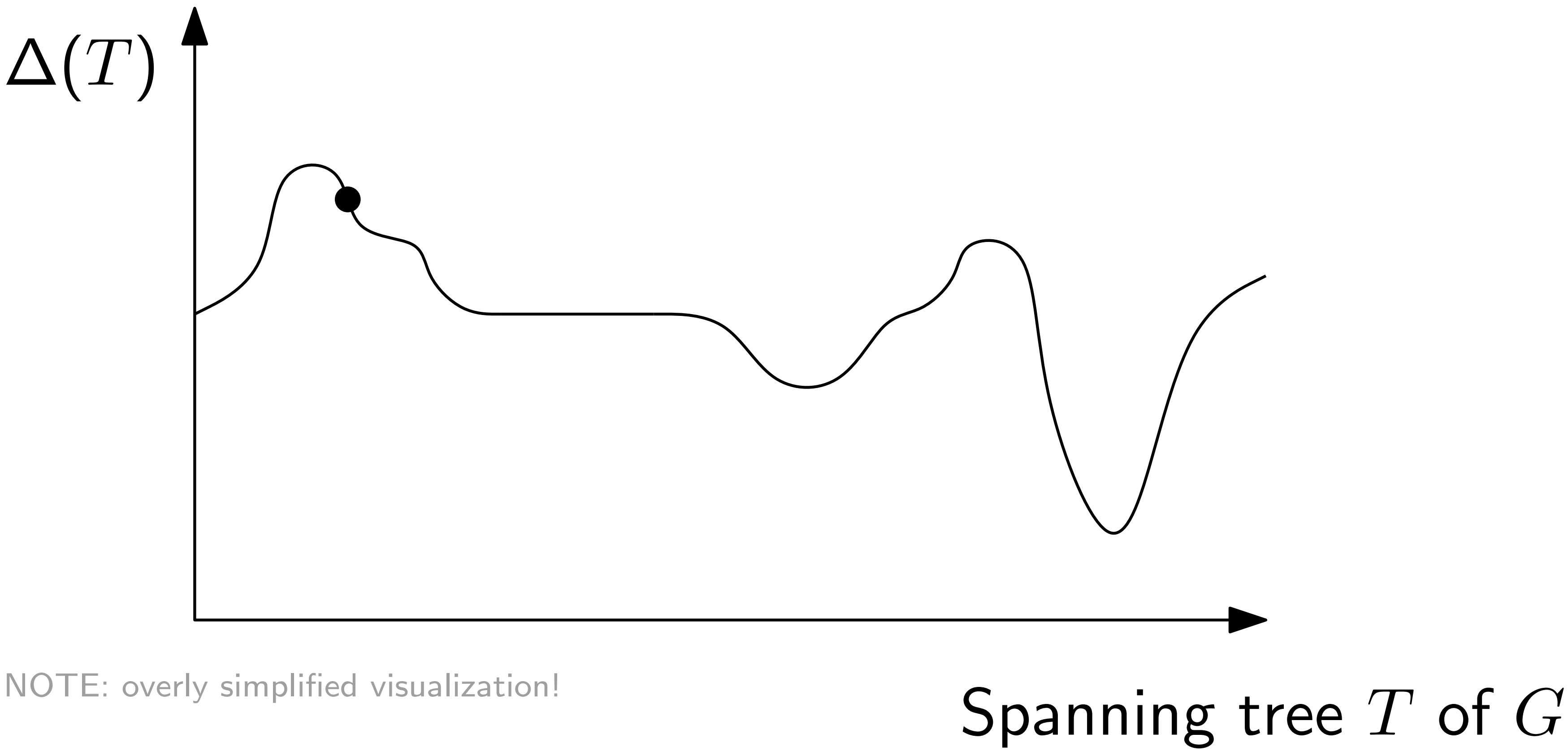

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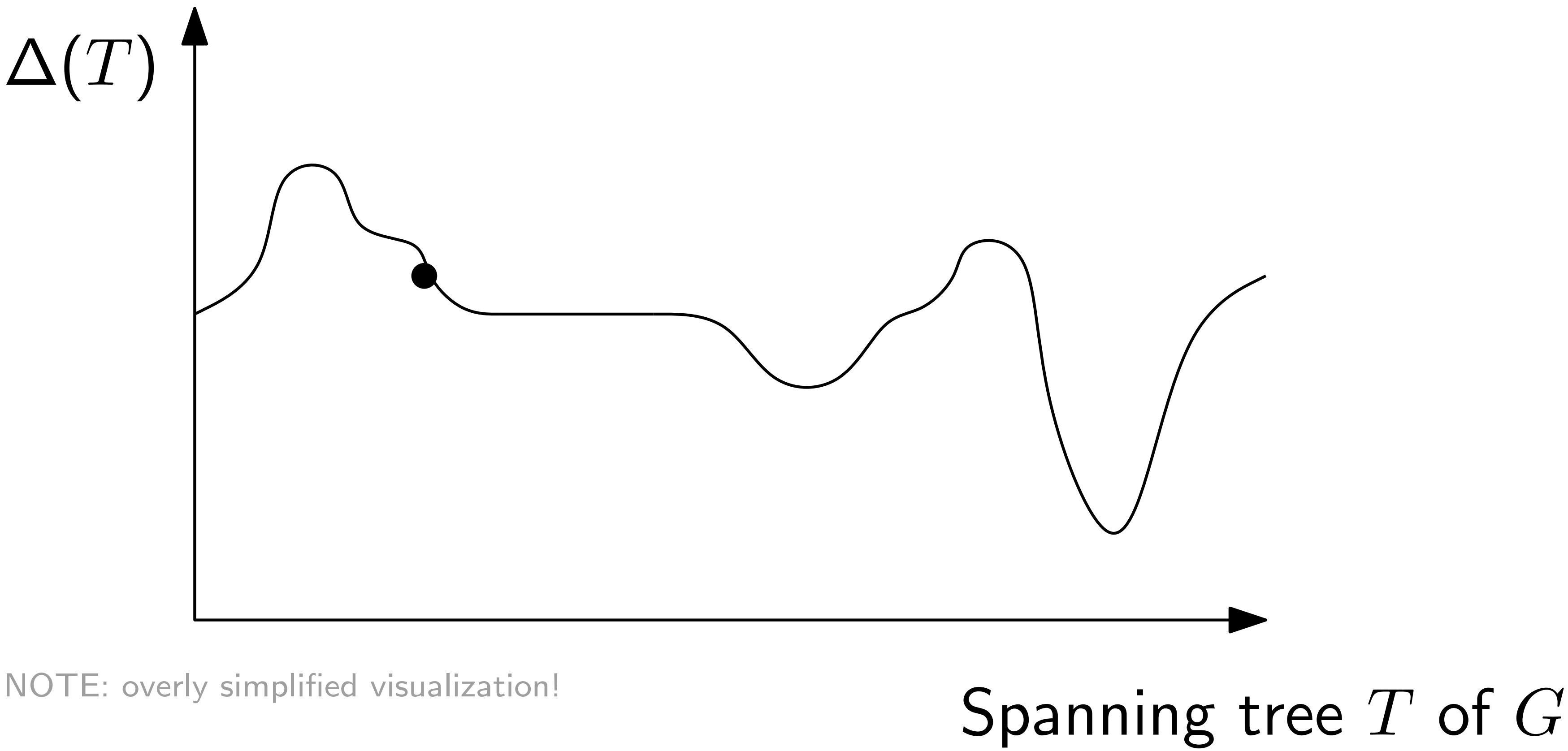


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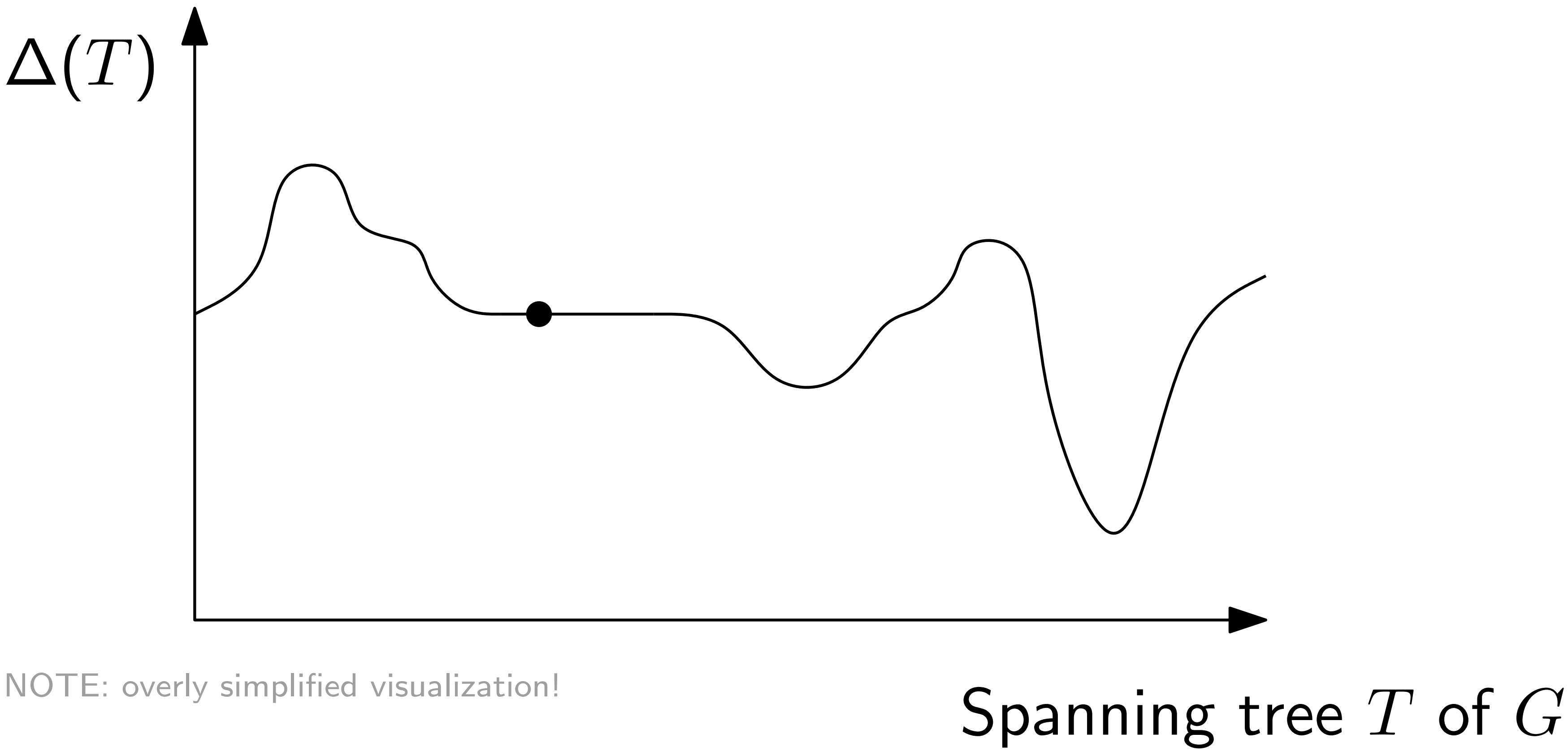


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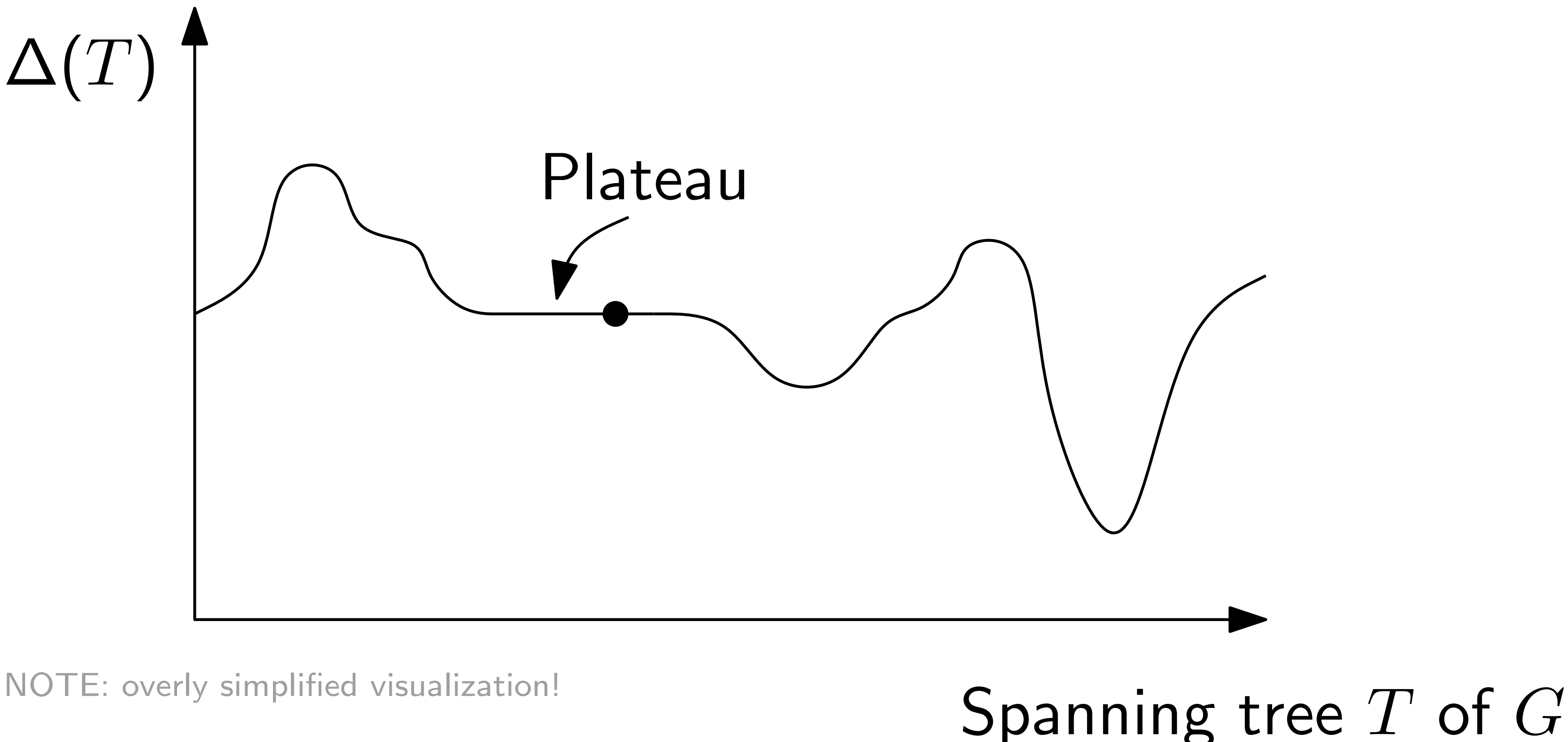


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Flips don't always improve $\Delta(T)$ !!

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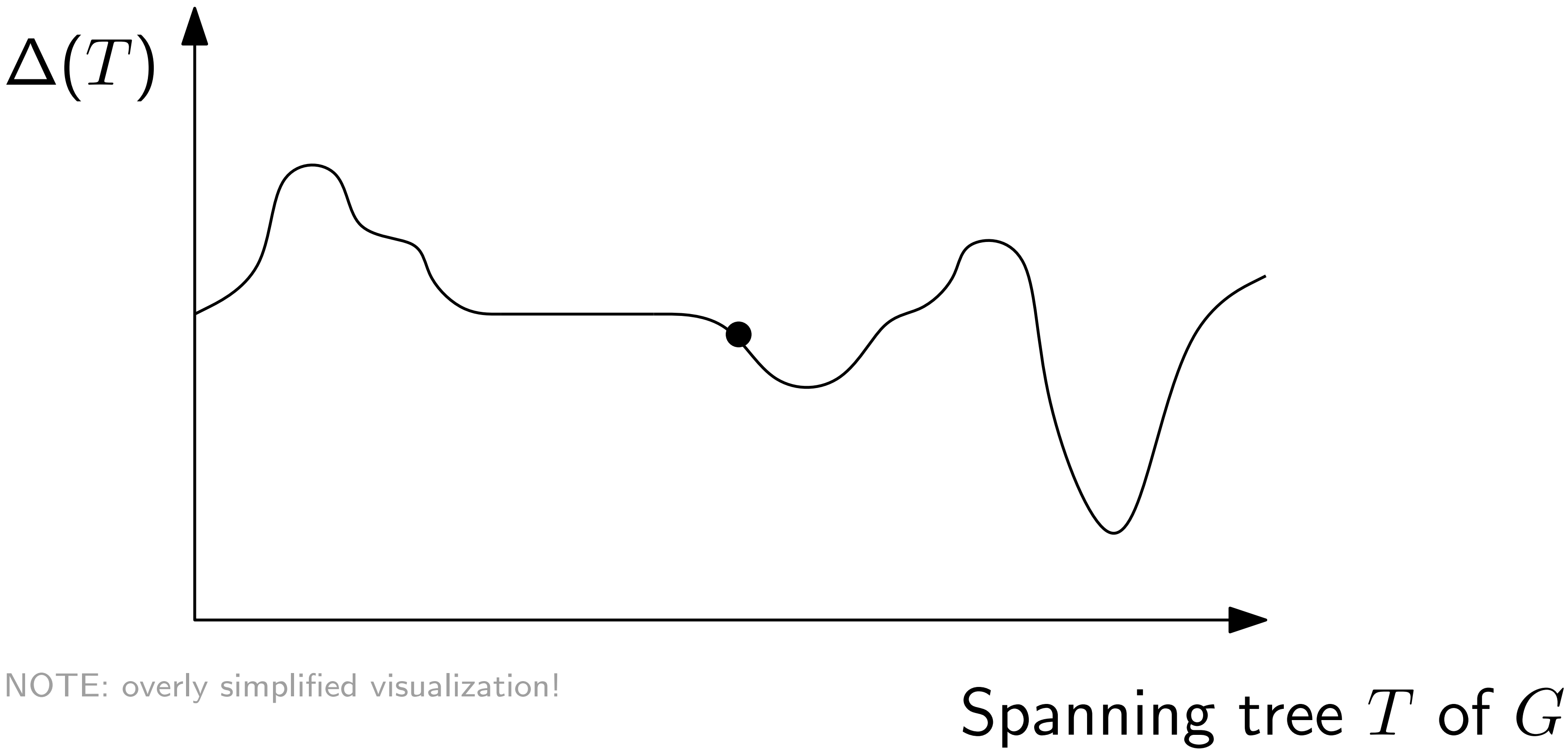


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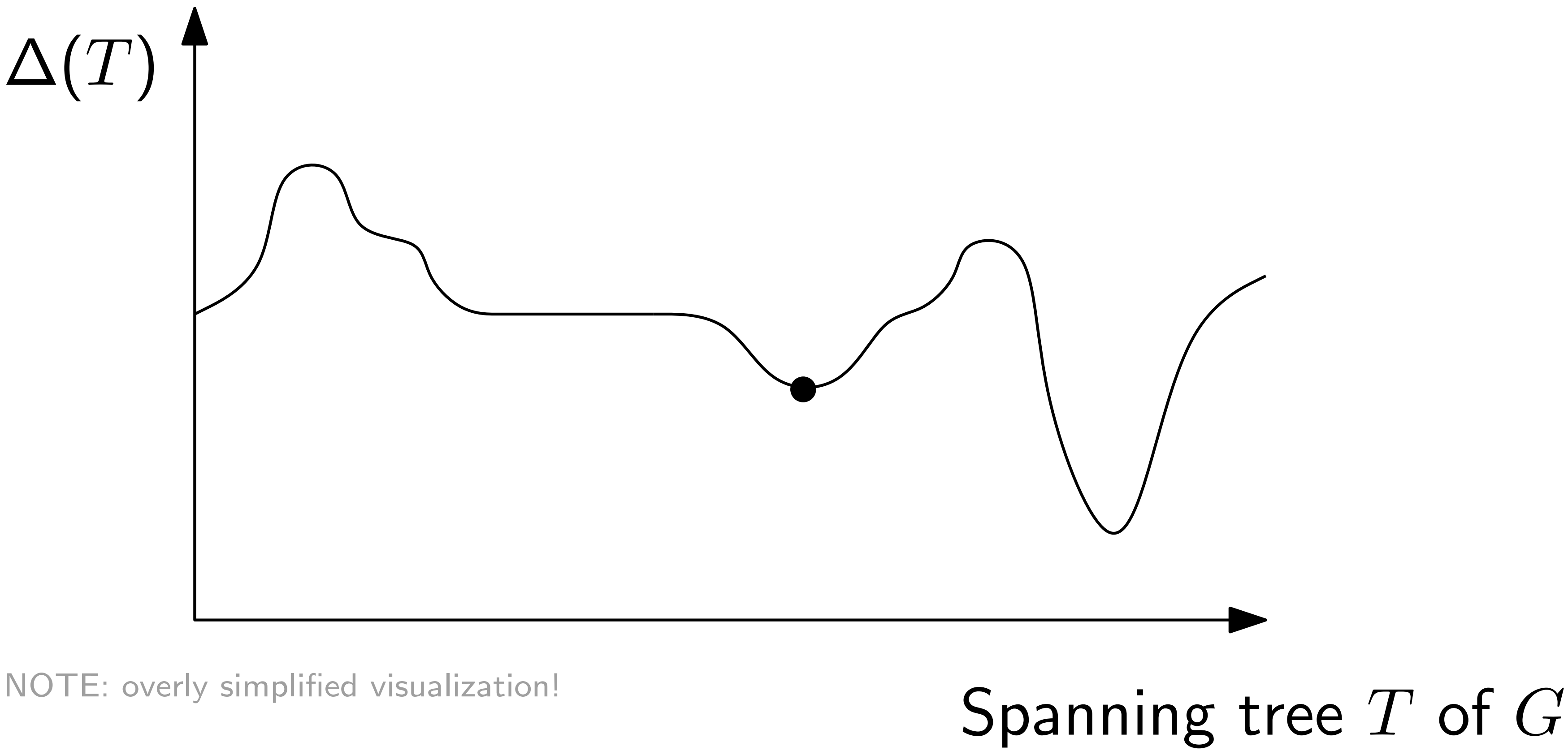


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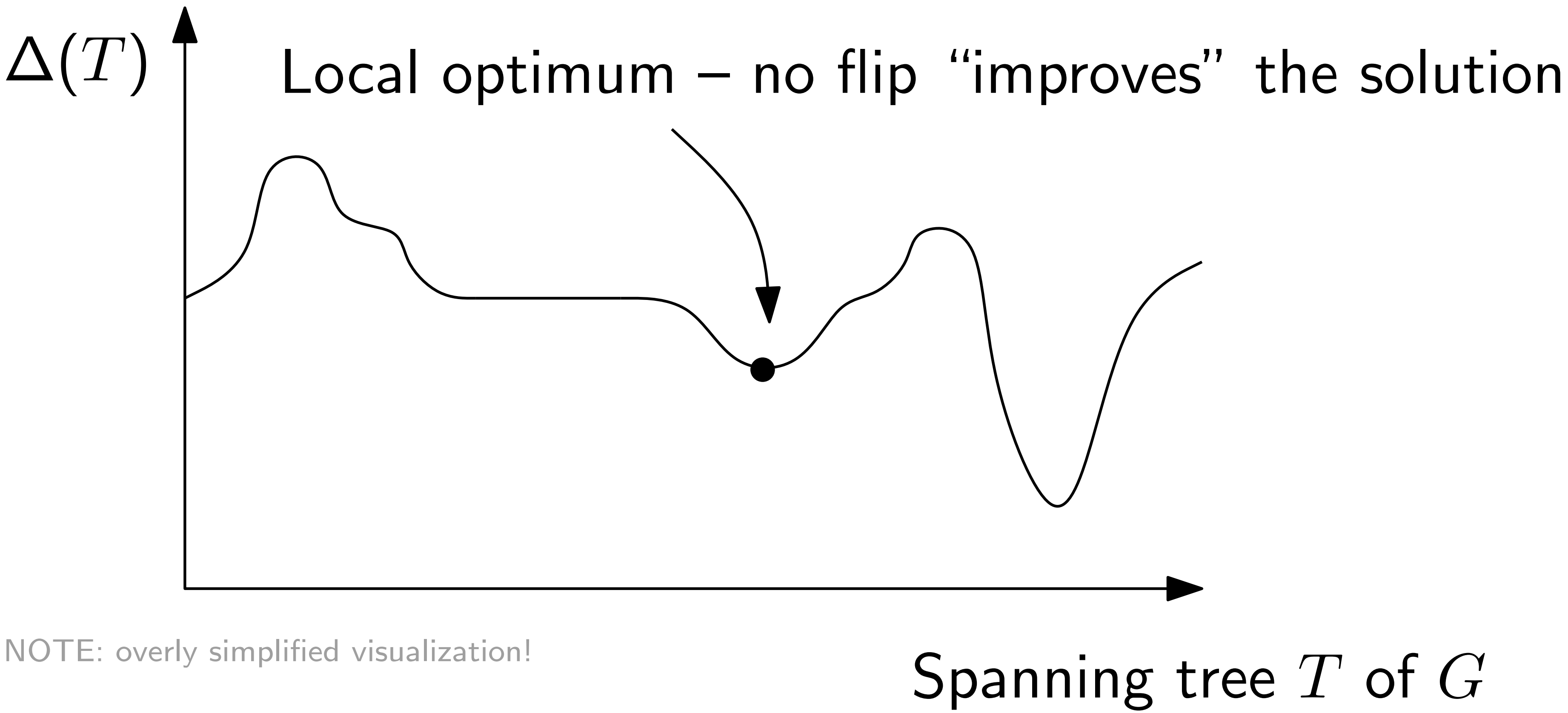


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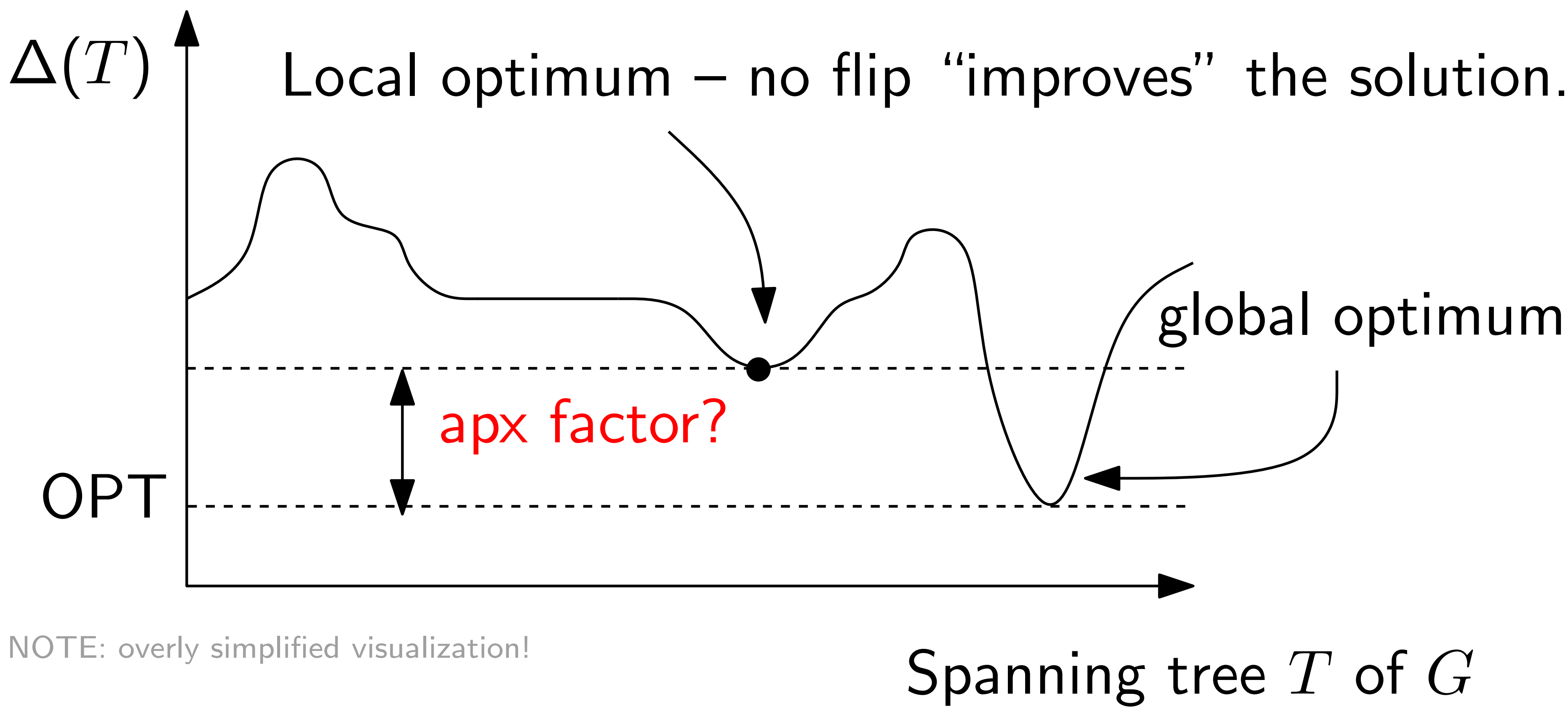

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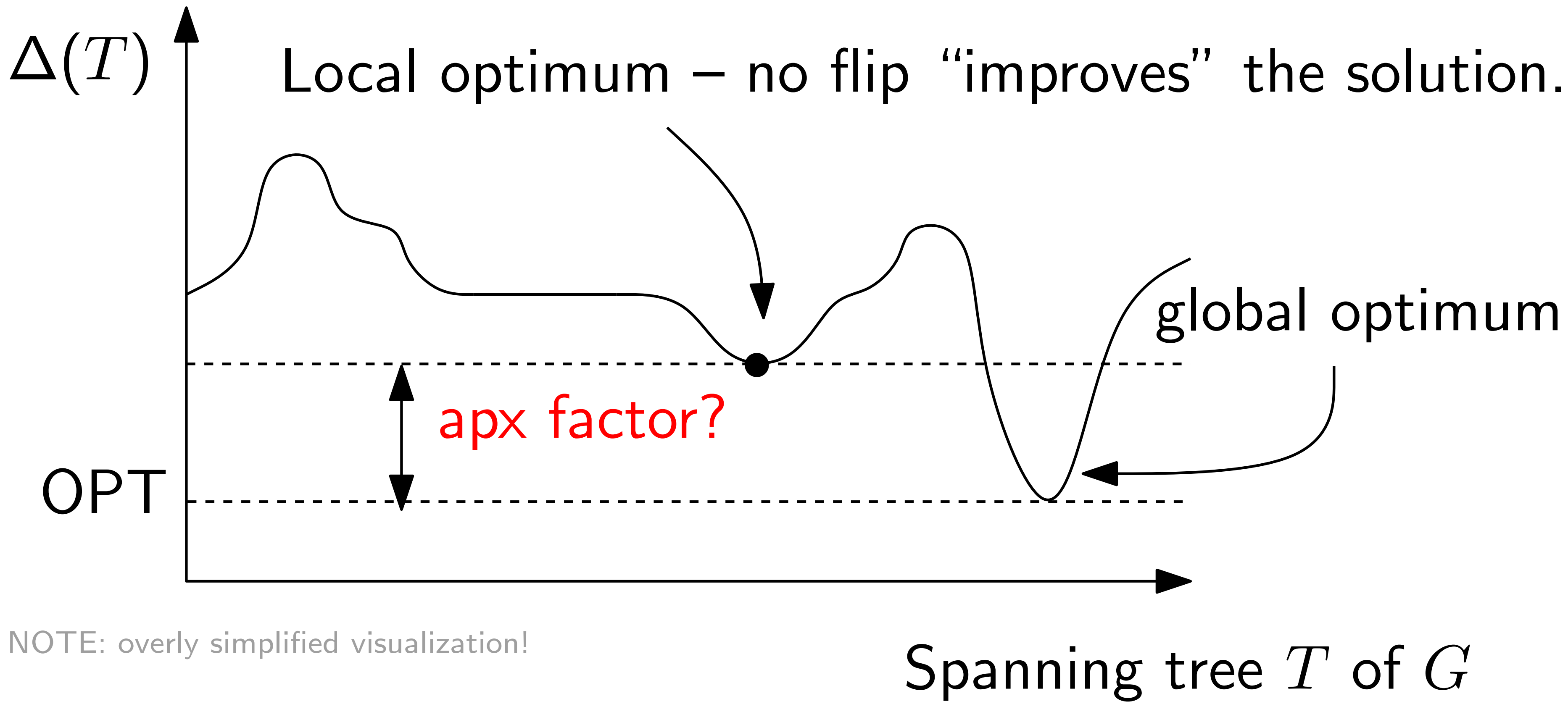


How to handle plateaus? What is the runtime?

# Local Search

Algorithm MinDegSTLocalSearch($T$)

**while** there is an "improving flip"(*) in $T$ for a vertex $v$ with $d_T(v) \geq \Delta(T) - \ell$ **do**

perform the flip.

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- first the approximation factor, then the runtime

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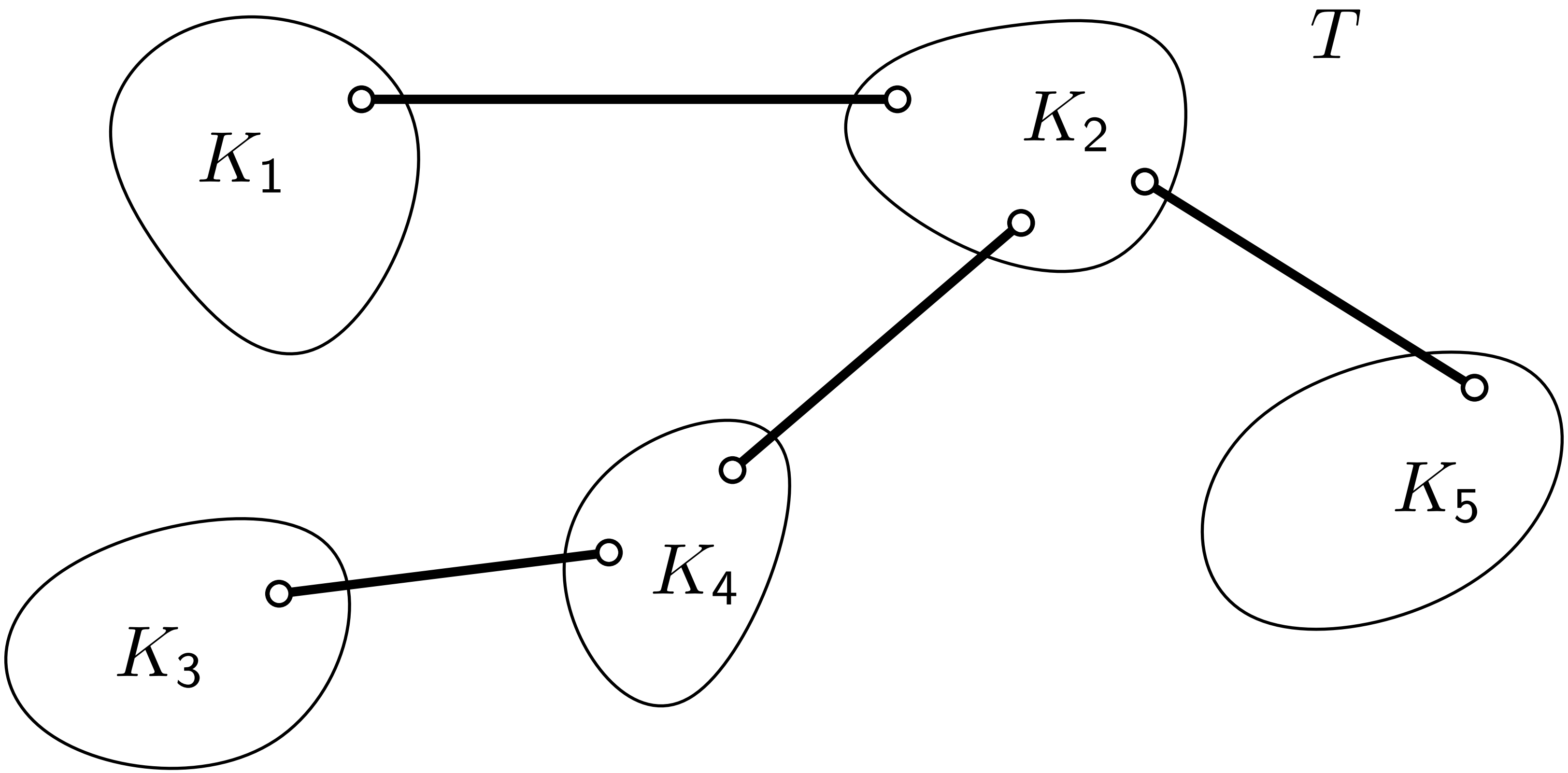


Removing $k$ edges partitions $T$ in $k+1$ components

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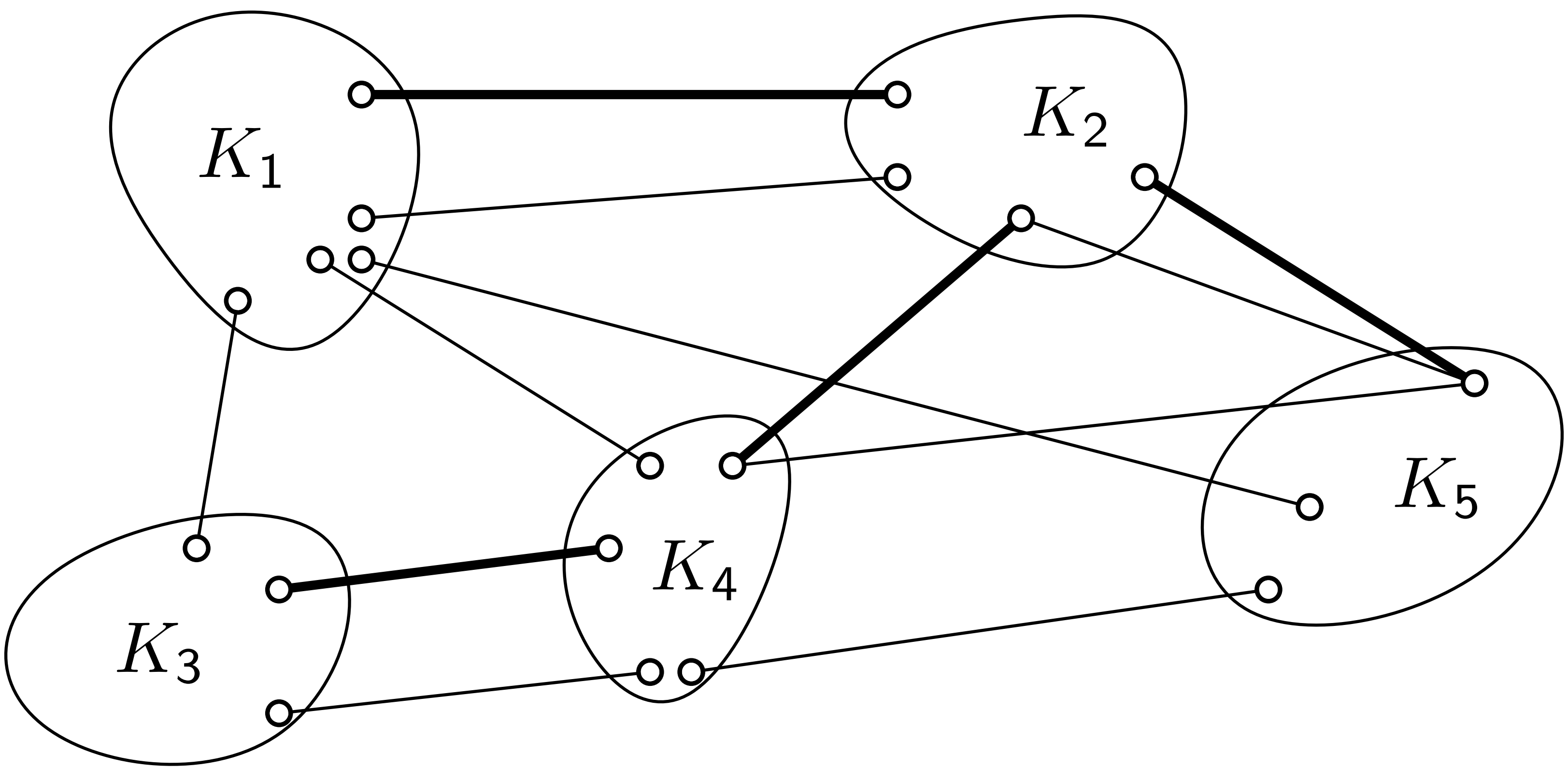

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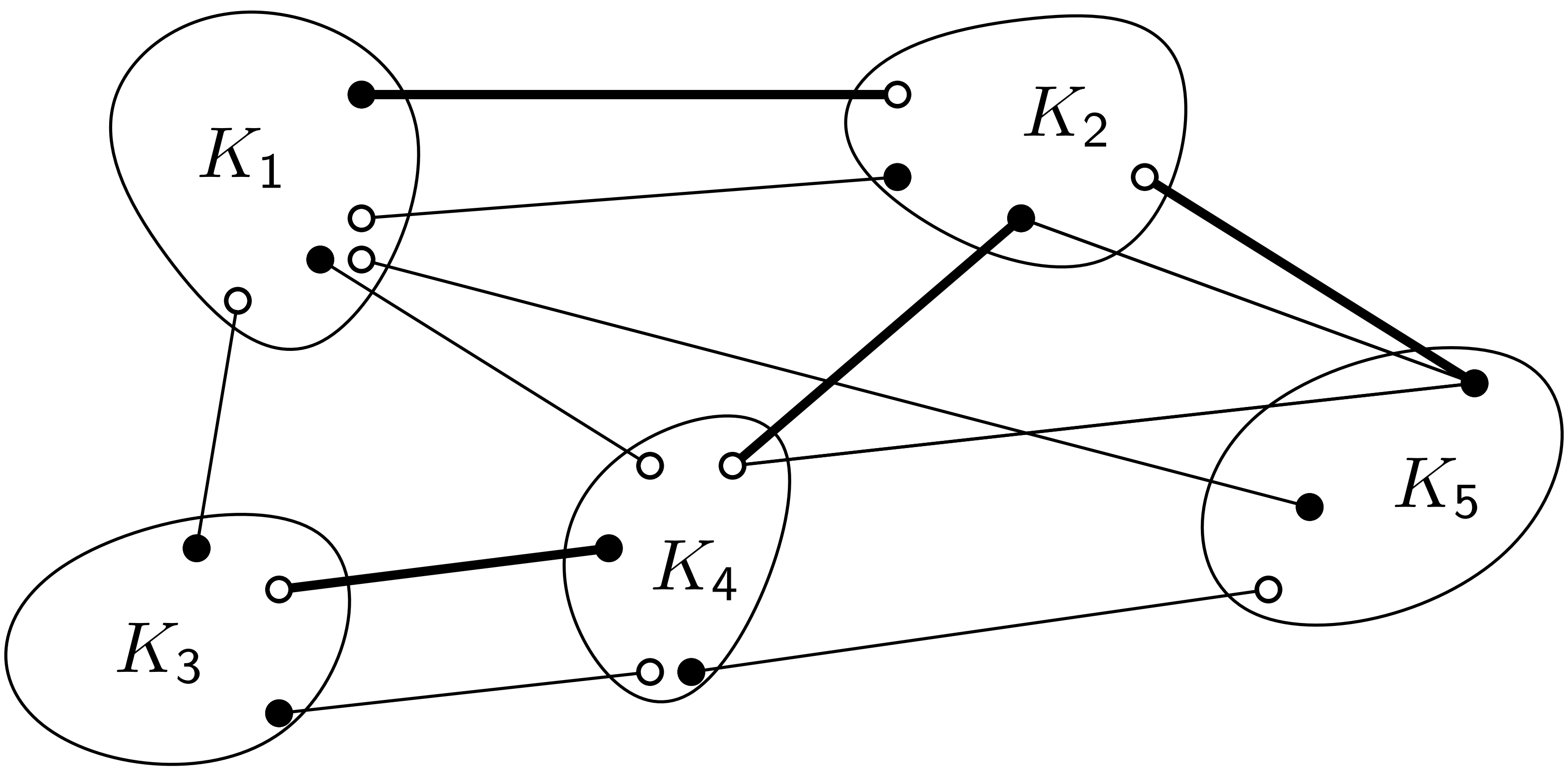


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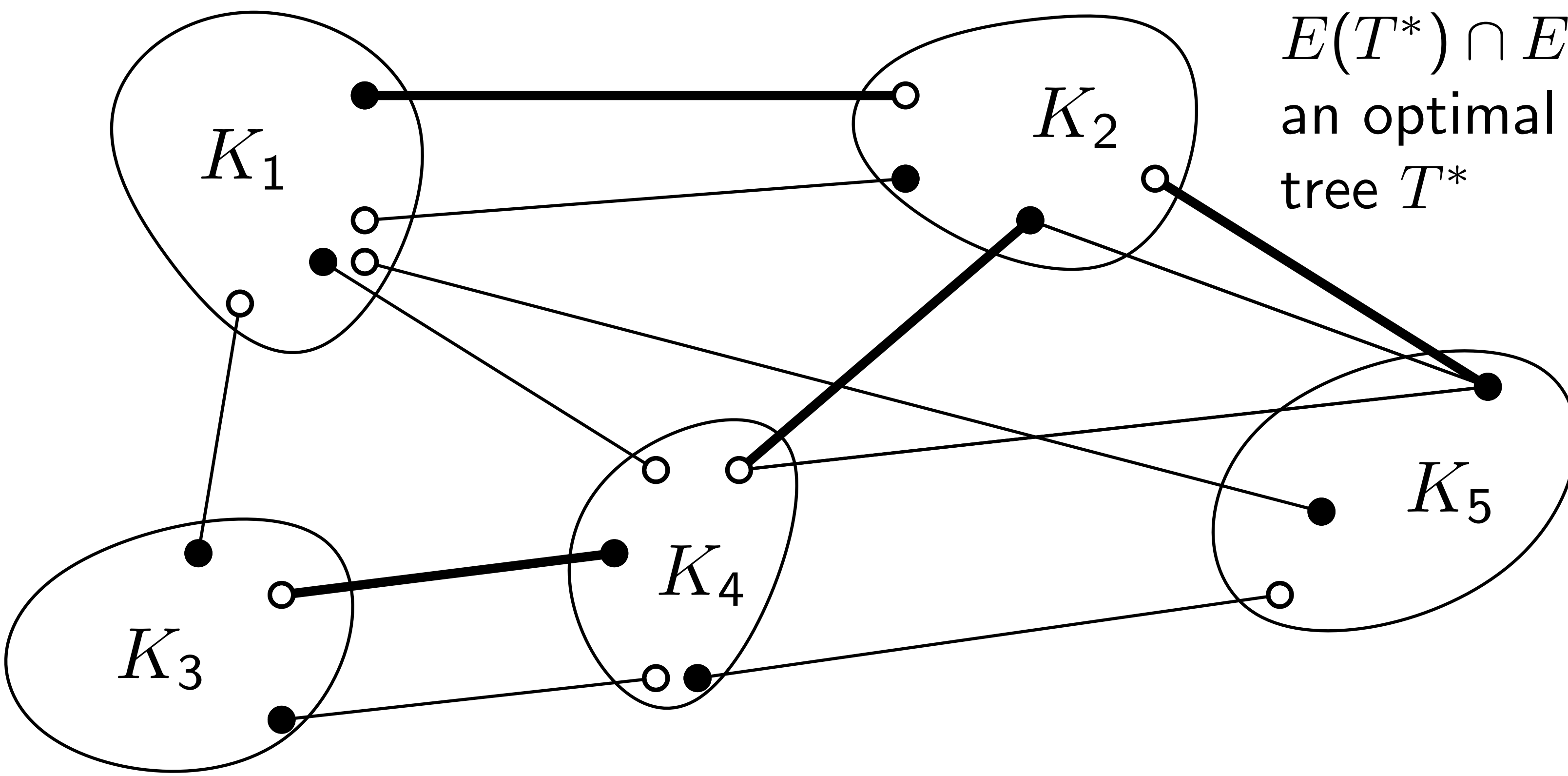


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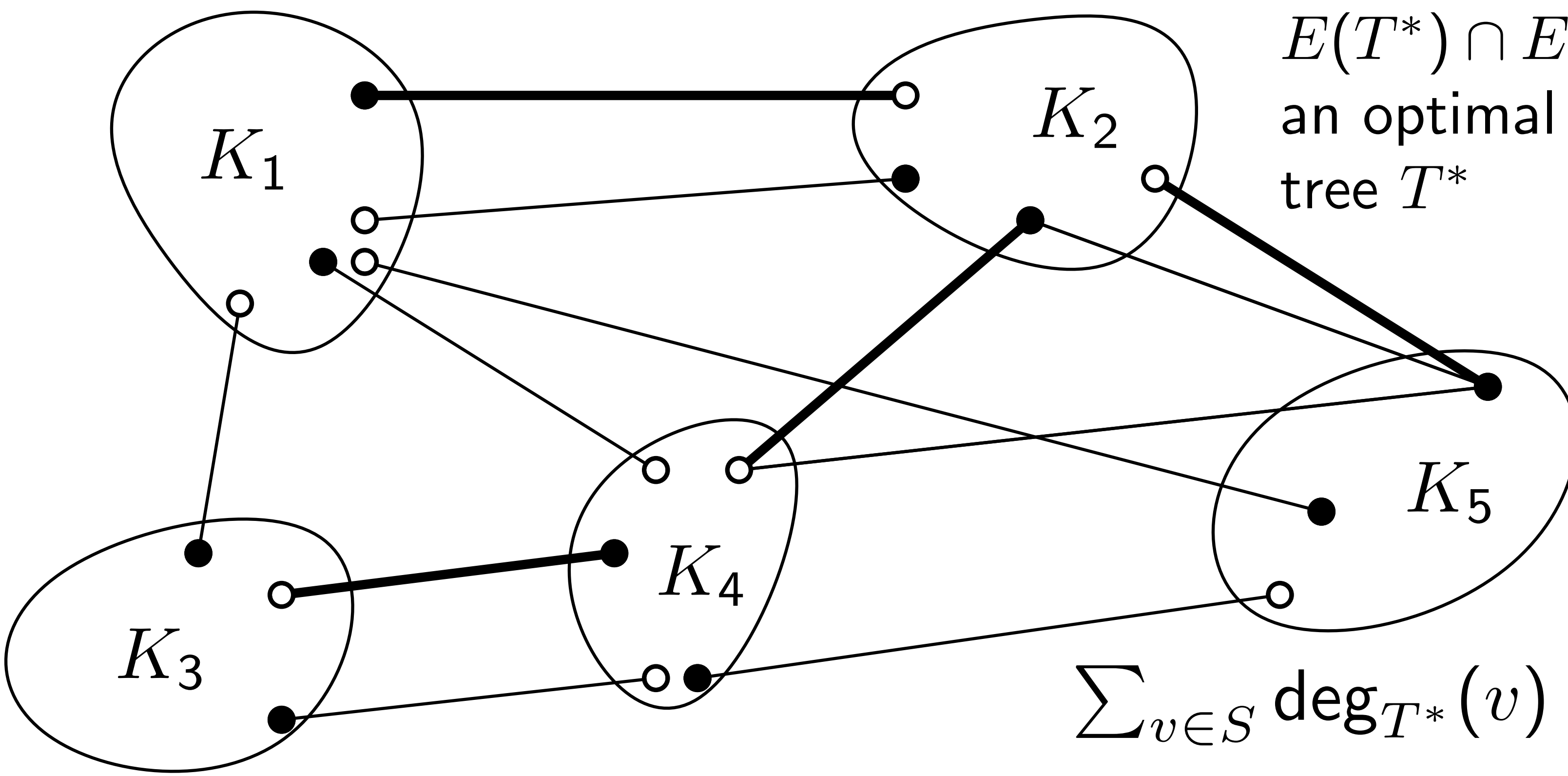


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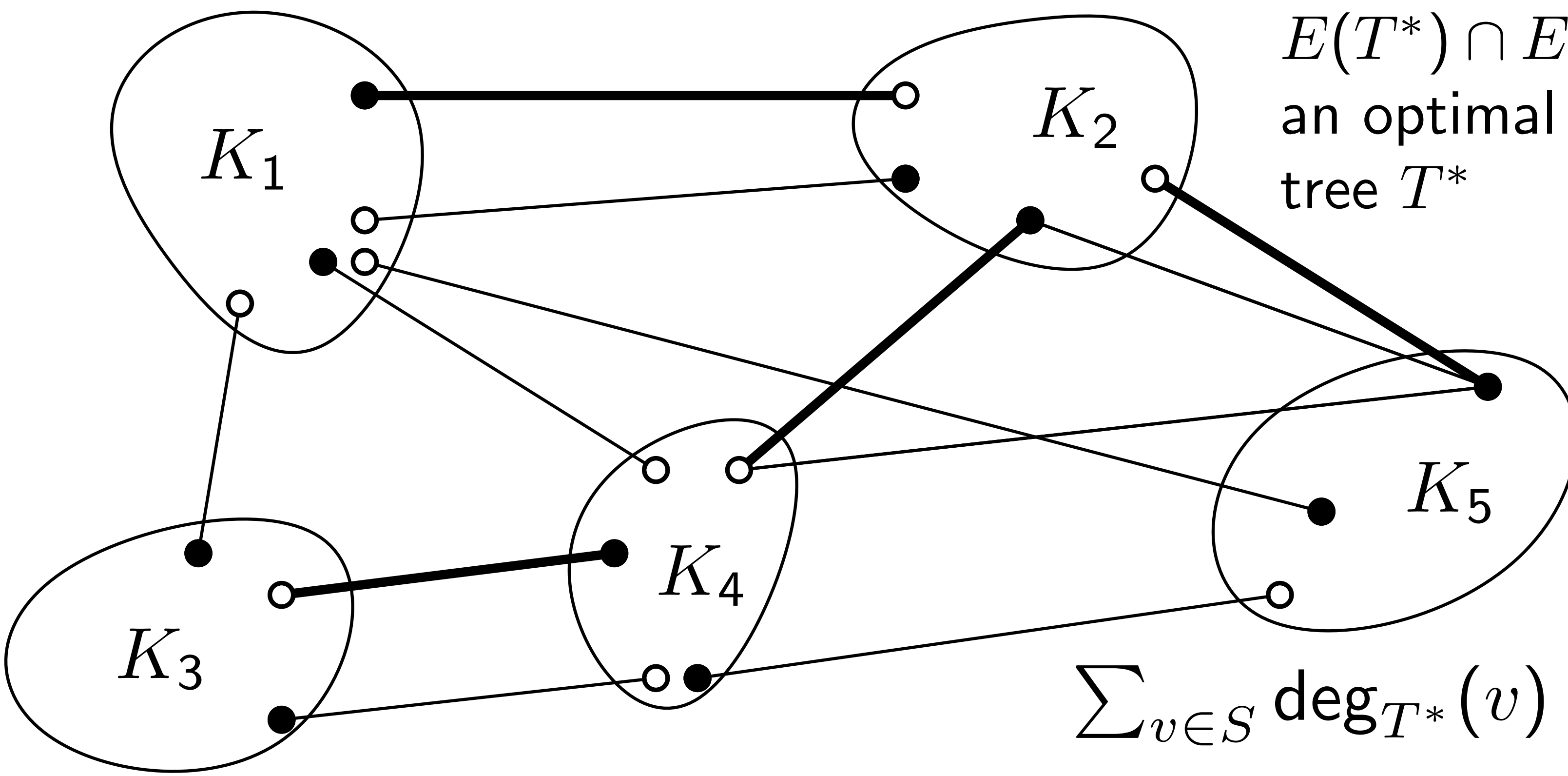


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**Proof.** Our potential function: $\Phi(T) = \sum_{v \in V(G)} 3^{d_T(v)}$

Via potenial function $\Phi(G, T)$. $\rightsquigarrow$ a function measuring the value of a solution where, e.g., :

- each iteration decreases the potential of a solution.

**Lemma:** each iteration $\Phi(T') \leq (1 - \frac{2}{27n^3})\Phi(T)$.

- the function is bounded both from above and below.

For any spanning tree $T$, $\Phi(T) \in [3n, n3^n]$.

- executing $f(n)$ iterations would execeed this lower bound.

How does $\Phi(T)$ change?

shrinks by: $(1 - \frac{2}{27n^3})^{f(n)} \leq (e^{-\frac{2}{27n^3}})^{f(n)}$

Goal $\rightsquigarrow$ after $f(n)$ iterations $\Phi(T) = n < 3n$

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# Extensions

**Cor.** For a constant $b > 1$, and $\ell = \lceil \log_b n \rceil$, the local search algorithm runs in polynomial time and produces a spanning tree $T$ where $\Delta(T) \leq b \cdot \mathsf{OPT} + \lceil \log_b n \rceil$.

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**Next Class:**
**Approximation Schemes:**
**$(1 + \epsilon)$-approximation**